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Constrained Minimum-Power Control of Spiking Neuron Oscillators

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Abstract— **In this article, we study optimal control problems of spiking neurons whose dynamics are described by a phase model. We design minimum-power current stimuli (controls) that lead to desired spiking times. In particular, we consider bounded control amplitude and characterize the range of possible spiking times according to the bound. The design of such bounded optimal controls is of fundamental importance as phase models are accurate under weak forcing. We show that for a given bound, the corresponding feasible spiking times are optimally achieved by switching controls. We present analytical expressions with numerical simulations of these minimum-power stimuli for various phase models of neurons.**

I. Introduction

Optimal control of neurons and hence the nervous system by external current stimuli (controls) has received increased scientific attention in recent years for its wide range of applications from deep brain stimulation to oscillatory neurocomputers [1], [2], [3]. Traditionally, neuron oscillators are modeled by phase-reduced models, which form standard nonlinear systems [4], [5]. Intensive studies using phase models have been carried out, for example, on investigation of the patterns of synchrony that results from the type and architecture of coupling [6], [7] and on the response of large groups of oscillators to external stimuli [8], [9], where the inputs to the neuron systems were initially defined and the dynamics of neural populations were analyzed in detail.

Recently, control theoretic approaches have been employed to design external stimuli that drive neurons to behave in a desired way. For example, a multilinear feedback control technique has been used to control the individual phase relation between coupled oscillators [10] and geometric control theory has been adopted to study controllability and optimal control of a network of neurons with different natural oscillation frequencies [11]. It is feasible to change the spiking periods of oscillators or tune the individual phase relationship between coupled oscillators by an electric stimulus [12], [10]. Minimum-power stimuli that elicit spikes of a neuron at specified times close to the natural spiking time have been analyzed previously [8]. Optimal waveforms for the entrainment of weakly forced oscillators that maximize the locking range have been calculated, where first and second harmonics were used to approximate the phase response curve (PRC) [13]. These optimal controls were found mainly based on the calculus of variation. That restricts the optimal solutions to the class of smooth controls, also the bound of the control amplitude has not been taken into account.

In this article, we apply Pontryagin's maximum principle [14], [15] to derive minimum-power controls that spike a neuron at desired time instants. We consider bounded control amplitude and fully characterize the range of feasible spiking times determined by the bound. In particular, our optimal control strategies are general so that the bound can be chosen sufficiently small within the range that the phase models are valid. The design of such minimum-power stimuli to elicit spikes of neuron oscillators is of clinical importance, notably in deep brain stimulation therapy for Parkinson's disease and essential tremor [16], where mild stimulations are required. In addition, the demand to reduce the energy consumption in neurological implants such as cardiac pacemakers makes such optimal designs imperative.

This paper is organized as follows. In Section II, we introduce the phase model for spiking neurons and formulate the related optimal control problem. In Section III, we derive the minimum-power controls associated with specified spiking times for the sinusoidal phase model in the absence and presence of a control amplitude constraint. The optimal control strategies derived here are applied to find minimum-power controls for practical PRC's, including Morris-Lecar and Hodgkin-Huxley phase models, presented in Section IV.

II. Optimal Control of Spiking Neuron Oscillators

A periodically spiking or firing neuron can be considered as a periodic oscillator governed by a nonlinear dynamical equation of the form

$$\frac{d\theta}{dt} = f(\theta) + Z(\theta)I(t), \tag{1}$$

where θ is the phase of the oscillation, $f(\theta)$ and $Z(\theta)$ are real-valued functions giving the neuron's baseline dynamics and its phase response, respectively, and $I(t)$ is an external current stimulus [4]. This nonlinear dynamical system described in (1) is referred to as the phase model for the oscillation. The assumption that $Z(\theta)$ vanishes only on isolated points and that $f(\theta) > 0$ are made so that a full revolution of the phase is possible. By convention, neuron spikes occur when $\theta = 2n\pi$, where $n \in \mathbb{N}$. In the absence of any input $I(t)$, the neuron spikes periodically at its natural frequency, while the spiking time can be advanced or delayed in a desired manner by an appropriate choice of $I(t)$.

