

Encouraging Attacker Retreat through Defender Cooperation

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Abstract—This paper is motivated by a desire to develop analytic formulations for adversarial interactions between an attacker and a defensive team. We analyze a multi-stage, two-player game in which one player represents an attacker with superior dynamic characteristics and the other player represents a team consisting of a mobile, high-value target and N protective agents. At the start of the game, the attacker must decide whether to engage the target or retreat. The defending team must then decide whether to maximize or minimize the attacker's cost in response. These decisions are referred to as the players' intent. After each side has selected an intent, a differential pursuit-evasion game is played in which the value represents the integral cost to the attacker. Within the differential game, the terminal conditions and the players' optimal control strategies are dictated by the previous intent selections. We obtain the optimal intent strategies in terms of the differential game values and relevant bonus and penalty values. We solve the differential games by developing the optimality conditions for the equilibrium control strategies. We show that for certain conditions, the defenders should cooperate with the attacker so that retreat becomes the most attractive option; thereby, fulfilling the defensive goal of protecting the high-value target.

I. INTRODUCTION

The protection of vulnerable, high-value assets has been a challenge throughout history, but it is especially relevant in contemporary insurgent based conflicts. Using guerrilla tactics, insurgents often take advantage of easy opportunities to attack soft targets. These targets may be fixed (stockpiles, factories, and population centers) or mobile (transports, supply convoys, and VIPs). In either case, it is necessary to deploy defensive assets in an attempt to neutralize an attack if it occurs, or better yet, make the prospect of further engagement so unappealing that the attackers stand down and retreat. When the primary goal of a defensive team is to protect the high-value target, an early disengagement by the attacker is attractive because it will allow the defenders to reduce the risk of injury or damage to the high-value target as well as conserve resources for any future confrontations. Therefore, defenders must not only leave retreat as an option to the attacking forces, but should make retreat as attractive of an option as possible through cooperation when retreat occurs.

Game theory provides a powerful framework to analyze the conflicting interests of the defensive team's desire to prevent attack and the attackers' desire to successfully engage

the target with minimum cost. In this paper, we examine a two-player game in which one player represents an *attacker*, and the other player represents a *defensive team* that consists of a mobile, high-value *target* and N *protective agents*. It is assumed that the protective agents generate a cost to the attacker, which can represent casualties incurred, resources used, or the risk of injury or damage. It is also assumed that the attacker possesses superior performance capabilities, allowing it to successfully capture the target from all initial conditions. At the start of the game, the attacker must choose between engagement or retreat. After the attacker has made its decision, the defending team must then decide whether to maximize or minimize the attacker's cost in response. We will refer to the attacking and defending teams' choices as their intent. We also discuss the scenario in which the attacker and defensive team are allowed to update their intents throughout the game.

Once each side has selected an intent, a differential pursuit-evasion game is played in which the terminal conditions and the players' optimal control strategies are dictated by the intent selections. There are four variants of the differential game based on the four possible combinations of intent. If the attacker chooses to engage, the differential game terminates when the distance between the attacker and high-value target is equal to a predefined capture distance. If retreat is chosen by the attacker, the differential game terminates when all distances between the attacker and the protective agents are greater than or equal to a defined retreat distance. In all variants of the differential game, the value of the differential game represents the integral cost to the attacker.

It is assumed that both the attacker and defensive team can calculate the resulting integral cost of the four possible differential games from any initial condition. Additionally, when the attacker chooses to engage the defensive team, the attacker is awarded a bonus and the defensive team assessed a penalty. Using the values from the possible differential games in conjunction with the given capture bonus and penalty values, the optimal intent strategies for each player can be calculated. For certain conditions, it will be shown that it is optimal for the defensive team to cooperate with the attacker in retreat so that retreat becomes a more attractive option than engagement from the attackers perspective.

Differential game theory has been used for several decades to analyze pursuit-evasion games since its formal introduction by Isaacs [1]. In particular, there have been several

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papers that address combat using realistic dynamics [2], [3]. In these papers, pursuit and capture is the only condition in which the game terminates. There has been some work that allows the players to switch roles between evader or pursuer depending on the initial conditions [4]. In [5], a game is analyzed in which a team of UAVs attempts to postpone an attack by an aerial jammer on the communication channel. There has also been work done on defensive strategies within sequential games. In [6], the author examines a sequential game in which the defensive player must distribute limited resources in preparation for the opposing player's attack. It is assumed that the attacker will always engage, but higher-value assets can be protected through the proper allocation of resources. In the context of this literature, the primary distinguishing feature of our work is that the attacker is capable of successful capture from every initial position, but through the selection of appropriate control strategies, the defensive team makes retreat a more attractive strategy for the attacker.

Previously, we have analyzed a single-pursuer, two-evader game using simple motion [7] where the pursuer was allowed to capture either evader and the cost function was dependent on both evader distances and the angle between them. By restricting capture to a specific evader that generates no cost and introducing a cost function with particular convergence properties, we introduce the possibility of retreat in this paper. This results in not only pursuit behaviors but also retreat behaviors. Also, this paper differs from our previous paper in that we develop the optimality conditions in terms of an arbitrary number, N , of protective agents.

We develop the optimal intent selection strategies of the attacker and defensive team in Section II. Next, we describe the coordinate systems, system kinematics, and cost function of the resulting pursuit-evasion games. Following that, we define the optimality conditions of the game of attack and game of retreat in Sections IV and Section V respectively. Using the intent selection strategies and optimality conditions, we examine the resulting behaviors for three illustrative examples in Section VI. We discuss the effects of allowing the attacker and defending team to update their intents throughout the game in Section VII. Concluding remarks and future directions are presented in Section VIII.

II. INTENT SELECTION STRATEGIES

At the start of the game, each player must determine their intent for the entirety of the game. The attacker and defensive team's intent are represented by the discrete control variables I_A and I_D respectively. Once the players have made their selection, they cannot switch. The relaxation of this restriction is discussed later in Section VII. The selection of intent is performed in a two-step sequence. The attacker must first determine whether to engage, $I_A = i_E$, or retreat, $I_A = i_R$. After the attacker makes its selection, it is assumed the defensive team knows the attacker's intent and must then choose to maximize, $I_D = i^+$, or minimize, $I_D = i^-$, the attacker's cost in response.

