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Robust $\mathscr{H}_\infty$ Filter Design for Polytopic Linear Discrete-Time Delay Systems via LMIs and Polynomial Matrices

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***Abstract*— This paper presents new robust linear matrix inequality conditions for full order robust $\mathscr{H}_\infty$ filter design of discrete-time polytopic linear systems affected by a time-varying delay. Thanks to the use of a larger number of slack variables, the proposed conditions are less conservative than the existing methods. Numerical experiments illustrate the better performance of the proposed filter design procedure when compared to other approaches available in the literature.**

I. Introduction

Time-delay systems have received intensive research efforts in the last years, mainly due to the increasing of digital systems that are affected by delays. As discussed in [1], time-delays can cause instability or performance degradation of control systems. There are many works dealing with control design [2–5] and with stability analysis [6–9] of time-delay systems. The filtering problem for time-delay systems has been investigated by many authors in different contexts [10–14]. It is also worth to mention the recent strategy proposed in [15], where the time-delay interval is partitioned in several segments.

The robust filter design for discrete-time uncertain systems has been addressed through several papers with different performance criteria, using quadratic stability [16–18] and affine parameter-dependent Lyapunov matrices [19, 20]. Parameter-dependent matrices with polynomial dependence of degree greater than one were used in [21–23] and also in [24], for discrete time-varying systems, improving the existing results.

This paper adresses the problem of robust $\mathscr{H}_\infty$ filter design for uncertain linear discrete delay systems with a time-varying delay. By using the Jensen's inequality [25] and Finsler's Lemma, new parameter-dependent and delay-dependent linear matrix inequality (LMI) conditions assuring the existence of a full order robust filter that minimizes a bound to the $\mathscr{H}_\infty$ norm of the transfer function from the noise to the estimation error are given. Thanks to the use of extra matrix variables, the proposed LMI conditions are more general than the others in the literature. By imposing a structure to the decision variables, LMI relaxations based on homogeneously polynomially parameter-dependent matrices of arbitrary degree are derived for the robust filter design. As illustrated by examples, the proposed conditions provide less conservative results than other existing methods.

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The paper is organized as follows. Section II presents the preliminary results. The main results are presented in Section III. Section IV presents numerical experiments that illustrate the advantages of the proposed method when compared to other techniques from the literature and Section V concludes the paper.

II. Preliminaries

Consider the discrete-time uncertain linear system with a time-varying delay affecting the state described by

$$\begin{aligned} x_{k+1} &= A(\alpha)x_k + A_d(\alpha)x_{k-d_k} + B_1(\alpha)w_k \\ z_k &= C_1(\alpha)x_k + C_{d1}(\alpha)x_{k-d_k} + D_{11}(\alpha)w_k \\ y_k &= C_2(\alpha)x_k + C_d(\alpha)x_{k-d_k} + D_{21}(\alpha)w_k \end{aligned} \quad (1)$$

with

$$A(\alpha) \in \mathbb{R}^{n\times n},\ A_d(\alpha) \in \mathbb{R}^{n\times n}\ B_1(\alpha) \in \mathbb{R}^{n\times r},$$

$$C_1(\alpha) \in \mathbb{R}^{p\times n},\ C_{d1}(\alpha) \in \mathbb{R}^{p\times n},\ D_{11}(\alpha) \in \mathbb{R}^{p\times r},$$

$$C_2(\alpha) \in \mathbb{R}^{q\times n},\ C_d(\alpha) \in \mathbb{R}^{q\times n},\ D_{21}(\alpha) \in \mathbb{R}^{q\times r}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^r$ is the noise input, $z_k \in \mathbb{R}^p$ is the signal to be estimated and $y_k \in \mathbb{R}^q$ is the measured output.

The matrices of the system are uncertain and belong to a polytopic domain parameterized in terms of a time-invariant vector α, being given by

$$Z(\alpha) = \sum_{i=1}^{N} \alpha_i Z_i\ , \quad \alpha \in \Delta_N \quad (2)$$

where $Z(\alpha)$ represents any matrix of the system in (1), Z_i, $i = 1,\dots,N$ are the vertices, N is the number of vertices of the polytope and Δ_N is the unit simplex, given by

$$\Delta_N = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1,\dots,N \right\} \quad (3)$$

The time delay d_k is a positive integer, constant or time-varying, such that

$$1 \leq \underline{d} \leq d_k \leq \overline{d} \quad (4)$$

where $\underline{d}$ and $\overline{d}$ are constant positive integers, respectively the lower and upper bound of d_k.

The problem addressed in this paper is: find a full order robust linear stable filter given by

$$\begin{aligned} x_{f_{k+1}} &= A_f x_{f_k} + B_f y_k, \\ z_{f_k} &= C_f x_{f_k} + D_f y_k \end{aligned} \quad (5)$$

with $A_f \in \mathbb{R}^{n_f \times n_f}$, $B_f \in \mathbb{R}^{n_f \times q}$, $C_f \in \mathbb{R}^{p \times n_f}$ and $D_f \in \mathbb{R}^{p \times q}$, where $x_{f_k} \in \mathbb{R}^{n_f}$, $n_f = n$, is the estimated state and $z_{f_k} \in \mathbb{R}^p$ is the estimated output, such that the error dynamics is asymptotically stable and the $\mathscr{H}_\infty$ norm of the transfer function from w to the error $e_k = z_k - z_{f_k}$ is minimized.

