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On Stability across a Gaussian Product Channel

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Abstract— **We present sufficient conditions for stabilizing a scalar discrete-time LTI plant in the mean squared sense when a sensor transmits the plant state information to a remotely placed controller across a Gaussian product channel. The Gaussian product channel models a continuous-time waveform Gaussian channel, where the encoder transmits information to the receiver across multiple noisy paths. It is known that linear coding schemes may lead to overly restrictive stabilizability conditions in such scenarios. We present a non-linear coding scheme and present the resulting stablizability conditions. When these conditions are satisfied with equality, the proposed coding scheme transmits data across the product channel at a rate equal to the capacity of the channel; thus, the conditions are conjectured to be necessary as well.**

I. Introduction

Networked control systems are now an active area of research. The performance of such systems is adversely affected by the detrimental effects such as random delays, data loss, data corruption, and so on introduced by the underlying communication network. The presence of various communication channel models in the control loop has been considered, including channels that introduce data loss (e.g., [5]), delay (e.g., [9]), digital noiseless channels (e.g., [10]), additive white Gaussian noise (AWGN) channels (e.g., [2]), etc.

In this paper, we are interested in stabilizability of a scalar unstable linear time invariant (LTI) discrete time system across a Gaussian product channel (also known as parallel Gaussian channels). Stability conditions in the presence of one AWGN channel are available (e.g., [2], [3]). Stabilizing the plant across a Gaussian relay [7], broadcast and multiple access channel [8] have also been considered. Interesting parallels of the problem with schemes achieving the capacity of a Gaussian channel with feedback through the Schalkwijk-Kailath (SK) scheme [12] are known [4].

The Gaussian product channel models a continuous-time waveform Gaussian channel in which the transmitter sends information to the receiver across multiple parallel channels, each parallel channel being individually modeled by an AWGN channel. The parallel channels may represent different frequency bands, time instances, or in general different "degrees of freedom". Control across such a channel is inherently more difficult than control across a single channel. For instance, while it is known that for a single AWGN channel, the optimal encoding policies are linear [1]–[3], for the parallel channel case, Yüksel and Taikonda [6] presented a counterexample which shows that in general linear encoding strategies may lead to overly restrictive stabilizability conditions[1]. Since design of optimal non-linear strategies is not trivial, analytical results and coding schemes for this setup are largely lacking. The inadequacy of linear controllers in achieving optimal stabilizability conditions or performance for a parallel Gaussian channel was also noted in [11]. However, neither [6] nor [11] proposed a non-linear controller and encoder structure.

By constructing one particular stabilizing non-linear encoder and decoder structure, we present sufficient conditions for stabilizing a scalar discrete-time LTI plant in the mean squared sense when the sensor transmits information to a remotely placed controller across a Gaussian product channel. When the sufficient conditions for stability using our coding scheme are satisfied with equality, data about the initial condition is transmitted at a rate equal to the capacity of a Gaussian product channel; thus, the conditions may be conjectured to be necessary as well.

II. Problem Setup

Consider the set-up shown in Fig. 1. The plant is described by an open loop unstable scalar linear time invariant process evolving as

$$S(k+1) = aS(k) + U(k), \tag{1}$$

where $S(k) \in \mathbb{R}$ is the state and $U(k) \in \mathbb{R}$ is the control value. We assume that the initial condition $S(0)$ is a random variable uniformly distributed in the interval $[c, d]$ with a finite variance $\sigma^2_{S(0)} = \frac{(d-c)^2}{12}$. For ease of exposition, and without loss of generality, we assume that at every time step a sensor observes the state of the process $S(k)$ and transmits information across the communication channel to the controller. The controller calculates a control input $U(k)$ and applies it to the process in (1). The communication channel from the plant to the controller is modeled as a Gaussian product channel, while the communication from the controller to the process is assumed to be perfect. The input and output of the i-th channel is denoted by X_i and Y_i respectively. The noise corrupting the i-th channel is denoted by Z_i. The output of the i-th channel at time k is given by

$$Y_i(k) = g_i X_i(k) + Z_i(k),$$

where g_i is attenuation due to path loss. The noises $Z_i(k)$ are modeled by a zero-mean AWGN with mean zero and variance σ_i^2. Moreover, the noises on the various links are assumed to be mutually independent and white. We impose

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[1]Note that while the general set-up in [6] considers a multi-sensor setting, the specific numerical example they provide is identical to the case when one sensor can transmit information across a Gaussian product channel with two parallel channels.