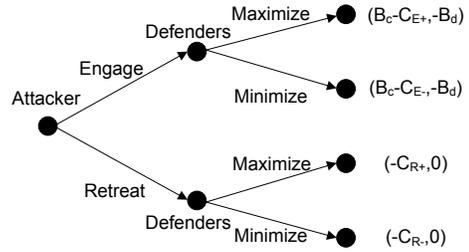


Fig. 1. Intent Selection Tree

Once the attacker and defensive team have made their intent selection, their respective utility functions, $U_A(I_A, I_D)$ and $U_D(I_A, I_D)$, are evaluated. The utility value pairs, (U_A, U_D) , are listed next to the right most nodes in Fig. 1. These values are based on the integral attacker cost in the four possible differential games as well as any relevant bonuses or penalties. The goal of each player is to maximize their respective utilities. The values C_{E+} and C_{E-} represent the cost of engagement to the attacker when the defensive team maximizes or minimizes the cost respectively. The values C_{R+} and C_{R-} represent the cost of retreat to the attacker when the defensive team maximize or minimize the cost respectively. It is assumed that $C_{E+} \geq C_{E-}$ and $C_{R+} \geq C_{R-}$. These assumptions will be verified, through the analysis of the differential games in the following sections. The quantity $B_c \geq 0$ represents the bonus the attacker receives for capturing the target, and $B_d \geq 0$ represents the penalty the defensive team receives when the target is captured. It is assumed that all of these values are known or can be calculated by both the attacker and the defensive team.

In this paper, the defending team's sole goal is to prevent capture of the high-value target in this one-shot game. We do not consider any future confrontations with the attacker. Since it is assumed that the attacker possesses superior capabilities that guarantee successful capture if engagement is selected, the defending team's only option is to make engagement so costly that it outweighs any bonus the attacker gains from capture. The defensive team gains no direct utility from inflicting cost on the attacker. Instead, the attacker's cost is used as a tool by the defensive team to discourage engagement. Also, it is assumed that any difference in resource usage by the defensive team between maximizing and minimizing attacker cost is negligible compared to the penalty incurred when the high-value target is captured. Therefore, defensive team utility is dependent only on the attacker's intent.

The following theorem delineates the optimal intent strategies for both the attacker and the defensive team as a function of the defined utilities.

Theorem 1: Let C_{E+} represent the cost of engagement to the attacker when the defensive team maximizes cost. Let C_{R-} represent the cost of retreat when the defensive team minimizes cost, and B_c denote the terminal bonus awarded to the attacker for engagement. The optimal intent strategies

for the attacker, I_A , and defensive team, I_D , are

$$I_P^* = \begin{cases} i_E & \text{if } B_c - C_{E^+} \geq -C_{R^-} \\ i_R & \text{if } B_c - C_{E^+} < -C_{R^-} \end{cases} \quad (1)$$

$$I_D^* = \begin{cases} i^+ & \text{if } I_A = i_E \\ i^- & \text{if } I_A = i_R \end{cases} \quad (2)$$

Proof: In order to calculate the optimal intent strategies, we begin by representing the intent selection process as the directed tree in Fig. 1. The upper branch of this tree represents the scenario in which the attacker has elected to engage and forms a subgame for the defensive team. In this subgame, choosing either to maximize or minimize yields the same utility for the defending team because the attacker is guaranteed successful capture due to its superior capabilities. Similarly, in the lower subgame formed when the attacker chooses to retreat, maximizing or minimizing yields equal utilities for the defending team. However, the defensive team does prefer that the attacker retreats since $0 \geq -B_d$. In order to encourage retreat, the defensive team must minimize attacker utility if engaged and maximize attacker utility in retreat, which is achieved by (2). The attacker assumes that the defending team poses a credible threat and will implement (2). The maximum attacker utility is then achieved by (1). ■

From Theorem 1, it can be seen that the values C_{E^+} , C_{R^-} , and B_c play a critical role in the calculation of the optimal player intent. The value C_{E^+} represents the value of the differential game of engagement when the defensive team is maximizing attacker cost, and the value C_{R^-} represents the differential game of retreat when the defensive team is minimizing attacker cost. For the remainder of the paper, we will formulate and define the optimality conditions of these differential games. The resulting solutions to these games will then be used to calculate the optimal intent strategies for given initial conditions and values of B_c .

III. SYSTEM AND DIFFERENTIAL GAME FORMULATION

In the system under consideration, the attacker is represented by a pursuer, and the defending team consists of the mobile, high-value target and N protective agents. The attacker, mobile target, and N protective agents will be denoted by P , E_0 , and E_i for $i = 1, \dots, N$, respectively. For the sake of brevity, we will often omit the clarification that $i = 1, \dots, N$. Therefore whenever a variable uses the subscript i , it is assumed that $i = 1, \dots, N$ unless explicitly stated otherwise.

