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Global Stabilization of Non–Globally Linearizable Triangular Systems: Application to Transient Stability of Power Systems

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***Abstract*— A general methodology to globally stabilize an equilibrium of a class of non-globally linearizable triangular systems is presented. The technique is applicable to all triangular systems described by analytic vector fields. The method is used to give an explicit solution to the challenging problem of transient stability of multimachine power systems with leaky transmission lines, for which only existence results are currently available.**

I. Introduction

Nonlinear dynamical systems in triangular forms have been widely studied in the control literature. In particular, the so-called backstepping technique [1], [2], [3] has proven to be successful to globally stabilize a given equilibrium of *globally feedback linearizable* triangular systems. The main objective of this paper is to propose a methodology for global stabilization of a general class of non–globally feedback linearizable triangular systems.

Several works are dedicated to stabilization of triangular systems that are only locally feedback linearizable, see e.g. [4], [5], [6], [7], [8], [9], [10], [11]. All these works either provide only existence results for a stabilizing control law, or require additional assumptions in order to compute an explicit control law. In this paper, we consider a particular class of triangular systems, but show that all triangular systems described by *analytic* vector fields can be written in this form, thus giving to the method a general validity.

As an application of the general theory developed in the paper we consider the *transient stability* problem for multi-machine power systems, consisting of N generators, non-linear loads and *leaky* transmission lines. Transient stability is concerned with the ability of the system to reach an acceptable steady-state following a fault, e.g., a short circuit or a generator outage, that is later cleared by the protective system operation. The fault modifies the circuit topology – driving the system away from the stable operating point – and the question is whether the trajectory remains in the basin of attraction of this (or another) equilibrium after the fault is cleared. The key analysis issue is then the evaluation of the domain of attraction of the system's operating equilibrium, while the control objective is the enlargement of the latter, see [12], [13], and the references therein, for more details and a literature review.

Similarly to [13], the full $3N$–dimensional model of the N–generator system with lossy transmission lines, loads and excitation controllers, is considered. In [13] the *existence* of a nonlinear static state feedback law that ensures asymptotic stability of the operating point with a well-defined estimate of the domain of attraction provided by a *bona fide* Lyapunov function is established. To the best of our knowledge, even in the lossless case, no *explicit* globally stabilizable state-feedback controller has yet been reported. Providing an affirmative answer to this problem, even for the lossy case, is a central contribution of this paper.

The remaining of the paper is organized as follows. To facilitate the understanding of the general methodology the material is presented in increasing degrees of complexity. First, a simple single-input, two-dimensional example is considered in Section II. Then, Sections III and IV are devoted to the extensions to general two- and three-dimensional systems, respectively. The final result is presented in Section V, where the n–dimensional, multi–input case is treated. Section VI is devoted to the application example, while Section VII contains some concluding remarks.

II. A motivating example

With the aim of providing the reader with an insight into the more general construction presented below, we begin by studying a simple triangular system. The system is described by the equations

$$\begin{aligned} \dot{x}_1 &= 1 - x_2^2\,, \\ \dot{x}_2 &= u\,. \end{aligned} \tag{1}$$

This system is not globally feedback linearizable, since the relative degree is not defined for $x_2 = 0$. However, the set of equilibrium points for this system is $\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| = 1\}$ and local feedback linearization is possible around each equilibrium point. Suppose that the equilibrium to be stabilized is $(x_1, x_2) = (0, 1)$; the classical backstepping approach provides a solution through the dynamics $\dot{z}_1 = z_2$ and $\dot{z}_2 = v$ for the variables $z_1 = x_1$ and $z_2 = 1 - x_2^2$ and the control $v = -2x_2 u$. Selecting the stabilizing control for the z–dynamics as $v = -z_1 - z_2$, yields the control law

$$u = -\frac{1}{2x_2}(-x_1 - 1 + x_2^2)\,.$$

This solution has two limitations; the first one, which is due to the fact that the system is only locally feedback

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linearizable, is that the control law is not defined at $x_2 = 0$. The second limitation is that the solution does not distinguish between the points $(0, 1)$ and $(0, -1)$, since we have both

$$\dot{x}_1|_{x_1=0,x_2=1} = 0\,, \qquad \dot{x}_2|_{x_1=0,x_2=1} = 0$$

and

$$\dot{x}_1|_{x_1=0,x_2=-1} = 0\,, \qquad \dot{x}_2|_{x_1=0,x_2=-1} = 0\,.$$

We now show a possible way to overcome both limitations. Note that other methods based on modified backstepping approach could be used to stabilize the equilibrium of system (1), for instance a saturated backstepping as in [14]; however, extending these techniques to n-dimensional systems (such as systems (16) and (23) below) may result in cumbersome structures. On the contrary, the extension of the method presented herein to MIMO system does not introduce further complexity (see, for comparison, the control laws (13), (22) and (26) below). To streamline our statements the following definition is introduced.