Defining the augmented system with $\tilde{x}_k' = \begin{bmatrix} x_k' & x_{f_k}' \end{bmatrix}$, one has

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}(\alpha)\tilde{x}_k + \tilde{A}_d(\alpha)T\tilde{x}_{k-d_k} + \tilde{B}(\alpha)w_k \\ e_k &= \tilde{C}(\alpha)\tilde{x}_k + \tilde{C}_d(\alpha)T\tilde{x}_{k-d_k} + \tilde{D}(\alpha)w_k \end{aligned} \tag{6}$$

where $T = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}$ and

$$\begin{aligned} \tilde{A}(\alpha) &= \begin{bmatrix} A(\alpha) & 0 \\ B_f C_2(\alpha) & A_f \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \\ \tilde{A}_d(\alpha) &= \begin{bmatrix} A_d(\alpha) \\ B_f C_d(\alpha) \end{bmatrix} \in \mathbb{R}^{2n \times n}, \\ \tilde{B}(\alpha) &= \begin{bmatrix} B(\alpha) \\ B_f D_{21}(\alpha) \end{bmatrix} \in \mathbb{R}^{2n \times r}, \\ \tilde{C}(\alpha) &= \begin{bmatrix} C_1(\alpha) - D_f C_2(\alpha) & -C_f \end{bmatrix} \in \mathbb{R}^{p \times 2n}, \\ \tilde{C}_d(\alpha) &= \begin{bmatrix} C_{d1}(\alpha) - D_f C_d(\alpha) \end{bmatrix} \in \mathbb{R}^{p \times n}, \\ \tilde{D}(\alpha) &= \begin{bmatrix} D_{11}(\alpha) - D_f D_{21}(\alpha) \end{bmatrix} \in \mathbb{R}^{p \times r} \end{aligned} \tag{7}$$

Before presenting the main contributions, Finsler's Lemma and the Jensen's inequality are reproduced below for the sake of completeness.

Lemma 1: Let $\xi \in \mathbb{R}^n$, $\mathscr{Q} \in \mathbb{R}^{n \times n}$ and $\mathscr{B} \in \mathbb{R}^{m \times n}$ with $\text{rank}(\mathscr{B}) < n$ and $\mathscr{B}^\perp$ such that $\mathscr{B}\mathscr{B}^\perp = 0$. Then, the following conditions are equivalent:

i) $\xi' \mathscr{Q} \xi < 0, \forall\ \xi \neq 0 : \mathscr{B}\xi = 0$
ii) $\mathscr{B}^{\perp\prime} \mathscr{Q} \mathscr{B}^\perp < 0$
iii) $\exists\ \mu \in \mathbb{R} : \mathscr{Q} - \mu \mathscr{B}'\mathscr{B} < 0$
iv) $\exists\ \mathscr{X} \in \mathbb{R}^{n \times m} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0$

For the proof, see [26]. The following lemma (Jensen's inequality) can be found in [25].

Lemma 2: For any constant matrix $0 < M = M' \in \mathbb{R}^{r \times r}$, $d_1 \in \mathbb{N}$, $d_2 \in \mathbb{N}$, $d_1 \leq d_2$, and a vector function $f : [d_1, d_2] \to \mathbb{R}^n$ such that the sums in the following are well defined, then

$$-(d_2 - d_1 + 1) \sum_{i=d_1}^{d_2} f(i)' M f(i) \leq - \left(\sum_{i=d_1}^{d_2} f(i) \right)' M \left(\sum_{i=d_1}^{d_2} f(i) \right) \tag{8}$$

III. Main Results

Lemma 3: Let $\tilde{A}(\alpha)$ be a Schur stable matrix. The inequality $||H(z)||_\infty < \sqrt{\mu}$ holds for all $\alpha \in \Lambda_N$ if there exist parameter-dependent symmetric positive definite matrices $P(\alpha) \in \mathbb{R}^{2n \times 2n}$, $Z_1(\alpha) \in \mathbb{R}^{n \times n}$, $Z_2(\alpha) \in \mathbb{R}^{n \times n}$, $Q_1(\alpha) \in \mathbb{R}^{n \times n}$, $Q_2(\alpha) \in \mathbb{R}^{n \times n}$, $Q_3(\alpha) \in \mathbb{R}^{n \times n}$, $Q_4(\alpha) \in \mathbb{R}^{n \times n}$ and parameter-dependent matrices $E(\alpha) \in \mathbb{R}^{2n \times 2n}$, $K(\alpha) \in \mathbb{R}^{2n \times 2n}$, $H(\alpha) \in \mathbb{R}^{n \times 2n}$, $M(\alpha) \in \mathbb{R}^{n \times 2n}$, $N(\alpha) \in \mathbb{R}^{n \times 2n}$, $X(\alpha) \in \mathbb{R}^{p \times 2n}$ and $V(\alpha) \in \mathbb{R}^{r \times 2n}$ such that[1]

$$\Theta(\alpha) + \Psi(\alpha) < 0, \quad \forall\ \alpha \in \Lambda_N \tag{9}$$

[1]The symbol $\star$ denotes a symmetric block.

with $\Theta(\alpha)$ as in (10), $\Psi(\alpha)$ as in (11) and $\delta = \overline{d} - \underline{d}$.