A. Agent Kinematics

Each agent moves with simple motion and constant speed on an obstacle free plane. We will use two coordinate systems. The first coordinate system represents the location of each agent using a pair of Cartesian coordinates. The attacker location is represented by the pair (x_p, y_p) while (x_0, y_0) and (x_i, y_i) represent the positions of the mobile target and protective agents. The state of the system is completely defined by the $(2N+4)$ -tuple $\mathbf{x}_G = (x_p, y_p, x_0, y_0, \dots, x_N, y_N)$. We

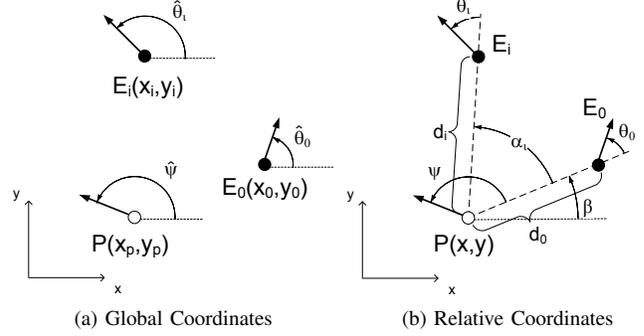


Fig. 2. Coordinate Systems

will refer to this representation as the *Global Coordinates*, and the resulting dynamic equations are as follows:

$$\dot{x}_p = v_p \cos \hat{\psi} \quad \dot{x}_0 = v_0 \cos \hat{\theta}_0 \quad \dot{x}_i = v_i \cos \hat{\theta}_i \quad (3)$$

$$\dot{y}_p = v_p \sin \hat{\psi} \quad \dot{y}_0 = v_0 \sin \hat{\theta}_0 \quad \dot{y}_i = v_i \sin \hat{\theta}_i \quad (4)$$

The constants v_p , v_0 , and v_i , represent the speeds of the attacker, the target, and the protective agents. The angles $\hat{\psi}$, $\hat{\theta}_0$, and $\hat{\theta}_i$ are the direction of travel and represent the control variables for each of their respective agents. All control variables are measured counterclockwise from the x-axis. This coordinate system is depicted graphically in Fig. 2a.

In order to simplify later optimality calculations, we will now introduce the *Reduced Coordinates*. In the reduced coordinate system, the state is completely defined by the $(2N+4)$ -tuple $\mathbf{x}_R = (d_0, d_1, \dots, d_N, \alpha_1, \dots, \alpha_N, \beta, x, y)$. The distance between the P and E_0 will be represented by d_0 . Similarly, the component d_i represents the distance between P and E_i . The angle measured counterclockwise from $\overrightarrow{PE_0}$ to $\overrightarrow{PE_i}$ is represented by the state component α_i . The angle β is measured counterclockwise from the x-axis to $\overrightarrow{PE_0}$ and represents the global rotation of the relative configuration of the agents. In order to simplify the dynamic equations, the heading angles ψ and θ_0 are measured counterclockwise from $\overrightarrow{PE_0}$, and the angles θ_i are measured counterclockwise from $\overrightarrow{PE_i}$. The remaining state components, x and y , are the global x and y position of the attacker and represent the global translational position of the $(N+2)$ -agent system. This coordinate system is depicted in Fig. 2b. The dynamic equations for the relative coordinate system are as follows:

$$\dot{d}_0 = v_0 \cos \theta_0 - v_p \cos \psi \quad (5)$$

$$\dot{d}_i = v_i \cos \theta_i - v_p \cos (\psi - \alpha_i) \quad (6)$$

$$\dot{\alpha}_i = \frac{v_i}{d_i} \sin \theta_i - \frac{v_0}{d_0} \sin \theta_0 + v_p \left(\frac{1}{d_0} \sin \psi - \frac{1}{d_i} \sin (\psi - \alpha_i) \right) \quad (7)$$

$$\dot{\beta} = \frac{1}{d_0} (\sin \theta_0 - v_p \sin \psi) \quad (8)$$

$$\dot{x} = v_p \cos (\psi + \beta) \quad (9)$$

$$\dot{y} = v_p \sin (\psi + \beta) \quad (10)$$

where $d_i > 0$ and $v_i < v_p$ for $i = 0, 1, \dots, N$. These inequalities require that all distances remain positive and

that P is faster than E_0 and E_i . The global and relative representations are related using the following:

$$x_p = x \quad y_p = y \quad (11)$$

$$x_p = d_0 \cos(\beta) + x \quad y_p = d_0 \sin(\beta) + y \quad (12)$$

$$x_i = d_i \cos(\beta + \alpha_i) + x \quad y_i = d_i \sin(\beta + \alpha_i) + y \quad (13)$$

The control variables are related as follows:

$$\hat{\theta}_0 = \theta_0 + \beta \quad \hat{\theta}_i = \theta_i + \beta + \alpha_i \quad \hat{\psi} = \psi + \beta \quad (14)$$

B. Instantaneous Cost Function

We now define the instantaneous cost to the attacker which is integrated over the total course of the game. This function can represent the risk of injury or the amount of damage that the attacker incurs at any instant in time. For this paper, we have chosen the following cost function:

$$C_T = c_0 + \sum_{i=1}^N c_i \frac{1}{d_i^2} \quad (15)$$

where c_i and c_0 are weighting parameters. As any $d_i \rightarrow 0$, the instantaneous cost explodes to infinity. As a result, it is impossible for the attacker to pass directly through any of the protective agents with finite cost. Therefore, the attacker must go around the protective agents in order to capture the target. Since it is assumed that the attacker has a speed advantage over all agents within the defensive team, the attacker will then guarantee that $d_i(t) > 0$ in order to maintain finite cost. Also, it will be shown in a later section that the game of retreat to an infinite distance has finite cost when $c_0 = 0$. Although this particular cost function was chosen because of its simplicity, more realistic and complex functions could be used in order to model particular attacker-defender interactions. If these more realistic cost functions possess characteristics similar to those just described, the resulting optimal agent behaviors will be similar to those developed in this paper.

C. Differential Game Formulation

Depending on the players' intent selection, various differential games can be formulated. In every game, the instantaneous cost function, (15), is integrated until the game terminates and represents the total cost to the attacker. The terminal conditions will be discussed in Section IV and Section V. The cost to the attacker for a game starting at t_0 and terminating at t_f is then defined as

$$V := \int_{t_0}^{t_f} C_T dt. \quad (16)$$

If the intent of the defending team is to maximize the attacker's cost, we can use the value function (16) to pose a two-player, zero-sum differential game. Although there are $N + 1$ agents within the defending team, they all share the same goal of maximizing the attacker's cost and can therefore be represented as one player with multiple control variables. If the intent of the defending team is to cooperate with the attacker and minimize cost, the differential game now reduces to a standard optimization problem with $N + 2$ control variables.