In this article, we study optimal design of neural inputs that lead to the spiking of neurons at a specified time T after spiking at time $t = 0$. In particular, we find the stimulus that

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fires a neuron with minimum power at desired time,T, which is formulated as the following optimal control problem,

$$\begin{aligned} \min_{I(t)} \quad & \int_0^T I(t)^2\, dt \qquad (2)\\ \text{s.t.} \quad & \dot{\theta} = f(\theta) + Z(\theta)I(t),\\ & \theta(0) = 0, \quad \theta(T) = 2\pi\\ & |I(t)| \le M, \quad \forall\, t \in [0,T], \end{aligned}$$

where $M > 0$ is the amplitude bound of the current stimulus $I(t)$. Here, we are optimizing over all possible current inputs including both hyper-polarizing and depolarizing inputs, i.e., $I(t)$ can be positive or negative. Note that if T is equal to the neuron's natural spiking time, then no input is needed. We first investigate the case when the control amplitude is unbounded, upon which the optimal control with bounded amplitude can be constructed.

III. Minimum-Power Stimulus for Sinusoidal PRC

Consider the sinusoidal PRC,

$$\dot{\theta} = \omega + z_d \sin\theta \cdot I(t), \qquad (3)$$

where ω is the natural oscillation frequency of the neuron and z_d is a model-dependent constant. The neuron described by this phase model spikes periodically with the period $T = 2\pi/\omega$ in the absence of any external input, i.e., $I(t) = 0$. Note that this type of PRC's with both positive and negative regions can be obtained by periodic orbits near the super critical Hopf bifurcation[4]. This type of bifurcation occurs for Type II neuron models like Fitzhugh-Nagumo model [17].

A. Spiking Neurons with Unbounded Control

The optimal current profile can be derived by Pontryagin's maximum principle [14]. Given the optimal control problem as in (2), we form the control Hamiltonian

$$H = I^2 + \lambda(\omega + z_d \sin\theta \cdot I), \qquad (4)$$

where λ is the Lagrange multiplier. The necessary optimality conditions according to the Maximum Principle gives

$$\dot{\lambda} = -\frac{\partial H}{\partial \theta} = -\lambda z_d I \cos\theta, \qquad (5)$$

and $\frac{\partial H}{\partial I} = 2I + \lambda z_d \sin\theta = 0$. Hence, the optimal current I satisfies

$$I = -\frac{1}{2}\lambda z_d \sin\theta. \qquad (6)$$

The maximum principle transforms the optimal control problem to a boundary value problem, which characterizes the optimal trajectories of $\theta(t)$ and $\lambda(t)$,

$$\dot{\theta} = \omega - \frac{z_d^2 \lambda}{2}\sin^2\theta, \qquad (7)$$

$$\dot{\lambda} = \frac{z_d^2 \lambda^2}{2}\sin\theta\cos\theta, \qquad (8)$$

with boundary conditions $\theta(0) = 0$ and $\theta(T) = 2\pi$ while $\lambda(0)$ and $\lambda(T)$ are unspecified.

Additionally, since the Hamiltonian is not explicitly dependent on time, the optimal triple (λ, θ, I) satisfies $H(\lambda, \theta, I) = c$, $\forall 0 \le t \le T$, where c is a constant. Together with (6), this yields

$$-\frac{z_d^2}{4}\sin^2\theta\lambda^2 + \omega\lambda = c. \qquad (9)$$

Since $\theta(0) = 0$, $c = \omega\lambda_0$, where $\lambda_0 = \lambda(0)$, which is undetermined. The optimal multiplier can be found by solving the above quadratic equation (9), which gives

$$\lambda = \frac{2\omega \pm 2\sqrt{\omega^2 - \omega\lambda_0 z_d^2 \sin^2\theta}}{z_d^2 \sin^2\theta}, \qquad (10)$$

and then, from (7), the optimal trajectory of θ follows

$$\dot{\theta} = \mp\sqrt{\omega^2 - \omega\lambda_0 z_d^2 \sin^2\theta}. \qquad (11)$$

Integrating the equation (11), we find the spiking time T in terms of the initial condition λ_0,

$$T = \int_0^{2\pi} \frac{1}{\sqrt{\omega^2 - \omega\lambda_0 z_d^2 \sin^2\theta}} d\theta. \qquad (12)$$

Note that we choose the positive sign in (11) since the negative velocity indicates the backward phase evolution. Therefore, given a desired spiking time T of the neuron, the initial value, λ_0, corresponding to the optimal trajectory of the multiplier can be found via the one-to-one relation in (12). Consequently, the optimal trajectories of θ and λ can be easily computed by evolving (7) and (8) forward in time. Plugging (10) into (6), we obtain the optimal feedback law for spiking the neuron at time T of the form

$$I^* = \frac{-\omega + \sqrt{\omega^2 - \omega\lambda_0 z_d^2 \sin^2\theta}}{z_d \sin\theta}, \qquad (13)$$

where λ_0 is to be calculated according to the desired spiking time from (12).