Definition 2.1: Let $X \subset \mathbb{R}^n$ be an open and path connected set. A function $f : X \to \mathbb{R}$ is *positive definite with respect to* $x_0 \in X$ if $f(x_0) = 0$ and $f(x) > 0$ for all $x \neq x_0$. It is *radially unbounded in* X if there exists a homeomorphism $\varphi : X \to \mathbb{R}^n$ such that $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded.

Introduce the variables ξ and x_2^* verifying the equality $1 - x_2^*\xi = -\tanh(x_1)$. An input $u(t)$ such that

$$|x_2^*(t) - x_2(t)| \to 0, \quad |\xi(t) - x_2(t)| \to 0, \quad x_2(t) \to 1,$$

as $t \to \infty$, renders the equilibrium $(0, 1)$ of (1) attractive. To find this control law, consider the Lyapunov function

$$V(x_1, x_2, \xi) = \frac{1}{2}(x_1^2 + (\xi - x_2)^2 + (x_2 - x_2^*)^2),$$

which is clearly motivated by our desire to drive x_1 to zero and by the asymptotic objectives given above. Computing its time-derivative we obtain

$$\dot{V} = x_1(1-x_2^2)+(\xi-x_2)(\dot{\xi}-u)+(x_2-x_2^*)(u-\dot{x}_2^*)\,. \quad (2)$$

The right hand side of (2) suggests the choice of the dynamics of ξ as

$$\dot{\xi} = u + x_2 - \xi - x_1 x_2^*, \quad (3)$$

which replaced in (2) yields

$$\begin{array}{rcl} \dot{V} & = & -x_1\tanh(x_1) - (\xi - x_2)^2 - x_1x_2(x_2 - x_2^*) + \\ & + & (x_2 - x_2^*)(u - \dot{x}_2^*) \\ & = & -x_1\tanh(x_1) - (\xi - x_2)^2 - (x_2 - x_2^*)^2\,, \end{array}$$

where the last equation is obtained selecting the control law

$$u = \dot{x}_2^* + x_1x_2 - (x_2 - x_2^*). \quad (4)$$

Hence, $\dot{V} \leq 0$ and $\dot{V} = 0$ when $x_1 = 0$, $\xi = x_2$ and $x_2^* = x_2$. Applying La Salle's invariance principle, when $x_1 \equiv 0$ we have $x_2 \in \{-1, 1\}$. Since V is radially unbounded in $\mathbb{R}^2 \times \mathbb{R}^+$ (and positive definite with respect to $(0, 1, 1)$), if the initial condition for ξ is chosen in $\mathbb{R}^+$, we have $\xi(t) \to 1$ and hence $x_2(t) \to 1$.

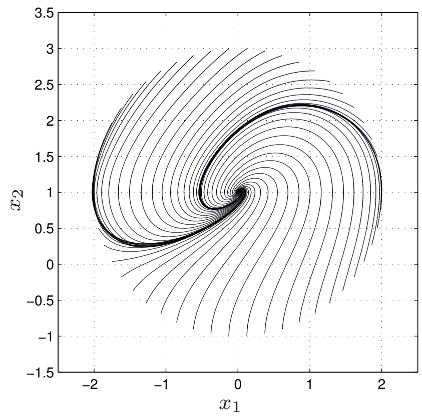

Fig. 1. Trajectories of the state (x_1, x_2) of system (1) with control (4) for initial conditions on the circle $x_1^2 + (x_2 - 1)^2 = 4$.

The first advantage obtained by introducing the variable ξ is that we can assure that x_2 tends to 1. Moreover, the choice of the hyperbolic tangent ensures that the input signal is defined for all possible trajectories. Indeed, replacing the definition of x_2^*, that is,

$$x_2^* = \frac{1 + \tanh(x_1)}{\xi}\,,$$

in (4), the control signal can be written in the form

$$u(x_1, x_2, \xi) = \frac{\xi^2 \widetilde{u}(x_1, x_2, \xi)}{\xi^2 + 1 + \tanh(x_1)}\,, \quad (5)$$

where $\widetilde{u}(x_1, x_2, \xi)$ is well-defined, and the denominator is always positive.

Simulations have been carried out to shown the effectiveness of the method. In Figure 1 the trajectory of the state (x_1, x_2) of system (1) with the control (4) are depicted for several initial conditions lying in the circle $x_1^2 + (x_2 - 1)^2 = 4$. The initial condition for ξ is the equilibrium value $\xi = 1$. In Figure 2 the time histories of x_1, x_2, ξ and u for the initial condition $(x_1(0), x_2(0), \xi(0)) = (2, 1, 1)$ are shown.