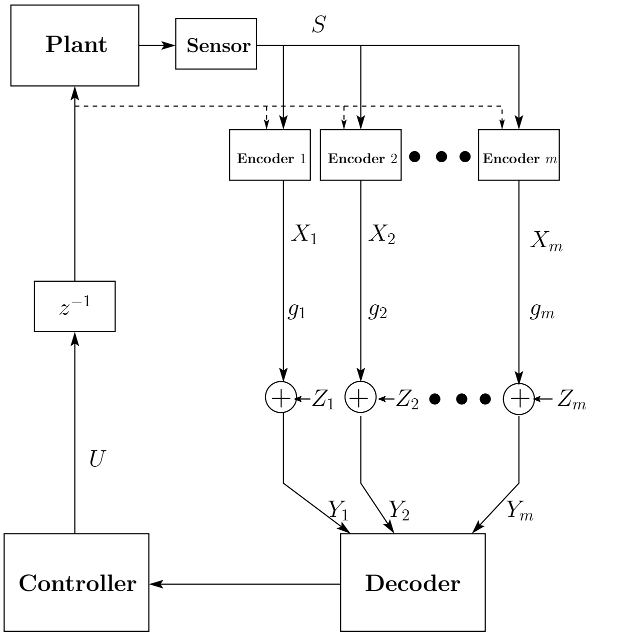

Fig. 1. Problem setup for an unstable plant being controlled by a controller across a Gaussian product channel

three constraints on the encoder and controller design:

- *Constraint C_1:* The control action must satisfy a controller cost constraint, $\sum_{k=0}^{\infty} \mathbb{E}[U(k)^2] < \infty$.
- *Constraint C_2:* There is an average power constraint imposed on the signals transmitted by the encoders on the different channels and a common power constraint on the total power used. Thus, the encoding schemes must be such that the transmitted signals satisfy

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}[X_i^2(k)] \le P_i,\ i = 1, 2, \dots m,$$
$$\sum_{i=1}^{m} P_i \le P.$$

- *Constraint C_3:* The encoders are assumed to be causal, but otherwise unconstrained in terms of computation and memory. The information structure at the encoders is as follows. If h_i be the encoding policy at the encoder for the i-th input, then

$$X_i(k) = h_i(S(0), \dots, S(k), U(0), \dots, U(k-1)).$$

The problem we are interested in this paper is to design the maps h_i's and controller $U(k)$ so that the process (1) is mean square stabilized, while satisfying the design constraints C_1, C_2 and C_3. The design of the encoder map involves designing a scheme to divide the total power amongst the various inputs $X_1, X_2, \dots X_n$ in an optimal way. Recall that a system is said to be stabilized in the mean squared sense if and only if irrespective of the initial state $S(0)$, the following conditions are satisfied:

$$\mathbb{E}[S(k)] = 0,$$
$$\lim_{k\to\infty} \mathbb{E}[S(k)S^T(k)] = 0. \tag{2}$$

III. Main Results

A. Preliminary Results

The encoders distribute the information about $S(0)$ amongst the various inputs $X_1(k), X_2(k), \dots, X_m(k)$. Let $S_i(k)$ be the information about $S(0)$ transmitted through the i-th ($i \in \{1, 2, \dots, m\}$) channel. $X_i(k)$ is calculated by scaling $S_i(k)$ to satisfy the power constraint C_2. The controller observes the outputs of the channels $Y_1(k), Y_2(k), \dots, Y_m(k)$ and extracts relevant information $\hat{S}_1(k), \hat{S}_2(k), \dots, \hat{S}_m(k)$. The estimate $\hat{S}(k)$ of the initial state $S(0)$ can be then calculated at the decoder as a function of the information collected from different links, i.e.,

$$\hat{S}(k) = f(k, \hat{S}_1(k), \hat{S}_2(k), \dots, \hat{S}_m(k)).$$

We define the overall estimation error as

$$\epsilon(k) := \hat{S}(k) - S(0),$$

and estimation error for information sent through the i-th channel as $\epsilon_i(k) := \hat{S}_i(k) - S_i(0)$. Let $\alpha(k)$ (resp. $\alpha_i(k)$) represent the variance of the estimation error $\epsilon(k)$ (resp. $\epsilon_i(k)$). The crucial property that needs to be satisfied for mean square stability is that the estimate $\hat{S}(k)$ converges to $S(0)$ at a rapid enough rate as shown by the following result.