The feasibility of spiking the neuron at a desired time T largely depends on the initial value of the multiplier, λ_0. It is not feasible to have a 2π revolution if $\lambda_0 \ge \omega/z_d^2$. This fact can be seen from Fig. 1, where the system evolution defined by (7) and (8) for $z_d = 1$ and $\omega = 1$ with respect to different λ_0 values ($\theta = 0$ axis) is illustrated. When $\lambda_0 = 0$, the spiking period is equal to the natural spiking period, $2\pi/\omega$, and no external stimulus needs to be applied, i.e., $I^*(t) = 0$, $\forall t \in [0, 2\pi/\omega]$. T is a monotonically increasing function of λ_0 for fixed ω and z_d and, the average phase velocity decreases when λ_0 increases, the spiking time $T > 2\pi/\omega$ for $\lambda_0 > 0$ and $T < 2\pi/\omega$ for $\lambda_0 < 0$. Fig. 2 shows variation of the spiking time T with the λ_0 corresponding to the optimal trajectories for different ω values with $z_d = 1$.

The relation between the spiking time T and required minimum power $E = \min\int_0^T I^2(t)dt$ is evident via a simple sensitivity analysis [18]. Since a small change in the initial condition, $d\theta$, and a small change in the initial time, dt, result in a small change in power according to $dE = \lambda(t)d\theta - H(t)dt$, it follows that $-\frac{\partial E}{\partial t} = H = c = \omega\lambda_0$ [18]. This implies that E increases with initial time t for $\lambda_0 < 0$

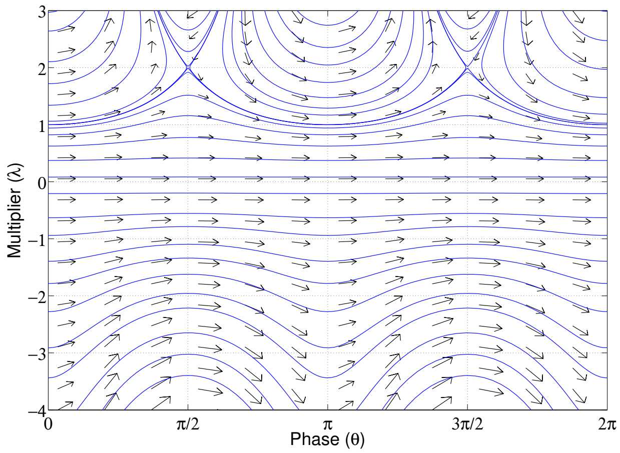


Fig. 1. Extremals of the sinusoidal PRC system with $z_d = 1$ and $\omega = 1$.

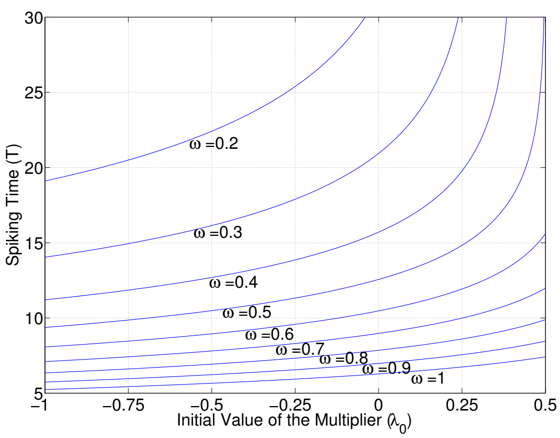


Fig. 2. Variation of the spiking time, T, with the initial multiplier, λ_0, for different ω values and $z_d = 1$.

and decreases for $\lambda_0 > 0$. Since the increment of the initial time is equivalent to the decrement of final time (i.e. spiking time T), $\partial E/\partial T = \omega\lambda_0$. Since $\lambda_0 < 0$ ($\lambda_0 > 0$) corresponds to $T < 2\pi/\omega$ ($T > 2\pi/\omega$), we see that the required minimum power increases if we move away from the natural spiking time.

The minimum-power stimulus I^* as in (13) plotted with respect to time and the phase for various spiking times $T = 3, 5, 10, 12$ with $\omega = 1$ and $z_d = 1$ are shown in Fig. 3(a) and 3(b), respectively. The respective optimal trajectories of $\lambda(\theta)$ and $\theta(t)$ for these spiking times are illustrated in Fig. 3(c) and 3(d).