In the following section we generalize the previous derivations, and propose a systematic method applicable to a generic planar system in triangular form.

III. The two-dimensional SISO case

Consider a two-dimensional system in triangular form described by the equations

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, x_2)\,, \\ \dot{x}_2 & = & u\,, \end{array} \quad (6)$$

where $x_1(t) \in \mathbb{R}$, $x_2(t) \in \mathbb{R}$, $u(t) \in \mathcal{U} \subset \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Suppose that $(x_1, x_2) = (0, 0)$ is an equilibrium. To formalize the method of the previous example we begin with the following definition.

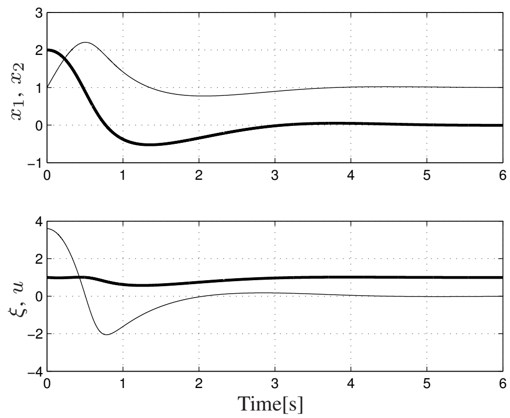

Fig. 2. Time histories of the states x_1 (top, bold line), x_2 (top, solid line) and ξ (bottom, bold line), u (bottom, solid line) of (1)-(4) for initial conditions $(x_1(0), x_2(0), \xi(0)) = (2, 1, 1)$.

Definition 3.1: A function $\sigma : \mathbb{R} \to \mathbb{R}$ has the *odd sign property* if $\sigma(x)x > 0$ for $x \neq 0$ and $\sigma(0) = 0$.

As shown in the example, the task of rendering the equilibrium attractive can be accomplished if there exists an input $u(t)$ such that, for some function $\sigma : \mathbb{R} \to \mathbb{R}$ with the odd sign property,

$$\lim_{t\to\infty} |f(x_1(t), x_2(t)) + \sigma(x_1(t))| = 0$$

and $x_2(t) \to 0$ when $t \to \infty$, thus assuring attractivity of the zero-equilibrium. We now show that this is possible if f fulfills the condition below.

Assumption 3.1: There exist two function $k : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$, having the same regularity properties as f, with $k(0) \geq \epsilon > 0$, such that f can be written as

$$f(x_1, x_2) = k(x_1) - (x_2 + \lambda)h(x_1, x_2), \qquad (7)$$

for some nonzero constant $\lambda \in \mathbb{R}$.

Note that the requirement $k(0) \geq \epsilon > 0$ for all x_1 implies $h(0, 0) \neq 0$. Introduce a new state ξ, the dynamics of which are specified below, and let x_2^* be the solution of

$$(x_2^* + \lambda)\xi = \sigma(x_1) + k(x_1)\,. \qquad (8)$$

Now, using (7), consider the augmented system

$$\begin{array}{rcl} \dot{x}_1 & = & -(x_2 + \lambda)h(x_1, x_2) + k(x_1)\,, \\ \dot{x}_2 & = & u\,, \\ \dot{\xi} & = & \dot{h} + h(x_1, x_2) - \xi - x_1(x_2^* + \lambda)\,. \end{array} \qquad (9)$$

Define the point

$$(x_1, x_2, \xi) = (0, 0, h(0, 0)) := \mathcal{E}_2,$$

and note that, if u is a static state feedback verifying $u(\mathcal{E}_2) = 0$, then $\mathcal{E}_2$ is an equilibrium of (9).

Analogously to the stability result described in the previous section, this equilibrium can be (locally) asymptotically stabilized in such a way that the region of attraction is[1] $\mathbb{R}^2 \times \mathbb{R}^{sgn(h)}$. The following lemmata are instrumental to establish this result.

Lemma 3.1: The controllability rank condition for system (6) at $(0, 0)$ is

$$h(0, 0) \neq -\lambda \frac{\partial h}{\partial x_2}(0, 0)\,. \qquad (10)$$

Lemma 3.2: If $\sigma(x_1) + k(x_1) \neq 0$ for all x_1 then the function

$$V(x_1, x_2, \xi) = \frac{1}{2}\left(x_1^2 + (x_2 - x_2^*)^2 + (\xi - h)^2\right)\,, \qquad (11)$$

with x_2^* and ξ defined by (8), is radially unbounded in $\mathbb{R}^2 \times \{\xi \in \mathbb{R} \;:\; \xi h(0, 0) > 0\}$ if and only if

$$\lim_{x_2\to\infty} h(x_1, x_2) = \infty, \qquad (12)$$

uniformly in x_1. □

Theorem 3.3: Suppose that system (6) verifies Assumption 3.1, the controllability rank condition (10) and the growth condition (12).