Lemma 1: The LTI system in (1) can be mean square stabilized over a communication channel if the following conditions are satisfied:

$$\mathbb{E}[\epsilon(k)] = 0,$$
$$\lim_{k\to\infty} a^{2k}\mathbb{E}[\epsilon^2(k)]] = 0. \tag{3}$$

Proof: Since the controller does not know the state value $S(0)$ exactly, the controller takes actions using the estimates $\hat{S}(k)$. The controller actions are defined as

$$U(k) := \begin{cases} -a\hat{S}(0), & k = 0 \\ -a^{k+1}(\hat{S}(k) - \hat{S}(k-1)), & k \ge 1. \end{cases} \tag{4}$$

Thus, the state $S(k)$ and its moments evolve as

$$S(k+1) = -a^{k+1}\left(\hat{S}(k) - S(0)\right) = -a^{k+1}\epsilon(k), \tag{5}$$
$$\mathbb{E}[S(k+1)] = -a^{k+1}\mathbb{E}[\epsilon(k)],$$
$$\mathbb{E}[S^2(k+1)] = a^{2(k+1)}\mathbb{E}[\epsilon^2(k)].$$

Thus, if the conditions in (3) are satisfied, the process is mean square stabilized. ■

The above result also presents the controller design. We will now present our coding scheme and show that it does satisfy the constraints in (3). We start with considering the special case when we have only one channel ($m = 1$), then extend it to the case when there are two channels ($m = 2$) and then generalize it for m channels.

B. Special Case m = 1

The code for a setting with $m = 1$ works as follows. Note that since there is only one channel, $S_1(0) = S(0), \hat{S}_1(k) = \hat{S}(k), \epsilon_1(k) = \epsilon(k)$ and $P_1 = P$.

Initialization: At time step $k = 0$, the encoder transmits

$$X_1(0) = \sqrt{\frac{P_1}{\sigma^2_{S_1(0)}}} S_1(0). \tag{6}$$

The decoder forms an estimate of $S_1(0)$ as follows:

$$\hat{S}_1(0) = \frac{1}{g_1}\sqrt{\frac{\sigma^2_{S_1(0)}}{P_1}}Y_1(0).$$

The estimation error $\epsilon_1(0)$ is given by

$$\epsilon_1(0) = \frac{1}{g_1}\sqrt{\frac{\sigma^2_{S_1(0)}}{P_1}}Z_1(0).$$

Clearly, $\epsilon_1(0)$ is zero-mean Gaussian with variance $\alpha_1(0)$, given by

$$\alpha(0) = \alpha_1(0) = \frac{\sigma^2_{S_1(0)}\sigma_1^2}{g_1^2 P_1}. \quad (7)$$

The controller calculates the control $U(0)$ according to (4) and transmits its to the process.

Update: At each time step $k \geq 1$, the encoder transmits

$$X_1(k) = \sqrt{\frac{P_1}{\alpha_1(k-1)}}\epsilon_1(k-1). \quad (8)$$

The decoder updates its estimate as follows. At time $k \geq 1$, the decoder calculates the linear minimum mean squared error (MMSE) estimate of $S_1(0)$ given $Y_1(k)$ and $\hat{S}_1(k-1)$ as

$$\hat{S}_1(k) = \hat{S}_1(k-1) - \frac{\mathbb{E}[Y_1(k)\epsilon_1(k-1)]}{\mathbb{E}[{Y_1}^2(k)]}Y_1(k). \quad (9)$$

The controller calculates the control $U(k)$ according to (4) and transmits its to the process. Note that the input $X_1(k)$ satisfies the respective power constraint and that subsequent transmissions are orthogonal to each other (since the MMSE estimation error $\epsilon_1(k)$ is orthogonal to all observations).

It can be seen that the estimation error $\epsilon_1(k)$ are Gaussian with zero mean and variance $\alpha_1(k)$. We now proceed to evaluate the recursive expression for $\alpha_1(k)$ as used in the coding scheme presented above. Since $\epsilon_1(k)$ is defined as $\hat{S}_1(k) - S_1(0)$, from (9) we obtain

$$\epsilon_1(k) = \epsilon_1(k-1) - \frac{\mathbb{E}[Y_1(k)\epsilon_1(k-1)]}{\mathbb{E}[{Y_1}^2(k)]}Y_1(k). \quad (10)$$