Proof: Choose a Lyapunov functional candidate as

$$V(\alpha, k) = \sum_{i=1}^{8} V_i(\alpha, k) > 0 \tag{12}$$

$$V_1(\alpha, k) = \tilde{x}_k' P(\alpha) \tilde{x}_k \tag{13}$$

$$V_2(\alpha, k) = \sum_{j=k-d_k}^{k-1} \tilde{x}_j' T' Q_1(\alpha) T \tilde{x}_j \tag{14}$$

$$V_3(\alpha, k) = \sum_{j=k-\overline{d}}^{k-1} \tilde{x}_j' T' Q_2(\alpha) T \tilde{x}_j \tag{15}$$

$$V_4(\alpha, k) = \sum_{j=k-\underline{d}}^{k-1} \tilde{x}_j' T' Q_3(\alpha) T \tilde{x}_j \tag{16}$$

$$V_5(\alpha, k) = \sum_{\ell=2-\overline{d}}^{1-\underline{d}} \sum_{j=k+\ell-1}^{k-1} \tilde{x}_j' T' Q_1(\alpha) T \tilde{x}_j \tag{17}$$

$$V_6(\alpha, k) = \delta \sum_{\ell=-\overline{d}}^{-1-\underline{d}} \sum_{m=k+\ell}^{k-1} y_m' T' Q_4(\alpha) T y_m \tag{18}$$

$$V_7(\alpha, k) = \overline{d} \sum_{\ell=-\overline{d}}^{-1} \sum_{m=k+\ell}^{k-1} y_m' T' Z_1(\alpha) T y_m \tag{19}$$

$$V_8(\alpha, k) = \underline{d} \sum_{\ell=-\underline{d}}^{-1} \sum_{m=k+\ell}^{k-1} y_m' T' Z_2(\alpha) T y_m \tag{20}$$

where $y_j = \tilde{x}_{j+1} - \tilde{x}_j$, $P(\alpha) = P(\alpha)' > 0$, $Q_i(\alpha) = Q_i(\alpha)' > 0$, $i = 1, \ldots, 4$, $Z_j(\alpha) = Z_j(\alpha)' > 0$, $j = 1, 2$.

Define $\Delta V = V(k+1) - V(k)$. Then, along the solutions of (6), one has

$$\Delta V_1(k) = \tilde{x}_{k+1}' P(\alpha) \tilde{x}_{k+1} - \tilde{x}_k' P(\alpha) \tilde{x}_k \tag{21}$$

$$\Delta V_2(k) \leq \tilde{x}_k' T' Q_1(\alpha) T \tilde{x}_k - \tilde{x}_{k-d_k}' T' Q_1(\alpha) T \tilde{x}_{k-d_k} + \sum_{i=k+1-\overline{d}}^{k-\underline{d}} \tilde{x}_i' T' Q_1(\alpha) T \tilde{x}_i \tag{22}$$

$$\Delta V_3(k) = \tilde{x}_k' T' Q_2(\alpha) T \tilde{x}_k - \tilde{x}_{k-\overline{d}}' T' Q_2(\alpha) T \tilde{x}_{k-\overline{d}} \tag{23}$$

$$\Delta V_4(k) = \tilde{x}_k' T' Q_3(\alpha) T \tilde{x}_k - \tilde{x}_{k-\underline{d}}' T' Q_3(\alpha) T \tilde{x}_{k-\underline{d}} \tag{24}$$

$$\Delta V_5(k) = \delta x_k' T' Q_1(\alpha) T x_k - \sum_{i=k+1-\overline{d}}^{k-\underline{d}} \tilde{x}_i' T' Q_1(\alpha) T \tilde{x}_i \tag{25}$$

$$\begin{aligned} \Delta V_6(k) =& \delta^2 y_k' T' Q_4(\alpha) T y_k - \delta \sum_{m=k-\overline{d}}^{k-\underline{d}-1} y_m' T' Q_4(\alpha) T y_m \\ =& \delta^2 y_k' Q_4 T'(\alpha) T y_k - \underbrace{\delta \sum_{m=k-\overline{d}}^{k-d_k-1} y_m' T' Q_4(\alpha) T y_m}_{S_1} \\ & \underbrace{- \delta \sum_{m=k-d_k}^{k-\underline{d}-1} y_m' T' Q_4(\alpha) T y_m}_{S_2} \end{aligned}$$

$$\Theta(\alpha) = \begin{bmatrix} P(\alpha)+T'\big(\overline{d}^2Z_1+\delta^2Q_4(\alpha)+\underline{d}^2Z_2(\alpha)\big)T & T'\big(-\overline{d}^2Z_1-\delta^2Q_4(\alpha)-\underline{d}^2Z_2(\alpha)\big)T & 0 & 0 & 0 & 0 & 0 \\ \star & \begin{matrix}-P(\alpha)+T'\big((\delta+1)Q_1(\alpha)+Q_2(\alpha)+Q_3(\alpha)+\overline{d}^2Z_1(\alpha) \\ +\delta^2Q_4(\alpha)+\underline{d}^2Z_2(\alpha)-Z_1(\alpha)-Z_2(\alpha)\big)T\end{matrix} & 0 & T'Z_1(\alpha) & T'Z_2(\alpha) & \tilde{C}(\alpha)' & 0 \\ \star & \star & -Q_1(\alpha)-2Q_4(\alpha) & Q_4(\alpha) & Q_4(\alpha) & \tilde{C}_d(\alpha)' & 0 \\ \star & \star & \star & -Q_2(\alpha)-Z_1(\alpha)-Q_4(\alpha) & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_3(\alpha)-Z_2(\alpha)-Q_4(\alpha) & 0 & 0 \\ \star & \star & \star & \star & \star & -\mathbf{I} & \tilde{D}(\alpha) \\ \star & \star & \star & \star & \star & \star & -\mu\mathbf{I} \end{bmatrix} \quad (10)$$