There exists a function σ with the odd sign property and a neighborhood $\mathcal{U}$ of the equilibrium $\mathcal{E}_2$ of the closed–loop system (9) such that the control law

$$u = \dot{x}_2^* + x_1 h(x_1, x_2) - (x_2 - x_2^*)\,, \qquad (13)$$

with x_2^* given by (8), is well-defined, and asymptotically stabilizes $\mathcal{E}_2$ with region of attraction $\mathcal{U}$. □

Remark 3.1: The controllability rank condition in Theorem 3.3 is necessary since the explicit computation of the control law (13) yields, after some manipulations,

$$u(x_1, x_2, \xi) = \frac{\xi^2 \tilde{u}(x_1, x_2, \xi)}{\xi^2 + (\sigma + k)\dfrac{\partial h}{\partial x_2}}\,,$$

where $\tilde{u}$ is a well-defined fuunction. This allows to give a better description of $\mathcal{U}$. More precisely, the equation

$$\xi^2 = -(\sigma(x_1) + k(x_1))\frac{\partial h}{\partial x_2} \qquad (14)$$

defines a surface in $\mathbb{R}^3$, that divides the space $\mathbb{R}^2 \times \mathbb{R}^+$ into two sets. Letting X denote the one containing the equilibrium point, it is easy to see that $\mathcal{U}$ corresponds to the largest level set of V contained in X.

Corollary 3.4: If $\partial h/\partial x_2 > 0$ for all x_1 and all x_2 and if, for some $\eta \in \mathbb{R}^+$, $k(x_1) > \eta$ for all x_1, then there exists a function σ – with the odd sign property – such that, with u given by (13), the equilibrium $\mathcal{E}_2$ of (9) is locally asymptotically stable and the region of attraction is $\mathbb{R}^2 \times \mathbb{R}^{sgn(h)}$ and hence the equilibrium $x_1 = 0, x_2 = 0$ is globally asymptotically stable.

[1]The notation $\mathbb{R}^{sgn(h)}$ is a compact form to denote $\mathbb{R}^+$ when $h(0, 0) > 0$ and $\mathbb{R}^-$ when $h(0, 0) < 0$.

Example 3.1: Consider again the example of Section II. Setting $\widehat{x}_2 = x_2 - 1$, system (1) is transformed into

$$\begin{aligned} \dot{x}_1 &= 1 - (\widehat{x}_2 + 1)^2 \,, \\ \dot{\widehat{x}}_2 &= u \,, \end{aligned} \tag{15}$$

which is of the form (6) with $k(x_1) = 1$, $\lambda = 1$, $h(x_1, \widehat{x}_2) = (\widehat{x}_2 + 1)$. Moreover, $h(0,0) = 1 \neq 0$ and the controllability rank condition holds. Finally, we have $\partial h / \partial \widehat{x}_2 = 1 > 0$ and the hypotheses of Corollary 3.4 are fulfilled. Therefore, the function $\sigma(x_1) = \tanh(x_1)$ is such that the control law locally asymptotically stabilizes the equilibrium and the region of attraction is $\mathbb{R}^2 \times \mathbb{R}^+$.

IV. The Three-dimensional SISO case

We now extend the result described in the previous section to a system with three states in triangular form. More precisely, consider the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \,, \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \,, \\ \dot{x}_3 &= u \,, \end{aligned} \tag{16}$$

where $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^3 \to \mathbb{R}$ are smooth vector fields and $(0,0,0)$ is an equilibrium. As in a classic backstepping procedure, the main idea is to use x_3 to control x_2 and, in turn, to use x_2 to control x_1.

Assumption 4.1: There exist functions $k_1 : \mathbb{R} \to \mathbb{R}$ and $k_2 : \mathbb{R}^2 \to \mathbb{R}$, verifying $k_1(0) \geq \epsilon_1 > 0$ and $k_2(0,0) \geq \epsilon_2 > 0$, such that

$$\begin{aligned} f_1(x_1, x_2) &= k_1(x_1) - (x_2 + \lambda_1) h_1(x_1, x_2) \\ f_2(x_1, x_2, x_3) &= k_2(x_1, x_2) - (x_3 + \lambda_2) h_2(x_1, x_2, x_3), \end{aligned}$$

for some constants $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and for some smooth functions $h_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_2 : \mathbb{R}^3 \to \mathbb{R}$.