The variance of $\epsilon_1(k)$ can be obtained as

$$\alpha_1(k) = \mathbb{E}[\epsilon_1^2(k)] = \alpha_1(k-1) - \frac{\mathbb{E}^2[Y_1(k)\epsilon_1(k-1)]}{\mathbb{E}[{Y_1}^2(k)]}, \quad (11)$$

with the initial condition in (7). The terms in (11) can be further evaluated to be

$$\mathbb{E}[{Y_1}^2(k)] = g_1^2 P_1 + \sigma_1^2, \quad (12)$$

and

$$\mathbb{E}[Y_1(k)\epsilon_1(k-1)] = g_1\sqrt{P_1\alpha_1(k-1)}. \quad (13)$$

Using (12) and (13) in (11), we obtain

$$\alpha_1(k) = \alpha_1(k-1)r_1, \quad (14)$$

where $r_1 = \left(\frac{\sigma_1^2}{g_1^2 P_1 + \sigma_1^2}\right)$. Note that since $\alpha(k) = \alpha_1(k)$ and $P_1 = P$,

$$\alpha(k) = \frac{\sigma^2_{S(0)}\sigma_1^2}{g_1^2 P}\left(\frac{\sigma_1^2}{g_1^2 P + \sigma_1^2}\right)^k. \quad (15)$$

We now present the stability conditions when the coding scheme described above is used to stabilize the process (1).

Theorem 2: Consider the problem formulation presented in Section II with the coding scheme presented above in Section III-B for $m = 1$. The process (1) is mean square stabilized over the point-to-point channel if

$$\log(a) < \frac{1}{2}\log\left(1 + \frac{g_1^2 P}{\sigma_1^2}\right). \quad (16)$$

Proof: It is easy to see that $\mathbb{E}[\epsilon(0)] = 0$. It is known that the linear minimum mean squared error is an unbiased estimator. Thus, $\mathbb{E}[\epsilon(k)] = 0$ for all $k \geq 0$, which is the first condition in (3). The theorem follows using the second condition in (3) and (15). ∎

Note that the right hand side of the condition in (16) is also the maximum rate at which information can be transmitted over a Gaussian point-to-point channel.

C. Special Case m=2

For pedagogical ease, we consider first the case when $m = 2$ before we present the scheme for arbitrary m. To develop a coding scheme for the case when more than one channel is present, we revisit a relevant result from information theory [13], and recognized also in [6]. For a distributed source-channel coding to be optimal in the information-theoretic sense, two conditions need to be satisfied:

- The information transmitted on all the channels should be independent.
- Capacity is utilized by all the channels (source-channel needs to be matched).

It is not possible to make the signals transmitted on different channels independent when linear schemes are used, which implies that linear schemes are not optimal [6]. We develop a non-linear encoding scheme which will ensure that we transmit independent information over the two channels.

For transmission over the Gaussian product channel with $m = 2$, consider the following construction. Recall that $S(0)$ is uniformly distributed over $[c, d]$. Divide the interval $[c, d]$ into M_1 (we will define how to choose M_1 later) disjoint, equal-length message intervals as shown in Fig. 2. Let c_j where $j \in \{1, 2, \ldots, M_1 - 1\}$ be the partition points. Also define $c_0 := c$ and $c_{M_1} := d$. A point x is said to be in the j-th interval I_j if $x \in [c_{j-1}, c_j]$. The message to be sent on the first channel corresponds to the output of a quantizer $Q_1(\cdot)$ which maps each point of the j-th interval ($j \in \{1, 2, \ldots, m\}$), to the midpoint of that particular interval. The message to be sent on the second channel corresponds to the quantization error $S(0) - Q_1(S(0))$. Thus we design the quantizer $Q_1(\cdot)$ as follows:

$$\begin{aligned}\tilde{S}_1(0) &= Q_1(S(0)) \\ &= a + \left(j - \frac{1}{2}\right)\frac{1}{M_1} \text{ if } S(0) \in I_j, \\ \tilde{S}_2(0) &= S(0) - \tilde{S}_1(0). \end{aligned} \quad (17)$$

Lemma 3: The random variables $\tilde{S}_1$ and $\tilde{S}_2$ defined in (17) are independent.

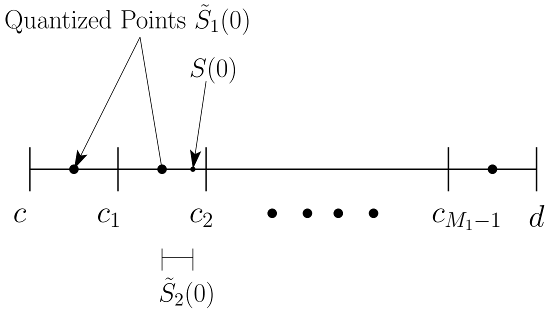

Fig. 2. Generation of $\tilde{S}_1$ and $\tilde{S}_2$

Proof: Consider the conditional probability $Pr(\tilde{S}_2 = \gamma|\tilde{S}_1 \in I_j)$.