$$\Psi(\alpha) = \begin{bmatrix} -E(\alpha)-E(\alpha)' & E(\alpha)\tilde{A}(\alpha)-K(\alpha)' & E(\alpha)\tilde{A}_d(\alpha)-H(\alpha)' & -M(\alpha)' & -N(\alpha)' & -X(\alpha)' & E(\alpha)\tilde{B}(\alpha)-V(\alpha)' \\ \star & K(\alpha)\tilde{A}(\alpha)+\tilde{A}(\alpha)'K(\alpha)' & K(\alpha)\tilde{A}_d(\alpha)+\tilde{A}(\alpha)'H(\alpha)' & \tilde{A}(\alpha)'M(\alpha)' & \tilde{A}(\alpha)'N(\alpha)' & \tilde{A}(\alpha)'X(\alpha)' & K(\alpha)B(\alpha)+\tilde{A}(\alpha)'V(\alpha)' \\ \star & \star & H(\alpha)\tilde{A}_d(\alpha)+\tilde{A}_d(\alpha)'H(\alpha)' & \tilde{A}_d(\alpha)'M(\alpha)' & \tilde{A}_d(\alpha)'N(\alpha)' & \tilde{A}_d(\alpha)'X(\alpha)' & H(\alpha)\tilde{B}(\alpha)+\tilde{A}_d(\alpha)'V(\alpha)' \\ \star & \star & \star & 0 & 0 & 0 & M(\alpha)\tilde{B}(\alpha) \\ \star & \star & \star & \star & 0 & 0 & N(\alpha)\tilde{B}(\alpha) \\ \star & \star & \star & \star & \star & 0 & X(\alpha)\tilde{B}(\alpha) \\ \star & \star & \star & \star & \star & \star & V(\alpha)\tilde{B}(\alpha)+\tilde{B}(\alpha)'V(\alpha)' \end{bmatrix} \quad (11)$$

Applying Lemma 2, one gets

$$S_1 \le -(\overline{d}-d_k)\sum_{m=k-\overline{d}}^{k-d_k-1} y_m'T'Q_4(\alpha)Ty_m \le -(\tilde{x}_{k-d_k}-\tilde{x}_{k-\overline{d}})'Q_4(\alpha)(\tilde{x}_{k-d_k}-\tilde{x}_{k-\overline{d}})$$

$$S_2 \le -(d_k-\underline{d})\sum_{m=k-d_k}^{k-\underline{d}-1} y_m'T'Q_4(\alpha)Ty_m \le -(\tilde{x}_{k-\underline{d}}-\tilde{x}_{k-d_k})'Q_4(\alpha)(\tilde{x}_{k-\underline{d}}-\tilde{x}_{k-d_k})$$

Moreover,

$$\Delta V_6(k) \le \delta^2 y_k'T'Q_4(\alpha)Ty_k - (\tilde{x}_{k-d_k}-\tilde{x}_{k-\overline{d}})'T'Q_4(\alpha)T(\tilde{x}_{k-d_k}-\tilde{x}_{k-\overline{d}}) - (\tilde{x}_{k-\underline{d}}-\tilde{x}_{k-d_k})'T'Q_4(\alpha)T(\tilde{x}_{k-\underline{d}}-\tilde{x}_{k-d_k}) \quad (26)$$

$$\Delta V_7(k) = \overline{d}y_k'T'Z_1(\alpha)Ty_k \underbrace{-\overline{d}\sum_{j=k-\overline{d}}^{k-1} y_j'T'Z_1(\alpha)Ty_j}_{S_3} \quad (27)$$

Applying Lemma 2 again, in S_3, one has

$$S_3 \le -\left(\sum_{j=k-\overline{d}}^{k-1} y_j\right)' T'Z_1(\alpha)T\left(\sum_{j=k-\overline{d}}^{k-1} y_j\right) = -(\tilde{x}_k-\tilde{x}_{k-\overline{d}})'T'Z_1(\alpha)T(\tilde{x}_k-\tilde{x}_{k-\overline{d}}) \quad (28)$$

Then,

$$\Delta V_7 \le \overline{d}^2 y_k'T'Z_1(\alpha)Ty_k - (\tilde{x}_k-\tilde{x}_{k-\overline{d}})'T'Z_1(\alpha)T(\tilde{x}_k-\tilde{x}_{k-\overline{d}}) \quad (29)$$

and, similarly,

$$\Delta V_8 \le \underline{d}^2 y_k'T'Z_2(\alpha)Ty_k - (\tilde{x}_k-\tilde{x}_{k-\underline{d}})'T'Z_1(\alpha)T(\tilde{x}_k-\tilde{x}_{k-\underline{d}}) \quad (30)$$

Finally,

$$\Delta V(\alpha,k) = \sum_{i=1}^{8} \Delta V_i \quad (31)$$

To establish the $\mathscr{H}_\infty$ performance for the filtering error system, consider the following criterion

$$J \triangleq \sum_{k=0}^{\infty}\left(e_k' e_k - \mu w_k' w_k\right) \tag{32}$$