Note that since $f_1(0,0) = 0$ and $f_2(0,0,0) = 0$, we have $k_1(0) = \lambda_1 h_1(0,0)$ and $k_2(0,0) = \lambda_2 h_2(0,0,0)$. Moreover the requirements on k_1 and k_2 imply $h_1(0,0) \neq 0$ and $h_2(0,0,0) \neq 0$, respectively. Introduce the new states ξ_1 and ξ_2, the dynamics of which are specified below, and let x_2^* and x_3^* be defined by

$$\begin{aligned} (x_2^* + \lambda_1)\xi_1 &= \sigma(x_1) + k_1(x_1) \,, \\ (x_3^* + \lambda_2)\xi_2 &= \sigma(x_2 - x_2^*) - x_1 h_1(x_1, x_2) + \\ &\quad + k_2(x_1, x_2) - \dot{x}_2^* \,, \end{aligned} \tag{17}$$

for some function[2] σ with the odd sign property. Now, consider the augmented system

$$\begin{aligned} \dot{x}_1 &= k_1 - (x_2 + \lambda_1) h_1 \,, \\ \dot{x}_2 &= k_2 - (x_3 + \lambda_2) h_2 \,, \\ \dot{x}_3 &= u \,, \\ \dot{\xi}_1 &= \dot{h}_1 + (h_1 - \xi_1) - x_1(x_2^* + \lambda_1) \,, \\ \dot{\xi}_2 &= \dot{h}_2 + (h_2 - \xi_2) - (x_3^* + \lambda_2)(x_2 - x_2^*) \,, \end{aligned} \tag{18}$$

Define the point

$$(x_1, x_2, x_3, \xi_1, \xi_2) = (0, 0, 0, h_1(0,0), h_2(0,0,0)) := \mathcal{E}_3 .$$

[2]For simplicity, we have used only one function σ.

and note that, if u is a static state feedback verifying $u(\mathcal{E}_3) = 0$, then $\mathcal{E}_3$ is an equilibrium of (18).

Lemma 4.1: If the rank condition for controllability at the equilibrium holds for system (16) then

$$\begin{aligned} h_1(0,0) &\neq -\lambda_1 \frac{\partial h_1}{\partial x_2}(0,0) \,, \\ h_2(0,0,0) &\neq -\lambda_2 \frac{\partial h_2}{\partial x_3}(0,0,0) \,. \end{aligned} \tag{19}$$

Lemma 4.2: Suppose that $\sigma(x_1) + k_1(x_1) \neq 0$, for all x_1, and that $\sigma(x_2 - x_2^*) - x_1 h_1(x_1, x_2) + k_2(x_1, x_2) - \dot{x}_2^* \neq 0$, for all x_1, x_2. The function

$$\begin{aligned} V(x_1, x_2, x_3, \xi_1, \xi_2) &= \frac{1}{2}(x_1^2 + (\xi_1 - h_1)^2) + \\ &+ \frac{1}{2}((\xi_2 - h_2)^2 + (x_2 - x_2^*)^2 + (x_3 - x_3^*)^2) \,, \end{aligned} \tag{20}$$

with x_2^* and x_3^* defined by (17), is radially unbounded in $\mathbb{R}^3 \times \{\xi_1 \in \mathbb{R} \,:\, \xi_1 h_1(0,0) > 0\} \times \{\xi_2 \in \mathbb{R} \,:\, \xi_2 h_2(0,0) > 0\}$ if and only if

$$\lim_{x_2 \to \infty} h_1(x_1, x_2) = \infty, \quad \lim_{x_3 \to \infty} h_2(x_1, x_2, x_3) = \infty \tag{21}$$

uniformly in x_1, and in x_1 and x_2, respectively.

Theorem 4.3: Consider the system (16) verifying Assumption 4.1, the controllability rank condition (19), and (21).

There exist a function σ, with the odd sign property, and a neighborhood $\mathcal{U}$ of the equilibrium $\mathcal{E}_3$ of the closed–loop system (18) such that the control law

$$u = \dot{x}_3^* + h_2(x_2 - x_2^*) - (x_3 - x_3^*) \,, \tag{22}$$

with x_2^* and x_3^* given by (17), is well-defined, and asymptotically stabilizes $\mathcal{E}_3$ with region of attraction $\mathcal{U}$. □

Corollary 4.4: If $\partial h_2 / \partial x_3 > 0$ and if, for some $\eta \in \mathbb{R}^+$, $k_2(x_1, x_2) - \dot{x}_2^* - x_1 h_1(x_1) > \eta$ for all x_1 and all x_2, then there exists a function σ with the odd property such that the control law (22) locally asymptotically stabilizes $\mathcal{E}_3$ and the region of attraction is $\mathbb{R}^3 \times \mathbb{R}^{sgn(h_1)} \times \mathbb{R}^{sgn(h_2)}$.