$$\begin{aligned}Pr(\tilde{S}_2{=}\gamma|\tilde{S}_1 \in I_j) &= \frac{Pr(\tilde{S}_2{=}\gamma, \tilde{S}_1 \in I_j)}{Pr(\tilde{S}_1 \in I_j)}\\ &= M_1 Pr(\tilde{S}_2{=}\gamma, \tilde{S}_1 \in I_j)\\ &= \sum_{j=1}^{M_1} Pr(\tilde{S}_2{=}\gamma, \tilde{S}_1 \in I_j)\\ &= Pr(\tilde{S}_2{=}\gamma).\end{aligned}$$

Thus, the random variables are independent. ∎

Now, we define the messages to be sent on the parallel channels as

$$S_1(0) \triangleq \tilde{S}_1(0), \;\; S_2(0) \triangleq M_1\tilde{S}_2(0).$$

It can be seen that both $S_1(0)$ takes values uniformly from a set with M_1 elements. The number of intervals M_1 is related the information rate R_1 and number of channel uses k over the first channel as

$$M_1 = 2^{kR_1}. \tag{18}$$

Note also that $S_2(0)$ is uniformly distributed in the interval $[c, d]$ and thus has a variance $\sigma^2_{S_2(0)} = \frac{(d-c)^2}{12}$.

Now we send the messages $S_1(0)$ and $S_2(0)$ over the two channels recursively and independently of each other in the same way as we sent $S_1(0)$ in Section III-B (See (6) and (8)). The decoder forms estimates $\hat{S}_i(k)$ of $S_i(0)$, $i = 1, 2$, the variances of which can written using (15) as

$$\alpha_1(k) = \frac{\sigma^2_{S_1(0)}\sigma^2_1}{g_1^2 P_1}\left(\frac{\sigma_1^2}{g_1^2 P_1 + \sigma_1^2}\right)^k \triangleq \alpha_1(0) r_1^k, \tag{19}$$

$$\alpha_2(k) = \frac{\sigma^2_{S_2(0)}\sigma^2_2}{g_2^2 P_2}\left(\frac{\sigma_2^2}{g_2^2 P_2 + \sigma_2^2}\right)^k \triangleq \alpha_2(0) r_2^k. \tag{20}$$

Note that the estimation error does not depend on the control inputs, and hence does not effect the controller design. The control input in this case is calculated as

$$U(k) = \left[U_1(k), U_2(k), U_1(k) + \tfrac{U_2(k)}{M_1}\right]^T, \tag{21}$$

$$U_i(k) \triangleq \begin{cases} -a\hat{S}_i(0), & k = 0\\ -a^{k+1}(\hat{S}_i(k) - \hat{S}_i(k-1)), & k \geq 1.\end{cases} \tag{22}$$

Note that T represents a transpose. The third component of $U(k)$ as defined above is extracted and applied to the process (1), whereas the i-th component ($i = 1, 2$) is used by the encoder i to update the i-th input, as given by equation (8). Note that because of the construction described above (17), the information sent on the parallel channels $i = 1, 2, \ldots, m$ are mutually independent. Also, except at time step $k = 0$, the inputs to both channels have a Gaussian distribution and are thus matched to the respective Gaussian channels.

Theorem 4: Consider the problem formulation presented in Section II with the coding scheme presented above for $m = 2$. The process (1) is mean square stabilized over the Gaussian product channel with $m = 2$ if

$$\log(a) < \max_{\sum_{i=1}^2 P_i = P} \sum_{i=1}^{2} \frac{1}{2}\log\left(1 + \frac{g_i^2 P_i}{\sigma_i^2}\right). \tag{23}$$

Proof: It is easy to see that $\mathbb{E}[\epsilon(0)] = \mathbb{E}[\epsilon_1(0) + \frac{\epsilon_2(0)}{M_1}] = 0$. It is known that the linear minimum mean squared error is an unbiased estimator. Thus, $\mathbb{E}[\epsilon(k)] = 0$ for all $k \geq 0$, which is the first condition in (3). To evaluate the estimation error variance $\alpha(k)$, we write

$$\begin{aligned}\mathbb{E}[\epsilon^2(k)] &\overset{(a)}{=} Pr(\hat{S}_1(k){\neq}S_1(0))\mathbb{E}[\epsilon^2(k)|\hat{S}_1(k){\neq}S_1(0)]\\ &\quad + Pr(\hat{S}_1(k){=}S_1(0))\mathbb{E}[\epsilon^2(k)|\hat{S}_1(k){=}S_1(0)]. \end{aligned}\tag{24}$$

The terms above can be written as follows.