B. Spiking Neurons with Bounded Control

In practice, the amplitude of stimuli in physical systems are limited, for example [15]. Therefore, we consider spiking the sinusoidal neuron with bounded control amplitude, namely, in the optimal control problem (2), $|I(t)| \le M < \infty$ for all $t \in [0,T]$, where T is the desired spiking time. In this case, there exists a range of feasible spiking times depending on the value of M, in contrast to the previous case where

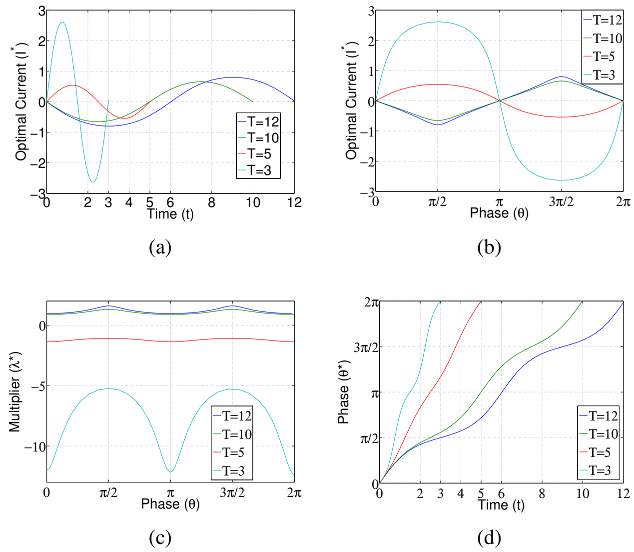


Fig. 3. (a) Variation of the minimum-power control, I^*, with time, t. (b) Variation of I^* with phase θ. (c) Variation of the optimal multiplier, λ^*, with θ. (d) Variation of the phase, θ^*, with t, for the sinusoidal PRC with $z_d = 1$ and $\omega = 1$ for the spiking times, T= 3, 5, 10 and 12.

any desired spiking time is feasible. We first observe that given this bound M, the minimum time control that spikes the neuron can be achieved by choosing the phase velocity $\dot{\theta}$ at its maximum over $t \in [0,T]$. Such a time-optimal control, for $z_d > 0$, can be characterized by a switching, i.e.,

$$I^*_{T_{min}} = \begin{cases} M & \text{for} \quad 0 \le \theta < \pi \\ -M & \text{for} \quad \pi \le \theta < 2\pi \end{cases} . \qquad (14)$$

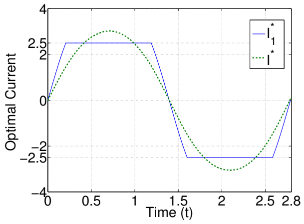
Consequently, the spiking time with $I^*_{T_{min}}$ can be computed using (3) and (14), which yields

$$T^M_{min} = 2\pi\sqrt{\frac{1}{-z_d^2M^2+\omega^2}} - \frac{4\tan^{-1}\left\{z_dM/\sqrt{-z_d^2M^2+\omega^2}\right\}}{\sqrt{-z_d^2M^2+\omega^2}}, \qquad (15)$$

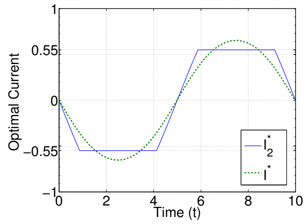
for $z_dM \neq \omega$. It follows that I^*, derived in (13), is the minimum-power stimulus that spikes the neuron at a desired spiking time T if $|I^*| \le M$ for all $t \in [0,T]$. However, there exists a shortest possible spiking time by I^* given the bound M. Simple first and second order optimality conditions applied to (13) find that the maximum value of I^* occurs at $\theta = \pi/2$ for $\lambda_0 < 0$ and at $\theta = 3\pi/2$ for $\lambda_0 > 0$. Therefore, the λ_0 for the shortest spiking time with control I^* satisfying $|I^*(t)| \le M$ can be calculated by substituting $I^* = M$ and $\theta = \pi/2$ to the equation (13), and then from (12) we obtain this shortest spiking period

$$T^{I^*}_{min} = \int_0^{2\pi} \frac{1}{\sqrt{\omega^2 + z_dM(z_dM + 2\omega)\sin^2(\theta)}} d\theta, \qquad (16)$$

Note that $T^M_{min} < T^{I^*}_{min}$. According to (3) when $M \ge \omega/z_d$, arbitrarily large spiking times can be achieved by making $\dot{\theta}$ arbitrary close to zero. Therefore we consider two cases for $M \ge \omega/z_d$ and $M < \omega/z_d$.