Under zero initial conditions, that is, $\tilde{x}_k = 0$, $V(\alpha,0)=0$ and $V(\alpha,\infty)\geq 0$, one has

$$J \leq \sum_{k=0}^{\infty}\left(e_k' e_k - \mu w_k' w_k + \Delta V(\alpha,k)\right) \tag{33}$$

that can be rewritten as

$$J \leq \sum_{k=0}^{\infty} \xi_k' \Theta(\alpha)\xi_k \tag{34}$$

with $\Theta(\alpha)$ given by (10) and

$$\xi_k = \begin{bmatrix} \tilde{x}_{k+1}' & \tilde{x}_k' & \tilde{x}_{k-d_k}' T' & \tilde{x}_{k-\overline{d}}' T' & \tilde{x}_{k-\underline{d}}' T' & z_k & w_k \end{bmatrix}'$$

By applying condition *i)* of Lemma 1 in $\xi_k'\Theta(\alpha)\xi_k$ and selecting

$$\mathscr{X} = \begin{bmatrix} E(\alpha) \\ K(\alpha) \\ H(\alpha) \\ M(\alpha) \\ N(\alpha) \\ X(\alpha) \\ V(\alpha) \end{bmatrix}, \mathscr{B} = \begin{bmatrix} -\mathbf{I} & \tilde{A}(\alpha) & \tilde{A}_d(\alpha) & 0 & 0 & 0 & \tilde{B}(\alpha) \end{bmatrix}$$

in condition *iv)* one has (9). ■

It is important to note that Lemma 3 has been established without defining a particular structure for the parameter-dependent matrix variables. Moreover, the decision variables of interest (i.e. A_f, B_f, C_f and D_f) appear in sub-matrices multiplying other matrices. As it has been presented, the robust filter design is a nonconvex problem of infinite dimension (since the parameter-dependent inequalities need to be verified for all $\alpha \in \Delta_N$).

Note that the parameter-dependent inequalities in Lemma 3 have parameter-dependent matrices $E(\alpha)$, $K(\alpha)$, $H(\alpha)$, $M(\alpha)$, $N(\alpha)$, $X(\alpha)$ and $V(\alpha)$ that can represent extra degrees of freedom when sufficient LMI conditions are derived.

In order to derive numerically tractable LMI conditions for the filter design, structural constraints are imposed to the parameter-dependent matrices $E(\alpha)$, $K(\alpha)$, $H(\alpha)$, $M(\alpha)$, $N(\alpha)$, $X(\alpha)$ and $V(\alpha)$, similarly to what has been done in [20, 23]:

$$\begin{aligned} E(\alpha) &= \begin{bmatrix} E_{11}(\alpha) & \hat{K} \\ E_{21}(\alpha) & \hat{K} \end{bmatrix},\ K(\alpha) = \begin{bmatrix} K_{11}(\alpha) & \lambda_1\hat{K} \\ K_{21}(\alpha) & \lambda_2\hat{K} \end{bmatrix}, \\ H(\alpha) &= \begin{bmatrix} H_1(\alpha) & \lambda_3\hat{K} \end{bmatrix},\ M(\alpha) = \begin{bmatrix} M_1(\alpha) & \lambda_4\hat{K} \end{bmatrix}, \\ N(\alpha) &= \begin{bmatrix} N_1(\alpha) & \lambda_5\hat{K} \end{bmatrix},\ X(\alpha) = \begin{bmatrix} X_1(\alpha) & 0 \end{bmatrix}, \\ V(\alpha) &= \begin{bmatrix} V_1(\alpha) & 0 \end{bmatrix} \end{aligned} \tag{35}$$

where $\hat{K} \in \mathbb{R}^{n\times n}$ is a matrix and λ_i, $i = 1,\ldots,5$ are scalar variables to be determined. For convenience, matrix $P(\alpha)$ is also partitioned in $n\times n$ blocks

$$P(\alpha) = \begin{bmatrix} P_{11}(\alpha) & P_{12}(\alpha) \\ P_{12}(\alpha)' & P_{22}(\alpha) \end{bmatrix} \tag{36}$$

and the following change of variables is adopted $K_1 = \hat{K}A_f$, $K_2 = \hat{K}B_f$. With this particular choice for the decision variables, a sufficient parameter-dependent LMI condition for the existence of a robust $\mathscr{H}_\infty$ filter is presented below.

Theorem 1: If there exist symmetric parameter-dependent positive definite matrices $Q_1(\alpha)$, $Q_2(\alpha)$, $Q_3(\alpha)$, $Q_4(\alpha)$, $Z_1(\alpha)$, $Z_2(\alpha)$ and $P(\alpha)$ as in (36) matrices $K(\alpha)$, $E(\alpha)$, $V(\alpha)$, $X(\alpha)$, $M(\alpha)$, $N(\alpha)$ and $H(\alpha)$ as in (35), $K_1 \in \mathbb{R}^{n\times n}$, $K_2 \in \mathbb{R}^{n\times q}$, $C_f \in \mathbb{R}^{p\times n}$, $D_f \in \mathbb{R}^{p\times q}$, $\mu > 0$ and scalars λ_1, λ_2, λ_3, λ_4 and λ_5 such that condition (37) holds for all $\alpha \in \Lambda_N$, then $A_f = \hat{K}^{-1}K_1$, $B_f = \hat{K}^{-1}K_2$, C_f and D_f are the matrices of the robust stable filter that assures a guaranteed cost $\mathscr{H}_\infty$ given by $\sqrt{\mu}$.

Proof: The proof follows straightforwardly the same steps of the proof of Lemma 3, with the structure presented in (35) to the slack variables and as in (36) to matrix $P(\alpha)$. ■

Theorem 1 is a parameter-dependent sufficient LMI condition for the existence of a robust $\mathscr{H}_\infty$ filter, obtained directly from Lemma 3 by imposing particular structures to the matrix variables. Moreover, the conditions depend on scalar variables λ_i, $i = 1,\ldots,5$ that need to be searched.