$$\begin{aligned}Pr(\hat{S}_1(k){\neq}S_1(0)) &\leq Pr\left[|\epsilon_1(n)| > \frac{1}{2M_1}\right]\\ &= 2Q\left(\frac{1}{2M_1\sqrt{\alpha_1(n)}}\right)\\ &\overset{(a)}{=} 2Q\left(\frac{2^{k\left(R_1 - \frac{1}{2}\log(1+\frac{g_1^2 P_1}{\sigma_1^2})\right)}}{2\sqrt{\frac{\sigma^2_{S_1(0)}\sigma_1^2}{g_1^2 P_1}}}\right),\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\epsilon^2(k)|\hat{S}_1(k){\neq}S_1(0)] &\overset{(b)}{\leq} (d-c)^2,\\ Pr(\hat{S}_1(k){=}S_1(0)) &\leq 1,\\ \mathbb{E}[\epsilon^2(k)|\hat{S}_1(k){=}S_1(0)] &= \frac{\alpha_2(k)}{M_1^2},\\ &\overset{(c)}{=} \frac{\sigma_2^2}{12 g_2^2 P_2 2^{2kR_1}}\left(\frac{\sigma_2^2}{g_2^2 P_2 + \sigma_2^2}\right)^k,\end{aligned}$$

where $Q(x) \triangleq \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})\,\mathrm{d}y$, (a) follows from (18) and (19), (b) follows using the fact the estimation error is upper bounded by the maximum distance between any two points on $[d, c]$ and (c) follows from (18) and (20). Thus, we can upper bound $a^{2k}\mathbb{E}[\epsilon^2(k)]$ as

$$\begin{aligned}a^{2k}\mathbb{E}[\epsilon^2(k)] &\leq a^{2k} Q\left(\frac{2^{k\left(R_1 - \frac{1}{2}\log(1+\frac{g_1^2 P_1}{\sigma_1^2})\right)}}{2\sqrt{\frac{\sigma^2_{S_1(0)}\sigma_1^2}{g_1^2 P_1}}}\right)(d-c)\\ &\quad + \frac{\sigma_2^2}{12 g_2^2 P_2}\frac{a^{2k}}{2^{2kR_1}}\left(\frac{\sigma_2^2}{g_2^2 P_2 + \sigma_2^2}\right)^k. \end{aligned}\tag{25}$$

Since $Q(x) \sim \exp(\frac{-x^2}{2})$ for large x, the $Q(\cdot)$ term in (25) decreases doubly exponentially in k. On the other hand, the term a^{2k} increases exponentially. This implies that if $R_1 <$

$\frac{1}{2}\log\left(1+\frac{g_1^2 P_1}{\sigma_1^2}\right)$, the first term in (25) goes to zero as $k\to\infty$ irrespective of the value of a. Moreover, if

$$\log a < R_1 + \frac{1}{2}\log\left(1+\frac{g_2^2 P_2}{\sigma_2^2}\right) < \frac{1}{2}\log\left(1+\frac{g_1^2 P_1}{\sigma_1^2}\right) + \frac{1}{2}\log\left(1+\frac{g_2^2 P_2}{\sigma_2^2}\right),$$

then the second term in (25) also approaches zero as $k\to\infty$. Now since we are allowed to choose P_1 and P_2, we can optimize the right hand side of the above equation to increase the stability region. Thus, if the condition in (23) is satisfied, then $a^{2k}\alpha(k)\to 0$ and mean square stability is obtained. ■
The optimization in (23) can be carried out using Lagrange multipliers and is a standard result in information theory [14, Chapter 10]. An interpretation of the optimization is presented in Section III-F, after discussing the coding scheme for the case when $m>2$ channels are present.

D. General Case: Arbitrary value of m

For transmission over the Gaussian product channel with $m>2$ channels, consider the following construction. Divide the interval $[d,c]$ into M_1 disjoint, equal-length message intervals. Then divide each of these M_1 intervals into a further M_2 subintervals and so on till M_{m-1}. Define I_j^i to be the j-th interval ($j\in\{1,2,\dots,M_i\}$) for the i-th ($i\in\{1,2,\dots,m-1\}$) level quantizer Q_i. The message to be sent on the i-th channel ($i=1,2,\dots,m-1$) corresponds to the output of the i-th quantizer $Q_i(\cdot)$, which maps each point of the interval I_j^i to the midpoint of that particular interval. The message to be sent on the m-th channel corresponds to the quantization error. Thus we design a set of $m-1$ quantizers as follows:

$$\begin{aligned}\tilde{S}_1(0) &= Q_1(S(0)),\\ \tilde{S}_2(0) &= Q_2(S(0)-\tilde{S}_1(0)),\\ &\vdots\\ \tilde{S}_{m-1}(0) &= Q_{m-1}(S(0)-\sum_{i=1}^{m-2}\tilde{S}_i(0)),\\ \tilde{S}_m(0) &= S(0)-\sum_{i=1}^{m-1}\tilde{S}_i(0).\end{aligned}$$

We have the following lemma which is a generalization of Lemma 3 for arbitrary m.