(a)



(b)

Fig. 4. Optimal bounded and unbounded controls for the sinusoidal PRC with parameters $z_d = 1$ and $\omega = 1$ (a) for $T = 2.8$ with the bound $M = 2.5$ and (b) for $T = 10$ with the bound $M = 0.55$.

1) Case I ($M \geq \omega/z_d$): Since $|I^*| < \omega/z_d \leq M$ for $\lambda_0 > 0$, I^* is the minimum-power control for any desired spiking time $T > 2\pi/\omega$, and hence for any spiking time $T \geq T_{min}^{I^*}$. Shorter spiking times $T \in [T_{min}^M, T_{min}^{I^*})$ are feasible but, due to the bound M, can not be achieved by I^* since it requires a control with amplitude greater than M for some $t \in [0,T]$. However, these spiking times can be optimally achieved by applying controls switching between I^* and $I_{T_{min}}^*$.

Let the desired spiking time $T \in [T_{min}^M, T_{min}^{I^*})$. Then, there exist two angles $\theta_1 = \sin^{-1}[-2M\omega/(z_d M^2 + z_d \omega\lambda_0)]$ and $\theta_2 = \pi - \theta_1$ where I^* meets the bound M. When $\theta \in (\theta_1, \theta_2)$, $I^* > M$ and we take $I(\theta) = M$ for $\theta \in [\theta_1, \theta_2]$. Due to the quadratic nature of the Hamiltonian with respect to I as in (4), if the minimum is not feasible then the boundary will be the optimal. The Hamiltonian of the system when $\theta \in [\theta_1, \theta_2]$ is, from (4), $H = M^2 + \lambda(\omega + z_d \sin\theta M)$. If the triple (λ, θ, M) is optimal, then H is a constant, which gives

$$\lambda = \frac{H - M^2}{\omega + z_d M \sin\theta}.$$

This multiplier satisfies the adjoint equation (5), and therefore $I(\theta) = M$ is optimal for $\theta \in [\theta_1, \theta_2]$. Similarly, by symmetry, $I^* < -M$ when $\theta \in [\theta_3, \theta_4]$, where $\theta_3 = \pi + \theta_1$ and $\theta_4 = 2\pi - \theta_1$, if the desired spiking time $T \in [T_{min}^M, T_{min}^{I^*})$. It can be easily shown by the same fashion that $I(\theta) = -M$ is optimal in the interval $\theta \in [\theta_3, \theta_4]$.

Therefore, the minimum-power optimal control that spikes the neuron at $T \in [T_{min}^M, T_{min}^{I^*})$ can be characterized by four switchings between I^* and M, i.e.,

$$I_1^* = \begin{cases} I^* & 0 \leq \theta < \theta_1 \\ M & \theta_1 \leq \theta \leq \theta_2 \\ I^* & \theta_2 < \theta < \theta_3 \\ -M & \theta_3 \leq \theta \leq \theta_4 \\ I^* & \theta_4 < \theta \leq 2\pi. \end{cases} \quad (17)$$

The initial value of the multiplier, λ_0, resulting in the optimal trajectory, can then be found according to the desired spiking time $T \in [T_{min}^M, T_{min}^{I^*})$ through the relation

$$T = \int_0^{\theta_1} \frac{4}{\sqrt{\omega^2 - \omega\lambda_0 z_d^2 \sin^2\theta}} d\theta + \int_{\theta_1}^{\frac{\pi}{2}} \frac{4}{\omega + z_d M \sin(\theta)} d\theta.$$

From (15) the minimum possible spiking time with the control bound $M = 2.5$ for $z_d = 1, \omega = 1$ is $T_{min}^M = 2.735$ and from (16) the minimum spiking time by I^* is $T_{min}^{I^*} = 3.056$. Thus, in this example, any desired spiking time $T > 3.056$ can be optimally achieved by I^* whereas any $T \in [2.735, 3.056)$ can be optimally obtained by I_1^* as in (17). FIG. 4(a) illustrates the bounded and unbounded optimal controls that fire the neuron at $T = 2.8$, where I^* is the minimum-power stimulus when the control amplitude is not limited and I_1^* is the minimum-power stimulus when the bound $M = 2.5$. I^* drives the neuron from $\theta(0) = 0$ to $\theta(2.8) = 2\pi$ with 13.54 units of power whereas I_1^* requires 14.13 units.