V. The general case

Motivated by the application to power systems described in the next section we extend the results of the previous sections to the case in which each row of the triangular structure is a vector, namely to the system

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) \,, \\ \dot{\mathbf{x}}_2 &= \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \,, \\ \dot{\mathbf{x}}_3 &= \mathbf{u} \,, \end{aligned} \tag{23}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$ for $i = 1, 2, 3$, $u(t) \in \mathbb{R}^n$ and $\mathbf{f}_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{f}_2 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are smooth maps. Suppose that $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ is an equilibrium.

Assumption 5.1: There exist functions $k_{1,i} : \mathbb{R}^n \to \mathbb{R}$ and $k_{2,i} : \mathbb{R}^{2n} \to \mathbb{R}$, for $i = 1, \ldots, n$, such that $k_{1,i}(0) > \epsilon_{1,i} >$

0 and $k_{2,i}(0,0) > \epsilon_{2,i} > 0$, and each component of $\mathbf{f}_1$ and $\mathbf{f}_2$ can be written as

$$f_{1,i}(\mathbf{x}_1,\mathbf{x}_2) = k_{1,i}(\mathbf{x}_1) - (x_{2,i}+\lambda_{1,i})h_{1,i}(\mathbf{x}_1,\mathbf{x}_2),$$

$$f_{2,i}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = k_{2,i}(\mathbf{x}_1,\mathbf{x}_2) - (x_{3,i}+\lambda_{2,i})h_{2,i}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$$

for some non-zero constants $\lambda_{1,i}$ and $\lambda_{2,i}$ and some functions $h_{1,i}$ and $h_{2,i}$.

Note that since $\mathbf{f}_1(\mathbf{0}) = \mathbf{0}$ and $\mathbf{f}_2(\mathbf{0}) = \mathbf{0}$, $k_{1,i}(\mathbf{0}) = \lambda_{1,i}h_{1,i}(\mathbf{0})$ and $k_{2,i}(\mathbf{0}) = \lambda_{2,i}h_{2,i}(\mathbf{0})$. Moreover, if Assumption 5.1 holds, then $h_{1,i}(\mathbf{0}) \neq 0$ and $h_{2,i}(\mathbf{0}) \neq 0$.

Introduce the variables $x^*_{2,i}$, $x^*_{3,i}$, $\xi_{2,i}$ and $\xi_{3,i}$ such that

$$\begin{array}{rcl} (x^*_{2,i}+\lambda_{1,i})\xi_{1,i} &=& \sigma(x_{1,i}) + k_{1,i}\,, \\ (x^*_{3,i}+\lambda_{2,i})\xi_{2,i} &=& \sigma(x_{2,i}-x^*_{2,i}) - x_{1,i}h_{1,i} + \\ && +k_{2,i} - \dot{x}^*_{2,i}\,, \end{array} \tag{24}$$

for some function σ with the odd sign property and consider the augmented system

$$\begin{array}{l} \dot{x}_{1,i} = k_{1,i} - (x_{2,i}+\lambda_{1,i})h_{1,i}\,, \\ \dot{x}_{2,i} = k_{2,i} - (x_{3,i}+\lambda_{2,i})h_{2,i}\,, \\ \dot{x}_{3,i} = u_i\,, \\ \dot{\xi}_{1,i} = \dot{h}_{1,i} + (h_{1,i}-\xi_{1,i}) - x_{1,i}(x^*_{2,i}+\lambda_{1,i})\,, \\ \dot{\xi}_{2,i} = \dot{h}_{2,i} + (h_{2,i}-\xi_{2,i}) - (x_{3,i}+\lambda_{2,i})^*(x_{2,i}-x^*_{2,i})\,. \end{array} \tag{25}$$

Define the point

$$(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{h}_1(\mathbf{0}),\mathbf{h}_2(\mathbf{0})) := \mathcal{E}\,,$$

and note that, if u is a static state feedback verifying $u(\mathcal{E}) = 0$, then $\mathcal{E}$ is an equilibrium of (25).

Theorem 5.1: Suppose that system (23) verifies Assumption 5.1, the controllability rank condition at the equilibrium and that h_1 and h_2 are such that

$$\lim_{x_{2,i}\to\infty} h_{1,i}(\mathbf{x}_1,\mathbf{x}_2) = \infty, \quad \lim_{x_{3,i}\to\infty} h_{2,i}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \infty$$

uniformly in $\mathbf{x}_1$ and $x_{2,j}$ with $j \neq i$, for $h_{1,i}$, and uniformly in $\mathbf{x}_1, \mathbf{x}_2$ and all $x_{3,j}$ for $j \neq i$, for $h_{2,i}$.