Lemma 5: The random variables $\tilde{S}_i$, $i=1,2,\dots,m$ defined in (17) are mutually independent.

Proof: The proof is similar to the proof for Lemma 3 and has been omitted. ■
Now, we define the messages to be sent on the i-th ($i=1,\dots,m$) parallel channel as

$$S_i(0) \triangleq \left(\prod_{j=1}^{i-1} M_j\right)\tilde{S}_i(0).$$

Note that $S_i(0)$, $i=1,2,\dots,m-1$ takes values uniformly from a set with M_i elements. As before, M_i is related to the information rate R_i and number of channel uses k over the i-th channel as

$$M_i = 2^{kR_i},\ i=1,2,\dots,m-1. \tag{26}$$

Also, $\tilde{S}_m(0)$ is uniformly distributed in the interval $[d,c]$ and thus has a variance of $\frac{(d-c)^2}{12}$.

We send the messages $S_i(0)$, $i=1,2,\dots,m$ over the m channels recursively in the same way as we sent $S_1(0)$ in Section III-B (See (6) and (8)). The decoder forms estimates $\hat{S}_i(k)$ of $S_i(0)$, $i=1,2,\dots,m$, the variances of which can written down using (15) as

$$\alpha_i(k) = \frac{\sigma^2_{S_i(0)}\sigma_i^2}{g_i^2 P_i}\left(\frac{\sigma_i^2}{g_i^2 P_i+\sigma_i^2}\right)^k \triangleq \alpha_i(0) r_i^k. \tag{27}$$

The controller design is as follow. The controller calculates and transmits the input

$$U(k) = \left[U_1(k),\dots,U_m(k),\sum_{i=1}^m \frac{U_i}{\prod_{j=1}^{i-1} M_j}\right]^T, \tag{28}$$

$$U_i(k) \triangleq \begin{cases} -a\hat{S}_i(0), & k=0\\ -a^{k+1}(\hat{S}_i(k)-\hat{S}_i(k-1)), & k\geq 1.\end{cases} \tag{29}$$

The $m+1$-th component of $U(k)$ defined above is extracted and applied to the process (1), whereas the i-th component ($1\leq i\leq m$) is used by the encoder i to update the i-th input. Note that because of the construction described above, the information sent on the parallel channels $i=1,2,\dots,m$ are mutually independent. Also, except at time step $k=0$, the inputs to both channels have a Gaussian distribution and are thus matched to the respective Gaussian channels. We have the following theorem on stability.

Theorem 6: Consider the problem formulation presented in Section II with the coding scheme presented above for arbitrary m. The process (1) is mean square stabilized over the Gaussian product channel if

$$\log(a) < \max_{\sum_{i=1}^m P_i = P}\sum_{i=1}^m \frac{1}{2}\log\left(1+\frac{g_i^2 P_i}{\sigma_i^2}\right). \tag{30}$$

Proof: It is easy to see that $\mathbb{E}[\epsilon(0)]=0$. It is known that the linear minimum mean squared error is an unbiased estimator. Thus, $\mathbb{E}[\epsilon(k)]=0$ for all $k\geq 0$, which is the first condition in (3). Define the event $E := (\hat{S}_1(k)=S_1(0),\hat{S}_2(k)=S_2(0),\dots,\hat{S}_{m-1}(k)=S_{m-1}(0))$. We can write the estimation error variance $\alpha(k)$ as

$$\begin{aligned}&\mathbb{E}[\epsilon^2(k)] = Pr(\bar{E})\mathbb{E}[\epsilon^2(k)|\bar{E}] + Pr(E)\mathbb{E}[\epsilon^2(k)|E]\\ &\Rightarrow a^{2k}\mathbb{E}[\epsilon^2(k)] = a^{2k}(Pr(\bar{E})\mathbb{E}[\epsilon^2(k)|\bar{E}]+Pr(E)\mathbb{E}[\epsilon^2(k)|E])\end{aligned} \tag{31}$$

Using arguments similar to the proof of Theorem 4, we can prove that the first term in (31) goes to zero irrespective of the value of a if the following conditions are satisfied simultaneously.