IV. OPTIMALITY CONDITIONS OF THE DIFFERENTIAL GAME OF ENGAGEMENT WITH DEFENDER MAXIMIZATION

In this section, we develop the solution for the game of engagement. In this game, the defensive team strives to maximize the attacker's integral cost over the course of the game. Simultaneously, the attacker attempts to minimize this cost. The game terminates when the distance between the attacker and high-value target, d_0 , is equal to the capture distance of d_c .

A. Hamiltonian and Adjoint Equations

We begin calculating the solution to the game of attack by constructing the Hamiltonian:

$$\begin{aligned} H &:= \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}, \boldsymbol{\psi}, \boldsymbol{\theta}) + C_T = 0 \quad (17) \\ &= \sum_{i=0}^N \lambda_{d_i} \dot{d}_i + \sum_{i=1}^N \lambda_{\alpha_i} \dot{\alpha}_i + \lambda_{\beta} \dot{\beta} + \lambda_x \dot{x} + \lambda_y \dot{y} + C_T. \end{aligned}$$

The vector $\boldsymbol{\lambda} := (\lambda_{d_0}, \dots, \lambda_{d_N}, \lambda_{\alpha_1}, \dots, \lambda_{\alpha_N}, \lambda_{\beta}, \lambda_x, \lambda_y)^T$ contains the adjoint variables conjugate to the kinematic equations. The adjoint equations are found by taking the partial derivative of the Hamiltonian with respect to each of the state components:

$$\dot{\lambda}_{d_0} = -\frac{\partial H}{\partial d_0} = -\sum_{i=1}^N \lambda_{\alpha_i} \frac{\partial \dot{\alpha}_i}{\partial d_0} - \lambda_{\beta} \frac{\partial \dot{\beta}}{\partial d_0} \quad (18)$$

$$\dot{\lambda}_{d_i} = -\frac{\partial H}{\partial d_i} = -\lambda_{\alpha_i} \frac{\partial \dot{\alpha}_i}{\partial d_i} - \frac{\partial C_T}{\partial d_i} \quad (19)$$

$$\dot{\lambda}_{\alpha_i} = -\frac{\partial H}{\partial \alpha_i} = -\lambda_{d_i} \frac{\partial \dot{d}_i}{\partial \alpha_i} - \lambda_{\alpha_i} \frac{\partial \dot{\alpha}_i}{\partial \alpha_i} \quad (20)$$

$$\dot{\lambda}_{\beta} = -\frac{\partial H}{\partial \beta} = -\lambda_x \frac{\partial \dot{x}_p}{\partial \beta} - \lambda_y \frac{\partial \dot{y}_p}{\partial \beta} \quad (21)$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x_p} = 0 \quad (22)$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y_p} = 0. \quad (23)$$

B. Boundary Conditions

Using the definition of capture, $d_0 = d_c$, the boundary conditions, $\boldsymbol{\Psi}_A$, for the game of attack are

$$\begin{aligned} \boldsymbol{\Psi}_A &:= (d_0(t_f) - d_c, \\ &d_0(t_0) - d_{00}, \dots, d_N(t_0) - d_{N0}, \\ &\alpha_1(t_0) - \alpha_{10}, \dots, \alpha_N(t_0) - \alpha_{N0}, \\ &\beta(t_0) - \beta_0, x(t_0) - x_0, y(t_0) - y_0)^T \quad (24) \end{aligned}$$

where d_{i0} , α_{i0} , β_0 , x_0 , and y_0 are the initial values of their respective state components at the start of the game. We can then construct a function of terminal conditions, $\boldsymbol{\Phi}_A = \boldsymbol{\nu}^T \boldsymbol{\Psi}_A$ where $\boldsymbol{\nu}$ is a vector of Lagrange multipliers corresponding to the boundary conditions.

The terminal values of the adjoint variables are found by taking the partial derivative of $\boldsymbol{\Phi}_A$ with respect to each of the state components:

$$\lambda_{d_0}(t_f) = \frac{\partial \Phi}{\partial d_0(t_f)} = \nu_1 \quad \lambda_{\beta}(t_f) = \frac{\partial \Phi}{\partial \beta(t_f)} = 0 \quad (25)$$

$$\lambda_{d_i}(t_f) = \frac{\partial \Phi}{\partial d_i(t_f)} = 0 \quad \lambda_x(t_f) = \frac{\partial \Phi}{\partial x(t_f)} = 0 \quad (26)$$

$$\lambda_{\alpha_i}(t_f) = \frac{\partial \Phi}{\partial \alpha_i(t_f)} = 0 \quad \lambda_y(t_f) = \frac{\partial \Phi}{\partial y(t_f)} = 0 \quad (27)$$

Combining the terminal values of $\lambda_\beta(t_f)$, $\lambda_x(t_f)$, and $\lambda_y(t_f)$ with their respective adjoint equations (21)-(23), it can be seen that

$$\lambda_\beta(t) = 0 \quad \lambda_x(t) = 0 \quad \lambda_y(t) = 0. \quad (28)$$

Using (28), we can simplify the Hamiltonian (17):

$$H = \sum_{i=0}^N \lambda_{d_i} \dot{d}_i + \sum_{i=1}^N \lambda_{\alpha_i} \dot{\alpha}_i + C_T = 0. \quad (29)$$

Using the reduced Hamiltonian (29), the optimal control strategies for each of the agents are calculated in the following theorem. Given $(a, b) \in \mathbb{R}^2$, the notation $\angle(a, b)$ is defined to be the unique value $\theta \in (0, \pi]$ such that

$$\cos \theta = \frac{a}{\sqrt{a^2+b^2}} \quad \sin \theta = \frac{b}{\sqrt{a^2+b^2}}. \quad (30)$$

Theorem 2: Suppose that the value function and the value function gradient are continuous. The control strategies for the agents are then given by