2) Case II ($M < \omega/z_d$): In contrast with Case *I* in the previous section, achieving arbitrarily large spiking times is not feasible with a bound $M < \omega/z_d$. In this case, the longest possible spiking time is achieved by control,

$$I_{T_{max}}^* = \begin{cases} -M & \text{for } 0 \leq \theta < \pi, \\ M & \text{for } \pi \leq \theta < 2\pi. \end{cases}$$

The spiking time of the neuron under this control is

$$T_{max}^M = 2\pi\sqrt{\frac{1}{-z_d^2 M^2 + \omega^2}} + \frac{4\tan^{-1}\{z_d M/\sqrt{-z_d^2 M^2 + \omega^2}\}}{\sqrt{-z_d^2 M^2 + \omega^2}}, \quad (18)$$

and the longest spiking time feasible with control I^* is given by,

$$T_{max}^{I^*} = \int_0^{2\pi} \frac{1}{\sqrt{\omega^2 + z_d M(z_d M - 2\omega)\sin^2(\theta)}} d\theta. \quad (19)$$

Similar to Case *I*, any spiking time $T \in [T_{min}^M, T_{min}^{I^*})$ for a given $M < \omega/z_d$ can be achieved with the minimum-power control I_1^* as given in (17), any $T \in [T_{min}^{I^*}, T_{max}^{I^*}]$ can be achieved with minimum power by I^* in (13), and moreover any $T \in (T_{max}^{I^*}, T_{max}^M]$ can be obtained by switching between I^* and $I_{T_{max}}^*$. The corresponding switching angles are $\theta_5 = \sin^{-1}[2M\omega/(z_d M^2 + z_d\omega\lambda_0)], \theta_6 = \pi - \theta_5, \theta_7 = \pi + \theta_5$ and $\theta_8 = 2\pi - \theta_5$, and the minimum-power optimal control for $T \in (T_{max}^{I^*}, T_{max}^M]$ is characterized by

$$I_2^* = \begin{cases} I^* & 0 \leq \theta < \theta_5 \\ -M & \theta_5 \leq \theta \leq \theta_6 \\ I^* & \theta_6 < \theta < \theta_7 \\ M & \theta_7 \leq \theta \leq \theta_8 \\ I^* & \theta_8 < \theta \leq 2\pi. \end{cases}$$

The λ_0 resulting in the optimal trajectory by I_2^* can be calculated according to the given $T \in (T_{max}^{I^*}, T_{max}^M]$ via the relation

$$T = \int_0^{\theta_5} \frac{4}{\sqrt{\omega^2 - \omega\lambda_0 z_d^2 \sin^2\theta}} d\theta + \int_{\theta_5}^{\frac{\pi}{2}} \frac{4}{\omega - z_d M \sin\theta} d\theta.$$

From (18) the maximum possible spiking time with $M = 0.55$ is $T_{max}^M = 10.312$ and from (19) the maximum spiking time feasible by I^* is $T_{max}^{I^*} = 9.006$. Therefore, in this example, any desired spiking time $T \in (9.006, 10.312]$ can be obtained with minimum power by the use of I_2^*. FIG. 4(b) illustrates the bounded and unbounded optimal controls that spike the neuron at $T = 10$, where I^* is the minimum-power stimulus when the control amplitude is not limited and I_2^*

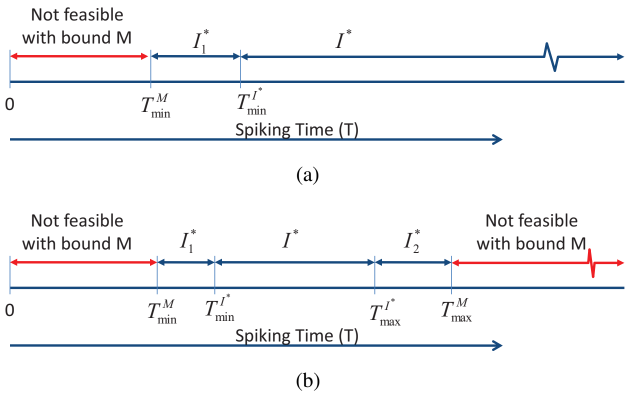


Fig. 5. (a) A summary of optimal controls for the case $M \geq \omega/z_d$, and (b) for $M < \omega/z_d$.

is the minimum-power stimulus when $M = 0.55$. I^* drives the neuron from $\theta(0) = 0$ to $\theta(10) = 2\pi$ with 2.193 units of power whereas I_2^* requires 2.327 units.