$$R_i < \frac{1}{2}\log\left(1+\frac{g_i^2 P_i}{\sigma_i^2}\right)\ \forall i=1,2,\dots,m-1. \tag{32}$$

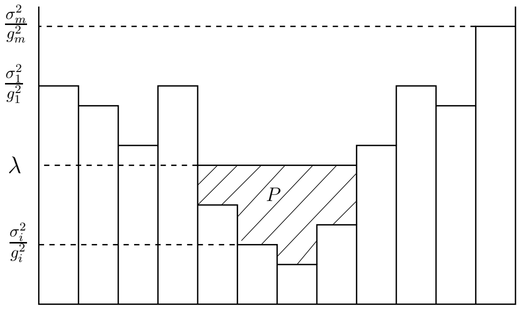

Fig. 3. Water-filling for the product channel. Area of the shaded region is equal to P.

To obtain a condition for the second term to approach zero, rewrite

$$a^{2k}Pr(E)\mathbb{E}[\epsilon^2(k)|E] \leq a^{2k}\mathbb{E}[\epsilon^2(k)|E] = \frac{a^{2k}\alpha_m(k)}{\prod_{i=1}^{m-1} M_i}$$
$$= \frac{\sigma_m^2}{12g_m^2 P_m}\frac{a^{2k}}{\prod_{i=1}^{m-1} 2^{2kR_i}}\left(\frac{\sigma_m^2}{g_m^2 P_m + \sigma_m^2}\right)^k.$$

Thus, a sufficient condition for the term to approach zero is that

$$\log a < \sum_{i=1}^{m-1} R_i + \frac{1}{2}\log\left(1 + \frac{g_m^2 P_m}{\sigma_m^2}\right)$$
$$< \sum_{i=1}^{m} \frac{1}{2}\log\left(1 + \frac{g_i^2 P_i}{\sigma_i^2}\right),$$

where the last inequality follows using (32). Now since we are allowed to choose P_i, we can optimize the right hand side of the above equation to increase the stability region. Thus, if the condition in (30) is satisfied, then $a^{2k}\alpha(k) \to 0$ and mean square stability is obtained. ■

E. Constraint C_1

The constraints C_2 and C_3 are satisfied by construction of the coding scheme. We can also show that the constraint C_1 is satisfied by the proposed design.

Proposition 7: The controller satisfies the cost constraint $\sum_{k=0}^{\infty}\mathbb{E}[U^2(k)] < \infty$.

Proof: From (29) and (9), we can write

$$U_i(k) = a^{k+1}\frac{\mathbb{E}[Y_i(k)\epsilon_i(k-1)]}{\mathbb{E}[{Y_i}^2(k)]}Y_i(k)$$
$$\Rightarrow \mathbb{E}[U_i^2(k)] = a^2\alpha(0)\frac{g_i^2 P_i}{\sigma_i^2}\prod_{j=0}^{k-1}\left[a^2\frac{\sigma_i^2}{g_i^2 P_i + \sigma_i^2}\right].$$

Using (30), it can seen that $\sum_{k=0}^{\infty}\mathbb{E}[U_i^2(k)] < \infty$ for all $i = 1, 2, \ldots, m$. Thus the result follows. ■

F. Water-filling Solution

The optimization problem in (23) or (30) can be solved using Lagrange multipliers and has a well known interpretation. The solution is given by [14, Chapter 10]

$$P_i = \left(\lambda - \frac{\sigma_i^2}{g_i^2}\right)^+ = \max\left\{\lambda - \frac{\sigma_i^2}{g_i^2}, 0\right\},$$

where the Lagrange multiplier λ is chosen to satisfy

$$\sum_{i=1}^{m}\left(\lambda - \frac{\sigma_i^2}{g_i^2}\right)^+ = P.$$

The optimal solution has the water-filling interpretation as shown in Fig. 3. The vertical levels indicate the noise levels in the various channels. As the power P is increased, power is first alloted to the channel with lowest noise, then the next lowest and so on. This power distribution is identical to the way water fills itself in a container, hence the name "water-filling".

IV. Conclusions and Future Work

In this paper, we derived sufficient conditions for mean square stabilizability of a scalar linear time invariant open loop unstable plant over a Gaussian product channel. When the sufficient conditions for stability using our coding scheme are satisfied with equality, data about the initial condition is being transmitted at a rate equal to the capacity of a Gaussian product channel, which suggests that our scheme might be optimal.

An immediate extension of this work would be to consider a vector plant. Similarly, the effect of process or (and) sensor noise in (1) can be considered. Generalizing the result for other probability distributions of the initial condition $S(0)$ is also an interesting direction for future work.

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