Optimal Maximizing Control Strategy of E_0 and E_i :

$$\theta_0^* = \angle(\lambda_{d_0}, -\sum_{i=1}^N \frac{\lambda_{\alpha_i}}{d_i}) \quad \theta_i^* = \angle(\lambda_{d_i}, \frac{\lambda_{\alpha_i}}{d_i}) \quad (31)$$

Optimal Minimizing Control Strategy of P:

$$\psi^* = \angle(-b_1, -b_2) \quad (32)$$

where

$$b_1 = \sum_{i=1}^N \left(\frac{\lambda_{\alpha_i}}{d_i} \sin \alpha_i - \lambda_{d_i} \cos \alpha_i \right) - \lambda_0 \quad (33)$$

$$b_2 = \sum_{i=1}^N \left(\frac{\lambda_{\alpha_i}}{d_0} - \lambda_{d_i} \sin \alpha_i + \frac{\lambda_{\alpha_i}}{d_i} \cos \alpha_i \right) \quad (34)$$

Proof: Along the optimal trajectories, the Hamiltonian must satisfy the following conditions [8]:

$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \psi^*) \leq H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\theta}^*, \psi^*) \leq H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\theta}^*, \psi) \quad (35)$$

$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\theta}^*, \psi^*) = 0 \quad (36)$$

where $\boldsymbol{\theta} = (\theta_0, \dots, \theta_N)$. From (35) we find that

$$\psi^* = \arg \min_{\psi} H \quad \theta_0^*, \dots, \theta_N^* = \arg \max_{\theta_0, \dots, \theta_N} H \quad (37)$$

Because the control variables are unbounded, the optimal strategies of (37) must satisfy the following conditions:

$$\frac{\partial H}{\partial \psi} = 0 \quad \frac{\partial H}{\partial \theta_i} = 0 \quad \text{for } i = 0, \dots, N \quad (38)$$

$$\frac{\partial^2 H}{\partial \psi^2} \geq 0 \quad \frac{\partial^2 H}{\partial \theta_i^2} \leq 0 \quad \text{for } i = 0, \dots, N. \quad (39)$$

The first set of conditions, (38), guarantees the Hamiltonian is stationary with respect to the control variables. The second set of equations (39), represent the necessary second-order conditions so that $\boldsymbol{\theta}$ maximizes and ψ minimizes. Solving (38) and (39), in terms of $\theta_0, \dots, \theta_N$, and ψ provide our optimal control strategies (31) and (32). ■

V. OPTIMALITY CONDITIONS OF THE DIFFERENTIAL GAME OF RETREAT WITH DEFENDER MINIMIZATION

In this game, the attacker is attempting to reach the retreat condition with minimal integral cost. Definition of the retreat condition requires the use of the minimum of N quantities. In principle, it is possible to do this, but since the minimum function is not differentiable everywhere, there is a large number of singular surfaces which make analysis of the game complicated. Instead, we define the retreat condition, $d_m(t_f) - d_r = 0$, using the p -norm with respect to the $\frac{1}{d_i}$'s corresponding to the protective agents where

$$d_m := \left(\sum \frac{1}{d_i^k} \right)^{-\frac{1}{k}} \quad \text{and } k > 1. \quad (40)$$

Since we restrict $d_i > 0$, the function is differentiable everywhere within the admissible state space. As $k \rightarrow \infty$, d_m converges to $(\max(\frac{1}{d_1}, \dots, \frac{1}{d_N}))^{-1} = \min(d_1, \dots, d_N)$ [9]. The attacker and defending team are both minimizing the cost function. Therefore, the differential game reduces to a standard optimal control problem with respect to all agents.

A. Hamiltonian and Adjoint Equations

Since the dynamics and cost function are the same as in the game of attack, the game of retreat has an identical Hamiltonian (17) and resulting adjoint equations (18)-(23).

B. Boundary Conditions

Using the condition of retreat, $d_m(t_f) - d_r = 0$, we can form the boundary conditions, $\boldsymbol{\Psi}_R$, for the game of retreat:

$$\begin{aligned} \boldsymbol{\Psi}_R := & (d_m(t_f) - d_r, \\ & d_0(t_0) - d_{00}, \dots, d_N(t_0) - d_{N0}, \\ & \alpha_1(t_0) - \alpha_{10}, \dots, \alpha_N(t_0) - \alpha_{N0}, \\ & \beta(t_0) - \beta_0, x(t_0) - x_0, y(t_0) - y_0)^T \end{aligned} \quad (41)$$

where $d_{00}, \dots, d_{N0}, \alpha_1, \dots, \alpha_N, \beta_0, x_0$, and y_0 are defined the same as in the previous section. After constructing a function of boundary conditions, $\boldsymbol{\Phi}_R := \boldsymbol{\nu}^T \boldsymbol{\Psi}_R$, and taking partials with respect to each of the state components, we have the terminal constraints on the adjoint variables:

$$\lambda_x(t_f) = \frac{\partial \Phi}{\partial x(t_f)} = 0 \quad \lambda_\beta(t_f) = \frac{\partial \Phi}{\partial \beta(t_f)} = 0 \quad (42)$$

$$\lambda_{\alpha_i}(t_f) = \frac{\partial \Phi}{\partial \alpha_i(t_f)} = 0 \quad \lambda_y(t_f) = \frac{\partial \Phi}{\partial y(t_f)} = 0 \quad (43)$$

$$\lambda_{d_0}(t_f) = \frac{\partial \Phi}{\partial d_0(t_f)} = 0 \quad (44)$$

$$\lambda_{d_i}(t_f) = \frac{\partial \Phi}{\partial d_i(t_f)} = \nu_1 \frac{1}{d_i(t_f)^{k+1}} \left(\sum \frac{1}{d_i(t_f)^k} \right)^{-\frac{1-k}{k}} \quad (45)$$

As in the game of attack, the adjoint variables corresponding to the β , x , and y components of the state are always zero (28) and we can further reduce the Hamiltonian as before. Using the reduced Hamiltonian, (29), we can now calculate the optimal control strategies for each of the agents in terms of the state and adjoint variables.