To solve the parameter-dependent LMI conditions of Theorem 1, the technique proposed in [27] to handle parameter-dependent LMIs with parameters in the unit simplex can be applied. To this end, the polynomial matrices (decision variables in the parameter-dependent LMIs, i.e. $P(\alpha)$, $K_{11}(\alpha)$, $K_{21}(\alpha)$, $E_{11}(\alpha)$, $E_{21}(\alpha)$, $H_1(\alpha)$, $M_1(\alpha)$, $N_1(\alpha)$, $X_1(\alpha)$ and $V_1(\alpha)$) are treated as homogeneous polynomials of arbitrary degree g and LMI conditions, more and more precise with the increase of g, are expressed only in terms of the vertices of the system. The LMI conditions were obtained with the Robust LMI Parser toolbox available at `http://www.dt.fee.unicamp.br/~agulhari/Doutorado/polynomial_parser.zip`.

IV. Numerical Experiments

The objective of the experiments is to compare the conditions proposed in this paper with other methods from the literature. The routines were implemented in MATLAB, version 7.1.0.246 (R14) SP 3 using the programs Yalmip [28] and SeDuMi [29]. Although line searches in λ_i could further improve the guaranteed costs, $\lambda_i = 0$, $i = 1,\ldots,5$ have been used in this paper with good results.

Consider the discrete-time system [22] given by

$$A = \begin{bmatrix} 0 & 0.3 \\ -0.2 & \rho \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 0 \\ 0.1 & \phi \end{bmatrix},\ 1 \leq d_k \leq \overline{d},$$

$$B_1^T = \begin{bmatrix} 0 & 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 0.2 & 0 \end{bmatrix},\ C_{d1} = D_{11} = 0,\ \ D_{21} = 1$$

where $|\phi| \leq 0.1$ and $|\rho| \leq \beta$. Table I shows the $\mathscr{H}_\infty$ costs obtained with different values of β for $\overline{d} = 5$. As can be seen, the proposed approach provides less conservative results than the one in [22].

As another experiment, Table II shows the $\mathscr{H}_\infty$ costs for $\beta = 0.7$ and $\overline{d} = 4,5,6,7$. It can be noticed that, in most cases, the $\mathscr{H}_\infty$ guaranteed costs obtained by Theorem 1 with

$$
\begin{bmatrix}
-E_{11}(\alpha)-E_{11}(\alpha)'+P_{11}(\alpha)+\overline{d}^2 Z_1+\delta^2 Q_4(\alpha)+\underline{d}^2 Z_2(\alpha) & -\hat{K}-E_{21}(\alpha)'+P_{12}(\alpha) & E_{11}(\alpha)A(\alpha)+K_2C_2(\alpha)-K_{11}(\alpha)'-\overline{d}^2 Z_1-\delta^2 Q_4(\alpha)-\underline{d}^2 Z_2(\alpha) & K_1-K_{21}(\alpha)' & E_{11}(\alpha)A_d(\alpha)+K_2C_d(\alpha)-H_1(\alpha)' & -M_1(\alpha)' & -N_1(\alpha)' & -X_1(\alpha)' & E_{11}(\alpha)B_1(\alpha)+K_2D_{21}(\alpha)-V_1(\alpha)' \\
\star & -\hat{K}-\hat{K}'+P_{22}(\alpha) & E_{21}(\alpha)A(\alpha)+K_2C_2(\alpha)-\lambda_1\hat{K}' & K_1(\alpha)-\lambda_2\hat{K}' & E_{21}(\alpha)A_d(\alpha)+K_2C_d(\alpha)-\lambda_3\hat{K}' & -\lambda_4\hat{K}' & -\lambda_5\hat{K}' & 0 & E_{21}(\alpha)B_1(\alpha)+K_2D_{21}(\alpha) \\
\star & \star & \begin{matrix} K_{11}(\alpha)A(\alpha)+A(\alpha)'K_{11}(\alpha)'+\lambda_1K_2C_2(\alpha)+\lambda_1C_2(\alpha)'K_2'+Q_1(\alpha)(\delta+1) \\ +Q_2(\alpha)+Q_3(\alpha)\overline{d}^2 Z_1+\delta^2 Q_4(\alpha)+\underline{d}^2 Z_2(\alpha)-Z_1(\alpha)-Z_2(\alpha)-P_{11}(\alpha)\end{matrix} & \lambda_1K_1+A(\alpha)'K_{21}(\alpha)'+\lambda_2C_2(\alpha)'K_2'-P_{12}(\alpha) & K_{11}(\alpha)A_d(\alpha)+\lambda_1K_2C_d(\alpha)+A(\alpha)'H_1(\alpha)'+\lambda_3C_2(\alpha)'K_2' & A(\alpha)'M_1(\alpha)'+\lambda_4C_2(\alpha)'K_2'+Z_1(\alpha)' & A(\alpha)'N_1(\alpha)'+\lambda_5C_2(\alpha)'K_2'+Z_2(\alpha)' & A(\alpha)'X_1(\alpha)'+C_1(\alpha)'-C_2(\alpha)'D_f' & K_{11}(\alpha)B_1(\alpha)+\lambda_1K_2D_{21}(\alpha)+A(\alpha)'V_1(\alpha)' \\
\star & \star & \star & \lambda_2K_1+\lambda_2K_1'-P_{22}(\alpha) & K_{21}(\alpha)A_d(\alpha)+\lambda_2K_2C_d(\alpha)+\lambda_3K_1' & \lambda_4K_1' & \lambda_5K_1' & -C_f' & K_{21}(\alpha)B_1(\alpha)+\lambda_2K_2D_{21}(\alpha) \\
\star & \star & \star & \star & H_1(\alpha)A_d(\alpha)+A_d(\alpha)'H_1(\alpha)'+\lambda_3K_2C_d(\alpha)+\lambda_3C_d(\alpha)'K_2'-Q_1(\alpha)-2Q_4(\alpha) & A_d(\alpha)'M_1(\alpha)'+Q_4(\alpha)' & A_d(\alpha)'N_1(\alpha)'+Q_4(\alpha)' & A_d(\alpha)'X_1(\alpha)'+C_{d1}(\alpha)'-C_d(\alpha)'D_f' & H_1(\alpha)B_1(\alpha)+\lambda_3K_2D_{21}(\alpha)+A_d(\alpha)'V_1(\alpha)' \\
\star & \star & \star & \star & \star & -Q_2(\alpha)-Z_1(\alpha)-Q_4(\alpha) & 0 & 0 & M_1(\alpha)B_1(\alpha)+\lambda_4K_2D_{21}(\alpha) \\
\star & \star & \star & \star & \star & \star & -Q_3(\alpha)-Z_2(\alpha)-Q_4(\alpha) & 0 & N_1(\alpha)B_1(\alpha)+\lambda_5K_2D_{21}(\alpha) \\
\star & \star & \star & \star & \star & \star & \star & -\mathbf{I}_p & X_1(\alpha)B_1(\alpha)+D_{11}(\alpha)-D_fD_{21}(\alpha) \\
\star & \star & \star & \star & \star & \star & \star & \star & V_1(\alpha)B_1(\alpha)+B_1(\alpha)'V_1(\alpha)'-\mu\mathbf{I}_r
\end{bmatrix} < 0 \quad (37)
$$