There exist a function σ with the odd sign property and a neighborhood $\mathcal{U}$ of the equilibrium $\mathcal{E}$ of the closed–loop system (25), such that the control law

$$u_i = \dot{x}^*_{3,i} + h_{2,i}(x_{2,i}-x^*_{2,i}) - (x_{3,i}-x^*_{3,i})\,, \tag{26}$$

with $x^*_{2,i}$ and $x^*_{3,i}$ given by (24), is well-defined, and locally asymptotically stabilizes $\mathcal{E}$ with region of attraction $\mathcal{U}$. □

VI. Transient stabilization of power systems

We finally come to the practical application that motivated this work, namely the problem of stabilizing the equilibrium of a system of N power interconnected machines described by the equations [13]

$$\begin{array}{rcl} \dot{\delta}_i &=& \omega_i\,, \\ \dot{\omega}_i &=& -D_i\omega_i + P_i - G_iE_i^2 - \\ && -E_i \displaystyle\sum_{k=1,k\neq i}^{n} E_kY_{ik}\sin(\delta_i-\delta_k+\alpha_{ik}) \\ \dot{E}_i &=& -a_iE_i + b_i \displaystyle\sum_{k=1,k\neq i}^{n} E_k\cos(\delta_i-\delta_k+\alpha_{ik}) + \\ && +\dfrac{1}{\tau_i}(E^*_{F_i}+\nu_i)\,, \end{array}$$

where δ_i, ω_i and E_i, $i = 1,\ldots,n$, are the states, ν_i, $i = 1,\ldots,n$, are the control inputs, D_i, P_i, G_i, a_i, b_i, τ_i and E^*_{Fi} are positive constants depending on the physical parameters of the i-th machine, and Y_{ik} and α_{ik} are constants depending on the topology of the connections. Let $(\overline{\boldsymbol{\delta}},\mathbf{0},\overline{\mathbf{E}})$ be the equlibrium. By setting $\mathbf{x}_1 = (\delta_1,\ldots,\delta_n)^\top - \overline{\boldsymbol{\delta}}$, $\mathbf{x}_2 = (\omega_1,\ldots,\omega_n)^\top$, $\mathbf{x}_3 = (E_1,\ldots,E_n)^\top - \overline{\mathbf{E}}$ and

$$\nu_i = \tau_i\left(u_i + a_iE_i - b_i\sum_{k=1,k\neq i}^{n} E_k\cos(\delta_i-\delta_k+\alpha_{ik})\right) - E^*_{F_i}\,,$$

for a new input u_i, we obtain the equations

$$\begin{array}{rcl} \dot{\mathbf{x}}_1 &=& \mathbf{x}_2\,, \\ \dot{\mathbf{x}}_2 &=& -\mathrm{diag}\{D_i\}\mathbf{x}_2 + \mathbf{P} - \mathrm{diag}\{(x_{3,i}+\overline{E}_i)^2\}\mathbf{G} - \\ && -\mathcal{F}(\mathbf{x}_1,\mathbf{x}_3)(\mathbf{x}_3+\overline{\mathbf{E}})\,, \\ \dot{\mathbf{x}}_3 &=& \mathbf{u}\,, \end{array} \tag{27}$$

where $\mathcal{F}(\mathbf{x}_1,\mathbf{x}_3) = \mathrm{diag}\{\mathcal{F}_i(\mathbf{x}_1,\mathbf{x}_3)\}$ with

$$\mathcal{F}_i = \sum_{k=1,k\neq i}^{n} (x_{3,k}+\overline{E}_k)Y_{ik}\sin(x_{1,i}+\overline{\delta}_i-x_{1,k}-\overline{\delta}_k+\alpha_{ik}), \tag{28}$$

$\mathbf{P} = (P_1,\ldots,P_n)^\top$ and $\mathbf{G} = (G_1,\ldots,G_n)^\top$.

The system (27) satisfies the conditions of Theorem 5.1. In particular, Assumption 5.1 is verified selecting

$$k_{1,i} = 1\,, h_{1,i} = -1\,, \lambda_{1,i} = -1$$

$$k_{2,i} = -D_ix_{2,i} + P_i\,,$$

$$h_{2,i} = G_i(x_{3,i}+\overline{E}_i) + \mathcal{F}_i\,, \lambda_{2,i} = \overline{E}_i\,.$$

The controller derived in Theorem 5.1 was tested in simulations for the two machine example of [13] with the parameters: $G_1 = 28.9008$, $G_2 = 20.3936$, $D_1 = 1$, $D_2 = 0.2$, $P_1 = 52.2556$, $P_2 = 48.4902$, $\alpha = 0.5430$, $Y_{12} = 51.2579$ and $Y_{21} = 36.6127$. The initial conditions have been set randomly to $\mathbf{x}_1(0) = (0.0462 \quad 0.0971)^\top$, $\mathbf{x}_2(0) = (0.3235 \quad 0.1948)^\top$, $\mathbf{x}_3(0) = (1.3171 \quad 1.9502)^\top$, $\boldsymbol{\xi}_1(0) = (-1 \quad -1)^\top$ and $\boldsymbol{\xi}_2(0) = (50.4327 \quad 45.9890)^\top$.