A summary of the optimal (minimum-power) spiking scenarios for a prescribed spiking time of the neuron governed by the sinusoidal phase model (3) is illustrated in Fig. 5.

IV. Minimum-Power Stimulus for Practical PRC

Many of experimentally determined PRC's for real neurons are not of sinusoidal, which is an approximation arising from the study of mathematical models of neuron oscillators close to certain bifurcations. In the following, we apply the optimal control strategies derived in Section III to practical PRC's including Morris-Lecar and Hodgkin-Huxley PRC.

A. Morris-Lecar PRC

The phase model of the Morris-Lecar neuron [19] is given by

$$\dot{\theta} = \omega + Z(\theta) I(t), \tag{20}$$

where the PRC, $Z(\theta)$, for a standard set of parameters given in Appendix I can be calculated by XPP [20] and is shown in Fig. 6. In this case, the neuron has the period $T = 22.212$ ms and natural frequency $\omega = 0.283$ rad/ms. Fig. 7 shows the optimal current stimuli without an amplitude constraint and the corresponding trajectories for various desired spiking times that are shorter, close, and longer than the natural spiking time, i.e., $T = 17, 22, 27$, respectively. With a bounded control amplitude, the feasible range of spiking times is limited. The possible range can be computed and it is $[19.623, 26.268]$ ms for the bound $M = 0.01\,\mu A$. FIG. 8(a) and 8(b) illustrate the unbound and bounded ($M = 0.01\,\mu A$) minimum-power controls for the spiking times $T = 20.0$ ms and $T = 25.5$ ms.

B. Hodgkin-Huxley PRC

The phase model of Hodgkin-Huxley neuron, describing the propagation of action potentials in a squid axon, is also of the form as (20) and is a canonical example of neural oscillator dynamics, with a nominal spiking period $T = 14.638$ ms and hence the frequency $\omega = 0.426$ rad/ms [21]. The PRC is

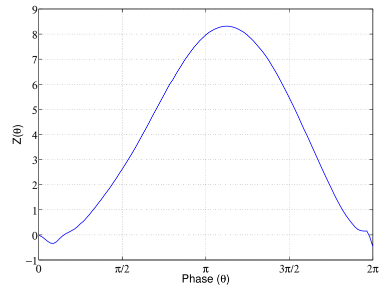


Fig. 6. Morris-Lecar PRC

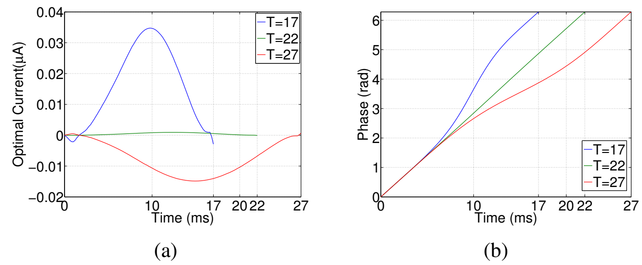


Fig. 7. (a) Optimal currents for various spiking times $T = 17, 22, 27$ *ms* for the Morris-Lecar PRC. (b) The corresponding phase trajectories.

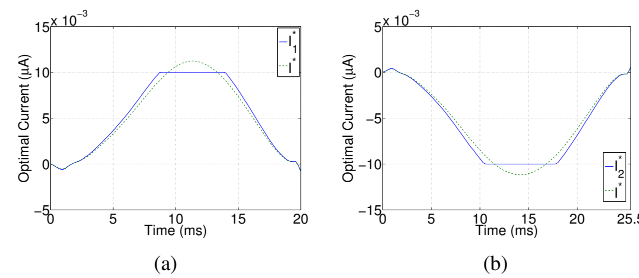


Fig. 8. Unbounded and bounded minimum-power controls for the Morris-Lecar model with $M = 0.01\,\mu A$ for the desired spiking time (a) $T = 20.0$ ms and (b) $T = 25.5$ ms.

shown in Fig. 9. The derived optimal current stimuli without an amplitude constraint and the corresponding trajectories for various desired spiking times are illustrated in Fig. 10. With a bounded control amplitude, the feasible range of spiking time is limited. The possible range can be computed and it is $[13.108, 17.492]$ ms for the bound $M = 1.0\,\mu A$. Fig. 11(a) and 11(b) illustrate the unbound and bounded minimum-power control for the spiking time $T = 13.3$ ms and $T = 16.7$ ms with bound $M = 1.0\,\mu A$.