TABLE I

$\mathscr{H}_\infty$ COSTS FOR EXAMPLE 1 USING THEOREM 1 (T1), $\lambda_i = 0, i = 1,\ldots,5$ AND [22], FOR $\overline{d} = 5$ AND DIFFERENT VALUES OF β.

β	0.5	0.6	0.7	0.75
[22] $(g=1)$	2.3691	3.0628	4.9838	9.1328
[22] $(g=2)$	2.3179	2.8175	3.9136	6.1137
T1 $(g=1)$	2.3179	2.7919	3.6249	6.2043
T1 $(g=2)$	2.3179	2.7919	3.6171	6.0030

TABLE II

$\mathscr{H}_\infty$ COSTS FOR EXAMPLE 1 USING THEOREM 1 (T1), $\lambda_i = 0, i = 1,\ldots,5$ AND [22] FOR $\beta = 0.7$ AND DIFFERENT VALUES OF $\overline{d}$.

$\overline{d}$	4	5	6	7
[22] $(g=1)$	3.9537	4.9838	6.6630	10.6396
[22] $(g=2)$	3.3236	3.9136	4.8960	7.7017
T1 $(g=1)$	3.1400	3.6249	4.6118	7.8222
T1 $(g=2)$	3.1400	3.6171	4.5222	7.6297

$g=1$ are smaller than the ones provided by [22] with $g=2$, illustrating clearly that the proposed approach can provide less conservative results with less computational effort. As an example, the robust filter provided by Theorem 1 with $g=2$, $\overline{d}=6$ and $\beta=0.7$ is given by

$$A_f = \begin{bmatrix} -0.6969 & 0.2431 \\ 0.2761 & -0.0828 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.2917 \\ -1.6307 \end{bmatrix}$$

$$C_f = \begin{bmatrix} -0.9541 & -1.9847 \end{bmatrix}, \quad D_f = \begin{bmatrix} -0.0042 \end{bmatrix}$$

V. CONCLUSIONS

This paper presented new parameter-dependent delay-dependent LMI conditions for the design of full order robust $\mathscr{H}_\infty$ filters for discrete-time uncertain polytopic linear systems with unknown time-varying delay. LMI relaxations based on homogeneous polynomials of arbitrary degrees provided less conservative results when compared to other existing techniques. The conditions could be extended to cope with $\mathscr{H}_2$ robust filter design as well.

REFERENCES

[1] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, ser. Lecture Notes in Control and Information Sciences. London: Springer-Verlag, 2001, vol. 269.

[2] J.-D. Chen, "Robust control for uncertain neutral systems with time-delays in state and control input via LMI and GAs," *Appl. Math. Comput.*, vol. 157, no. 2, pp. 535–548, September 2004.

[3] E. Fridman and U. Shaked, "Delay dependent $\mathscr{H}_\infty$ control of uncertain discrete delay system," *European J. Control*, vol. 11, no. 1, pp. 29–37, 2005.

[4] V. J. S. Leite, S. Tarbouriech, and P. L. D. Peres, "Robust $\mathscr{H}_\infty$ state feedback control of discrete-time systems with state delay: an LMI approach," *IMA J. Math. Control Inform.*, vol. 26, no. 3, pp. 357–373, September 2009.

[5] M. F. Miranda, V. J. S. Leite, and A. F. Caldeira, "Robust stabilization of polytopic discrete-time systems with time-varying delay in the states," in *Proc. 49th IEEE Conf. Decision Control*, Atlanta, GA, USA, December 2010, pp. 152–157.