Theorem 3: Suppose that the value function and the value function gradient are continuous. The control strategies for the agents are then given by

Optimal Minimizing Control Strategy of E_0 and E_i :

$$\theta_0^* = \angle(-\lambda_{d_0}, \sum_{i=1}^N \frac{\lambda_{\alpha_i}}{d_i}) \quad \theta_i^* = \angle(-\lambda_{d_i}, -\frac{\lambda_{\alpha_i}}{d_i}) \quad (46)$$

Optimal Minimizing Control Strategy of P:

$$\psi^* = \angle(-b_1, -b_2) \quad (47)$$

where the terms b_1 and b_2 are defined the same as in Theorem 2.

The proof is omitted due to space constraints. It follows the same approach as Theorem 2.

VI. ILLUSTRATIVE EXAMPLES

In this section, we will examine three specific cases. We will look at the optimal trajectories of the differential games of attack and retreat and the conditions for the resulting optimal intent strategies based on the games' values.

A. Numerical Analysis of Game of Engagement

In most cases, finding an analytic solution to the optimal trajectories for the differential subgames is not practical due to the nonlinear and coupled nature of the state and adjoint equations. In order to numerically generate the optimal trajectories that result from the previously developed optimality conditions, we first substitute the optimal control strategies into the kinematic equations (5)-(10) and the adjoint equations (18)-(23). This results in a system of $4N + 6$ ordinary differential equations in addition to the integral cost function. These equations can be numerically integrated backwards in time from any permissible point on the terminal surface for a defined timespan or until the trajectory crosses a singular or dispersal surface, which will be discussed in a later section.

In the game of engagement, we can completely define the terminal conditions. After substituting the optimal control strategies (31)-(32) into the Hamiltonian (17) and evaluating at the point of capture, we can solve directly for $\lambda_{d_0}(t_f) = \frac{C_T(t_f)}{v_p - 1}$. It can also be seen that the protective agents' terminal control angles are undefined at the moment of capture due to the fact that $\lambda_{d_i}(t_f) = \lambda_{\alpha_i}(t_f) = 0$. Conceptually this makes sense because at the moment of capture, the protective agents cannot prevent the capture of E_0 . Also, any increase in the cost function C_T will not be integrated because the game will terminate. It is still necessary to define a terminal control for E_i in order to take the first step of integration. For this value, we will use the limit of E_i 's control as t approaches t_f . Taking the limit of $\tan \theta_i$ yields

$$\lim_{t \rightarrow t_f} \tan \theta_i^*(t) = \lim_{t \rightarrow t_f} \frac{\lambda_{\alpha_i}}{d_i \lambda_{d_i}} = \lim_{t \rightarrow t_f} \frac{\lambda_{\alpha_i}}{d_i \lambda_{d_i} + d_i \lambda_{d_i}} = \frac{0}{2}. \quad (48)$$

The combination of (48) and the fact that $\dot{\lambda}_{d_i}(t_f) = \frac{2}{d_i(t_f)^3} > 0$ implies that $\lim_{t \rightarrow t_f} \theta_i^*(t_f) = \pi$. We now have a complete set of terminal values for the state, the adjoint variables, and control, which allow us to initialize the numerical integration. We can then use shooting techniques to solve for particular initial conditions.

B. Numerical and Analytic Solution to the Game of Retreat

For an arbitrary number of defending agents $N > 1$ and a finite retreat distance d_r , the same numerical shooting methods as in the previous section are used to solve for the optimal agent trajectories. In this case, the minimizing defender control (46) is substituted into the dynamic and adjoint equations in order to generate the system of $4N + 6$ differential equations. Additionally, the terminal retreat surface and corresponding adjoint conditions are used for the terminal constraints.

When $N = 1$, an analytic solution to the game of retreat can be calculated. First, the terminal condition of retreat reduces to $d_1(t_f) - d_r = 0$. After substituting the terminal constraints of the state and adjoint variables into the Hamiltonian, we can solve directly for $\lambda_{d_1}(t_f) = -\frac{c_0 + c_1 d_r^2}{d_r^2(1 + v_p)} \leq 0$. Using the terminal values of the adjoint variables and state, we can also find the terminal control of E_1 and the attacker: $\theta_1(t_f) = 0$ and $\psi(t_f) = \alpha + \pi$. Substituting the terminal control into the adjoint derivatives evaluated at the terminal surfaces, $d_2(t_f) = d_r$, yields $\dot{\lambda}_1(t_f) = 0$, $\dot{\lambda}_2(t_f) = \frac{2c_0}{d_r^3}$, and $\dot{\lambda}_\alpha(t_f) = 0$. After integrating backwards in time, we find that

$$\lambda_1(t) = 0 \quad \lambda_2(t) < 0 \quad \lambda_\alpha(t) = 0. \quad (49)$$

For the entire game of retreat, the optimal control of E_0 is undefined because E_0 has no effect on the cost function or when the game terminates. Therefore any control strategy is trivially optimal, and we will assume that $\theta_0(t) = 0$. From (49) we find the optimal control strategies of E_1 and P: $\cos \theta_1^*(t) = 0$ and $\cos \psi^*(t) = \alpha + \pi$. We can then calculate the optimal trajectory of the d_1 -component from any initial condition:

$$d_1^*(t) = d_{10} + (1 + v_p)t. \quad (50)$$

Assuming that the initial distance, d_{10} , is less than the retreat distance, the terminal time is calculated using (50): $t_f = \frac{d_r - d_{10}}{1 + v_p}$. We can then calculate the value of the game:

$$\begin{aligned} V(d_{10}) &= \int_{t_0}^{t_f} \frac{c_1}{d_1(t)^2} + c_0 = \int_0^{t_f} \frac{c_1}{(d_{10} + (1 + v_p)t)^2} + c_0 \\ &= t_f \left(c_0 + \frac{c_1}{d_{10}(d_{10} + (1 + v_p)t_f)} \right). \end{aligned} \quad (51)$$

For the special case where $c_0 = 0$ and $d_r \rightarrow \infty$, the value of the game of infinite retreat converges:

$$\begin{aligned} \lim_{d_r \rightarrow \infty} V(d_{10} | c_0 = 0) &= \lim_{t_f \rightarrow \infty} \frac{t_f c_1}{d_{10}(d_{10} + (1 + v_p)t_f)} \\ &= \frac{c_1}{d_{10}(1 + v_p)}. \end{aligned} \quad (52)$$

C. Singular Surfaces

Within this game, there are certain configurations in which either the attacker or defending agents' optimal control is not uniquely defined. This is typically a result of symmetry within the dynamics. For example, when the attacker, target, and defending agents are in a collinear arrangement, the attacker can choose to rotate clockwise or counterclockwise.