V. Conclusion

In this paper, we studied various phase-reduced models that describe the dynamics of neuron systems. We considered the design of minimum-power stimuli for spiking a neuron at a specified time instant that is different from the natural spiking time. We formulated this as an optimal control problem and investigated both cases when the control amplitude is unbounded and bounded, for which we found analytic expressions of optimal feedback control laws. In particular for the bounded control case, we characterized the range of possible spiking periods with respect to the control bound.

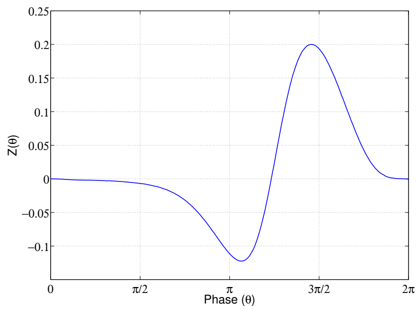
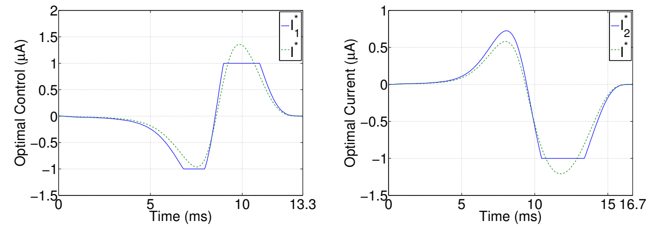
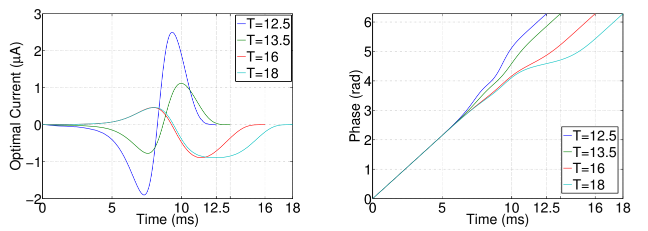


Fig. 9. Hodgkin-Huxley PRC



(a) (b)

Fig. 11. Unbounded and bounded minimum-power controls for the Hodgkin-Huxley model with $M = 1.0\,\mu A$ for the desired spiking time (a) $T = 13.3$ ms and (b) $T = 16.7$ ms.



(a) (b)

Fig. 10. (a) Optimal currents for various spiking times $T = 12.5, 13.5, 16, 18$ ms for the Hodgkin-Huxley PRC. (b) The corresponding phase trajectories following the optimal current stimuli.

The bound can be chosen sufficiently small within the range that the phase model of the neuron is valid.

Moreover, minimum-power stimulus for steering any nonlinear oscillator of the form as in (1) between desired initial and final states can be derived following the steps presented in this article. In addition, the charge-balanced constraint can be readily incorporated into this framework, which is of clinical importance especially in deep brain stimulations for Parkinson's disease [22]. The optimal control of a single neuron system investigated in this work illustrates the fundamental limit of spiking a neuron with external stimuli and provides a benchmark structure that enables us to study optimal control of spiking neuron populations.

Appendix I

The dynamics of the Morris-Lecar neuron is described by

$$\begin{aligned} C\dot{V} - I^b &= g_{Ca} m_\infty (V_{Ca} - V) + g_k w (V_k - V) + g_L (V_L - V) \\ \dot{w} &= \phi(\omega_\infty - w)/\tau_w(V) \\ m_\infty &= 0.5[1 + \tanh((V - V_1)/V_2)] \\ \omega_\infty &= 0.5[1 + \tanh((V - V_3)/V_4)] \\ \tau_\omega &= 1/\cosh[(V - V_3)/(2V_4)], \end{aligned}$$

where we consider the parameters

$$\begin{array}{lll} \phi = 0.5, & I^b = 0.09\ \mu A/cm^2, & V_L = -0.01\ mV, \\ v_2 = 0.15\ mV, & V_3 = 0.1\ mV, & v_4 = 0.145\ mV, \\ g_{Ca} = 1\ mS/cm^2, & V_k = -0.7\ mV, & V_L = -0.5\ mV, \\ g_k = 2\ mS/cm^2, & g_L = 0.5\ mS/cm^2, & C = 1\ \mu F/cm^2. \end{array}$$

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