[6] V. J. S. Leite, P. L. D. Peres, E. B. Castelan, and S. Tarbouriech, "On the robust stability of neutral systems with time-varying delays," in *Proc. 16th IFAC World Congr.*, Prague, Czech Republic, July 2005, in CD-rom.

[7] M. M. Peet, A. Papachristodoulou, and S. Lall, "Positive forms and stability of linear time-delay systems," *SIAM J. Control Optim.*, vol. 47, no. 6, pp. 3237–3258, January 2009.

[8] D. F. Coutinho and C. E. de Souza, "Delay-dependent robust stability and $\mathscr{L}_2$-gain analysis of a class of nonlinear time-delay systems," *Automatica*, vol. 44, no. 8, pp. 2006–2018, August 2008.

[9] J. Zhang, Y. Xia, P. Shi, and M. S. Mahmoud, "New results on stability and stabilisation of systems with interval time-varying delay," *IET Control Theory & Appl.*, vol. 5, no. 3, pp. 429–436, February 2011.

[10] R. M. Palhares, C. E. de Souza, and P. L. D. Peres, "Robust $\mathscr{H}_\infty$ filtering for uncertain discrete-time state-delayed systems," *IEEE Trans. Signal Process.*, vol. 49, no. 8, pp. 1096–1703, August 2001.

[11] S. Xu, "Robust $\mathscr{H}_\infty$ filtering for a class of discrete-time uncertain nonlinear systems with state delay," *IEEE Trans. Circuits Syst. I*, vol. 49, no. 12, pp. 1853–1859, December 2002.

[12] H. Gao and C. Wang, "A delay-dependent approach to robust $\mathscr{H}_\infty$ filtering for uncertain discrete-time state-delayed systems," *IEEE Trans. Signal Process.*, vol. 52, no. 6, pp. 1631–1640, June 2004.

[13] H. Gao, X. Meng, and T. Chen, "A parameter-dependent approach to robust $\mathscr{H}_\infty$ filtering for time-delay systems," *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2420–2425, November 2008.

[14] F. O. Souza, R. M. Palhares, and K. A. Barbosa, "New improved delay-dependent $\mathscr{H}_\infty$ filter design for uncertain neutral systems," *IET Control Theory & Appl.*, vol. 2, no. 12, pp. 1033–1043, December 2008.

[15] X. Li, Z. Li, and H. Gao, "Further results on $\mathscr{H}_\infty$ filtering for discrete-time systems with state delay," *Int. J. Robust Nonlinear Control*, vol. 21, no. 3, pp. 248–270, February 2011.

[16] J. C. Geromel, J. Bernussou, G. Garcia, and M. C. de Oliveira, "$\mathscr{H}_2$ and $\mathscr{H}_\infty$ robust filtering for discrete-time linear systems," *SIAM J. Control Optim.*, vol. 38, no. 5, pp. 1353–1368, May 2000.

[17] R. M. Palhares and P. L. D. Peres, "Robust filtering with guaranteed energy-to-peak performance — an LMI approach," *Automatica*, vol. 36, no. 6, pp. 851–858, June 2000.

[18] ——, "LMI approach to the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ filtering design for discrete-time uncertain systems," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 37, no. 1, pp. 292–296, January 2001.

[19] J. C. Geromel, M. C. de Oliveira, and J. Bernussou, "Robust filtering of discrete-time linear systems with parameter dependent Lyapunov functions," *SIAM J. Control Optim.*, vol. 41, no. 3, pp. 700–711, 2002.

[20] Z. S. Duan, J. X. Zhang, C. S. Zhang, and E. Mosca, "Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filtering for uncertain linear systems," *Automatica*, vol. 42, no. 11, pp. 1919–1926, November 2006.

[21] H. Gao, X. Meng, and T. Chen, "A new design of robust H_2 filters for uncertain systems," *Syst. Control Letts.*, vol. 57, no. 7, pp. 585–593, July 2008.

[22] ——, "$\mathscr{H}_\infty$ filter design for discrete delay systems: a new parameter-dependent approach," *Int. J. Control*, vol. 82, no. 6, pp. 993–1005, June 2009.

[23] M. J. Lacerda, R. C. L. F. Oliveira, and P. L. D. Peres, "Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for uncertain linear systems via LMIs and polynomial matrices," *Signal Process.*, vol. 91, no. 5, pp. 1115–1122, May 2011.

[24] C. E. de Souza, K. A. Barbosa, and A. Trofino, "Robust $\mathscr{H}_\infty$ filtering for discrete-time linear systems with uncertain time-varying parameters," *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 2110–2118, June 2006.

[25] X. L. Zhu and G. H. Yang, "Jensen inequality approach to stability analysis of discrete-time systems with time-varying delay," in *Proc. 2008 Amer. Control Conf.*, Seattle, WA, USA, June 2008, pp. 1644–1649.

[26] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control*, ser. Lecture Notes in Control and Information Science, S. O. Reza Moheimani, Ed. New York, NY: Springer-Verlag, 2001, vol. 268, pp. 241–257.

[27] R. C. L. F. Oliveira and P. L. D. Peres, "Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1334–1340, July 2007.

[28] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. 2004 IEEE Int. Symp. on Comput. Aided Control Syst. Des.*, Taipei, Taiwan, September 2004, pp. 284–289, http://control.ee.ethz.ch/~joloef/yalmip.php.

[29] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Method Softw.*, vol. 11, no. 1–4, pp. 625–653, 1999, http://sedumi.mcmaster.ca/.