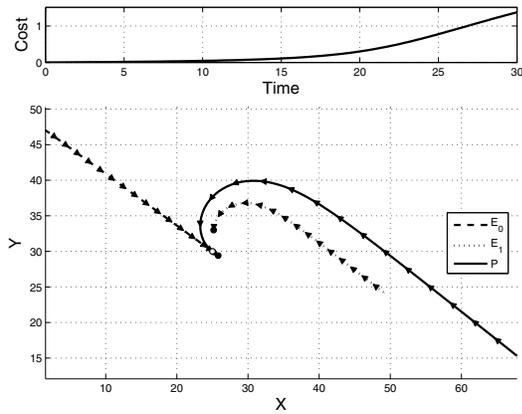


Fig. 3. Attack Trajectories Scenario 1

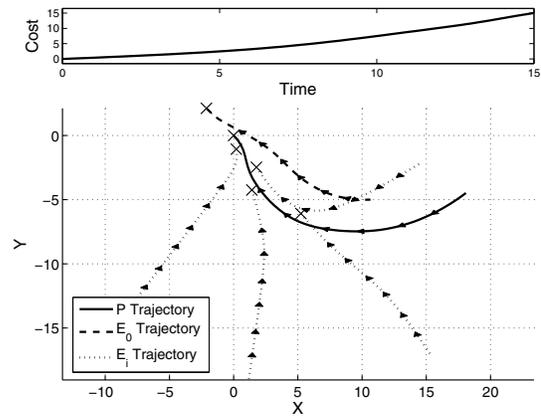


Fig. 5. Attack Trajectories Scenario 3

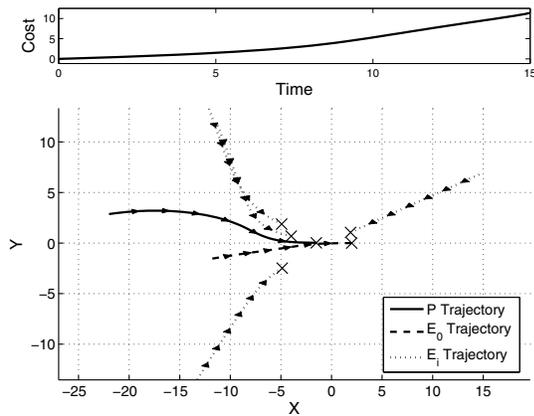


Fig. 4. Attack Trajectories Scenario 2

Either option is optimal. These regions of the state-space are referred to as singular surfaces and are characterized by discontinuities of the value function or its gradient. In this paper, we only consider initial conditions on singular surfaces known as dispersal surfaces in which the state instantaneously moves off and the optimal control is then uniquely defined. Full analysis of singular surfaces is beyond the scope of this paper and is topic of our current research.

D. Optimal Intent Selection

After the differential games are solved for particular initial conditions, we may use the values of game of engagement and game of retreat for C_{E+} and C_{R-} in (1) and (2). Also, we know that the assumptions $C_{E+} \geq C_{E-}$ and $C_{R+} \geq C_{R-}$ hold from Theorem 2 and Theorem 3. Therefore, for any given B_c , we can calculate the optimal intent strategies using Theorem 1. For a fixed value of B_c , the state space can be divided into two regions. In one region, the capture bonus, B_c , offsets the cost of attack, C_{E+} , and the attacker elects to engage despite the maximization of cost by the defensive team. In the remainder of the state space, the maximized cost of attack negates any benefit that the capture bonus grants when compared to the minimized cost of retreat. Therefore, the attacker elects to retreat and the defensive team cooperates to minimize the cost.

E. Illustrative Cases

In the following scenarios, we set the system parameters $d_c = 1$, $v_0 = v_i = 1$, $c_0 = 0$, and $c_i = 1$. A shooting method is used in order to solve for the initial value of the adjoint variables in the game of engagement as well as the game of finite retreat for $N > 1$. In Fig. 3 through Fig. 5 the trajectory of the attacker is the solid line, the trajectory of the mobile target is the dashed line, and the trajectory of the protective agent is the dotted line. The plots of the retreat trajectories are omitted due to space constraints. All trajectories are plotted in the global coordinates using (11)-(13) to convert from the relative coordinate system.

In Scenario 1, there is one defending agent, $N = 1$, and the retreat distance is taken to infinity. The attacker has a moderate speed advantage, $v_p = 2$. This scenario is shown in Fig. 3. Since the attacker starts far away from the protective agent, very little cost is generated early in the pursuit. The protective agent forces the attacker to come close mid-pursuit in order to outflank it and capture the target. The resulting cost of engagement and retreat for this scenario are 1.4 and .02 respectively. Engagement is optimal when $B_c \geq 1.38$.

In Scenarios 2 and 3, there are four defending agents, $N = 4$, and the retreat distance is taken to be $d_r = 20$. As can be seen in Figure 4 and Figure 5, the multiple protective agents converge on the pursuer from multiple directions forcing the attacker to weave through them. In Scenario 2, the costs of engagement and retreat are 10.4 and 1.7 respectively. Engagement is optimal when $B_c \geq 8.7$. In Scenario 3, the costs of engagement and retreat are 16.2 and 2.3 respectively. Engagement is optimal when $B_c \geq 13.9$.