In Figures 3 and 4 the time-histories of the states of the two machines and of the controller states $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are depicted. As seen from the figures, the transient is very fast.

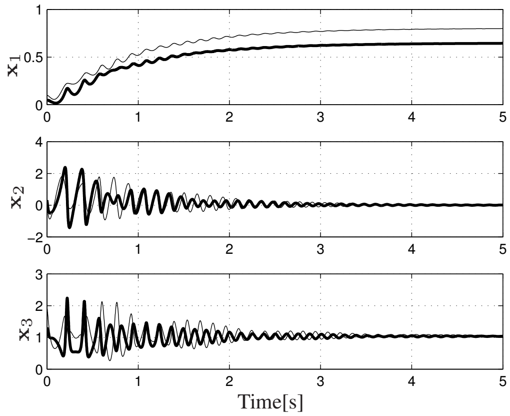

Fig. 3. Time histories of the states $\mathbf{x}_1(t)$ (top), $\mathbf{x}_2(t)$ (centre) and $\mathbf{x}_3(t)$ (bottom).

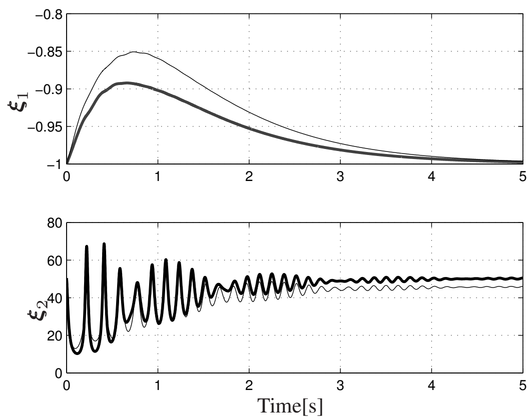

Fig. 4. Time histories of the states $\boldsymbol{\xi}_1(t)$ (top) and $\boldsymbol{\xi}_2(t)$ (bottom).

A. Simplified controller

In this subsection we show that the stabilization of the equilibrium of system (27) can be simplified with respect to the generic result provided in Theorem 5.1.

Theorem 6.1: Consider the system (27). Define the functions

$$h_i(\mathbf{x}_1, \mathbf{x}_3) := G_i(x_{3,i} + \overline{E}_i) + \mathcal{F}_i(\mathbf{x}_1, \mathbf{x}_3), \qquad (29)$$

where $\mathcal{F}_i$ is given in (28), and let $\boldsymbol{\xi}$ be the controller state with dynamics

$$\dot{\xi}_i = \dot{h}_i + (h_i - \xi_i) - (x^*_{3,i} + \overline{E}_i)x_{2,i} - \epsilon_i \sigma(\Delta_i)(x_{3,i} + \overline{E}_i)\,,$$

where we have defined

$$\Delta_i := x_{1,i} - \overline{\delta}_i,$$

$x^*_{3,i}$ is given by

$$(x^*_{3,i} + \overline{E}_i)\xi_i = \sigma(\Delta_i) + P_i\,, \qquad (30)$$

and the positive constants ϵ_i verify

$$D_i \geq \left(\sigma'(\Delta_i) + \frac{D_i^2}{4}\right)\epsilon_i. \qquad (31)$$

The control law

$$u_i = \dot{x}^*_{3,i} + h_i x_{2,i} - (x_{3,i} - x^*_{3,i}) + \epsilon_i\,\sigma(\Delta_i)\,\xi_i$$

globally asymptotically stabilizes the equilibrium of the system. □

Remark 6.1: Selecting, for instance, $\sigma(a) = \tanh(a)$, and noting that $0 \leq \sigma' \leq 1$, condition (31) is satisfied selecting

$$0 < \epsilon_i < \frac{4D_i}{4 + D_i^2}\,.$$

VII. Concluding remarks

Motivated by the problem of transient stabilization of power systems we have developed a procedure to construct asymptotically stabilizing controllers for systems in triangular form, which are not necessarily globally feedback linearizable. The result may, therefore, be seen as a non-trivial extension of the well-known backstepping procedure.

The application of the technique to power systems solves a longstanding problem of explicit derivation of asymptotically stabilizing controllers with a guaranteed domain of attraction defined by a *bona fide* Lyapunov function. Current research is under way to compare, in realistic multimachine simulations, the performance of both controllers.

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