VII. DISCUSSION ON THE RELAXATION OF INTENT SELECTION RESTRICTION

The problem solved in the previous sections was based on the assumption that the attacking agent and the defending team select an intent were not permitted to alter their intent selections for the remainder of the game. A natural extension would be to examine the case in which this restriction is relaxed, and the agents are allowed to re-evaluate or switch their respective intents throughout the game.

In this section, we propose a retaliatory defensive team control strategy dependent on attacker behavior and allow the attacker to switch intent at any time during the game. Let $V_E(\mathbf{x}_0)$ and $V_R(\mathbf{x}_0)$ represent the values of the differential games of engagement and retreat as described in Section IV and V with initial state \mathbf{x}_0 . Let ψ_E^* and ψ_R^* be the optimal attacker control for the game of engagement and retreat, and let θ_+^* and θ_-^* be the optimal defender control for the games of engagement and retreat respectively. Additionally, let $\mathbf{x}_R^*(t, \mathbf{x}_0)$ and $\mathbf{x}_E^*(t, \mathbf{x}_0)$ represent the state of the system at time t during these games when started at initial state \mathbf{x}_0 at time t_0 . Assume that the value functions, optimal control strategies, and optimal trajectories are known or can be calculated by the attacker and the defending team.

We now allow the attacker to change its intent at any moment during the game. We will assume that the defensive team is capable of detecting a deviation by the attacker from its optimal retreat strategy. A deviation by the attacker from its optimal retreat trajectory is represented in the variable ρ . A value of $\rho = 0$ indicates that the attacker has maintained its optimal retreat strategy while a value of $\rho = 1$ indicates that the attacker has deviated from its optimal retreat strategy. At initial state \mathbf{x}_0 if $V_E(\mathbf{x}_0) + B_c \leq V_R(\mathbf{x}_0)$, the variable ρ is set to zero with the intuitive meaning that the attacker is going to employ its optimal strategy of retreat. If this inequality does not hold, the variable ρ is set to one. If at anytime, the attacker deviates from its optimal retreat strategy, the variable ρ is set to one for the remainder of the game. The idea is that a deviation of the attacker from its optimal retreat strategy indicates that cooperation is inappropriate and the defending team should maximize attacker's cost.

Using the variable ρ , we propose the following retaliatory control strategy by the defensive team: Retaliatory Defensive Control Law

$$\boldsymbol{\theta} = \begin{cases} \boldsymbol{\theta}_-^* & \rho = 0 \\ \boldsymbol{\theta}_+^* & \rho = 1 \end{cases} . \quad (53)$$

In the following theorems, we will assume that the defensive team implements this retaliatory control law and the attacking agent is free to re-evaluate at any moment. The proofs of Theorem 4 and Theorem 5 are omitted due to space constraints.

Theorem 4: If

$$V_E(\mathbf{x}_R^*(t, \mathbf{x}_0)) + B_c < V_R(\mathbf{x}_R^*(t, \mathbf{x}_0)) \quad \forall t \in [t_0, t_f], \quad (54)$$

the optimal attacker strategy is to retreat for the entire game. This strategy will produce the optimal retreat trajectories $\mathbf{x}_R^*(t, \mathbf{x}_0)$ and game value $V_R(\mathbf{x}_0)$.

Theorem 5: Let \mathbf{x}_0 be the initial state. If

$$V_E(\mathbf{x}_0) + B_c > V_R(\mathbf{x}_0), \quad (55)$$

the attackers optimal strategy is to engage for the entire game. This strategy will produce the optimal engagement trajectories $\mathbf{x}_E^*(t, \mathbf{x}_0)$ and game value $V_E(\mathbf{x}_0) + B_c$.

Using $V_E(\mathbf{x})$ and $V_R(\mathbf{x})$, we can divide the admissible regions of state space into two regions: $\mathbf{X}_E := \{\mathbf{x} : V_E(\mathbf{x}) + B_c > V_R(\mathbf{x})\}$ and $\mathbf{X}_R := \{\mathbf{x} : V_E(\mathbf{x}) + B_c <$

$V_R(\mathbf{x})\}$. From Theorem 5, we know that if the defending team implements the modified retaliatory control law, any game that starts in \mathbf{X}_E will follow the original optimal trajectory $\mathbf{x}_E^*(t, \mathbf{x}_0)$. Thus, our original solution technique is sufficient. Similarly, if we find that a generated optimal retreat trajectory $\mathbf{x}_R^*(t, \mathbf{x}_0)$ stays within \mathbf{X}_R , we can conclude that $V_E(\mathbf{x}_R^*(t, \mathbf{x}_0)) < V_R(\mathbf{x}_R^*(t, \mathbf{x}_0)) \quad \forall t \in [t_0, t_f]$. Therefore, we know from Theorem 4 that the original retreat trajectory $\mathbf{x}_R^*(t)$ will still be optimal and our restricted solution technique remains valid.

This leaves the situation in which the initial state starts in \mathbf{X}_R , but the optimal retreat trajectory $\mathbf{x}_R^*(t, \mathbf{x}_0)$ passes into \mathbf{X}_E . The moment the state moves into this region, the attacker could switch its control strategy to ψ_E^* since the value of engagement would now be greater than the value of retreat. Therefore, neither our original retreat nor engagement solutions would be valid. A modified solution technique for this region is the focus of our current work.

VIII. CONCLUSION

In this paper, we have posed a multi-stage game in which an attacker must choose whether or not to engage a target protected by defensive assets. The defending team must determine whether to maximize or minimize the attacker's cost. The developed optimal intent strategies are a function of the resulting values of the differential game of engagement and the cooperative game of retreat. We then examined various optimal trajectories and values generated by different combinations of attacker and defensive team intent. Using the solutions from the case of restricted intent selection, we introduced a retaliatory control strategy for the defensive team, which in combination with the optimal attacker strategy yields equivalent solutions for particular initial conditions. In the future, we would like to incorporate more realistic dynamics and an arbitrary number of attacking agents.

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