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An adaptive hybrid robust regulator

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***Abstract*— An adaptive robust regulator based on a recently proposed hybrid frequency estimator is described. The main features of the proposed regulator are its clear structure, the simplicity of its design and its robustness, largely inherited from the mentioned frequency estimator even when the number of frequencies is not known a priori or change in time. The effectiveness of the proposed regulator is shown on simulation examples.**

I. Introduction

The problem of asymptotic regulation, including as special cases both tracking of references and rejection of disturbances belonging to known classes of exogenous signals generated by a finite dimensional exosystem, is a classical and fundamental problem in control theory. In the classical problem the exosystem is supposed to be known, and solutions have been described both in the geometric setting [8], [17] and in the algebraic setting [6], [5], [7], just to name a few.

When the exosystem is unknown, the problem becomes much more complicate since it becomes necessary either to directly adapt the internal model of the exosystem contained in the regulator (see for instance [16], [13], [15], often considering the case of nonlinear system), or to identify the frequency characteristics of the exogenous signals entering the plant and then either redesign the regulator according to the obtained information or directly cancel the exogenous disturbances using the estimated signal (see for instance [2], [14], [10], [12], [1]). In this paper, we present the application of the multifrequency estimator in [3], [4] to the problem of asymptotic regulation with an unknown exogenous system. The advantage of using the proposed approach consists in the simplicity of the construction and the good convergence properties achieved also in the presence of multiple frequencies that might change in number and values over time.

The structure of the paper is as follows: in Section II the problem is defined and classical results (to be used later) on regulator design are recalled; in Section III, the architecture of the proposed solution and its components are described; in Section IV, the main properties of the proposed solution are described; finally in Section V an example is shown to substantiate the effectiveness of the method.

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II. Problem definition and preliminaries

Consider the linear plant $\mathcal{P}$:

$$\dot{x} = Ax + Bu + Pw, \tag{1a}$$

$$e = Cx + Du + Qw, \tag{1b}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^p$ and performance output $e \in \mathbb{R}^q$. The signal $w \in \mathbb{R}^m$ is assumed to be generated by the following exosystem $\mathcal{S}$:

$$\dot{w} = A_w w, \tag{2}$$

where w contains both reference signals and disturbances.

Let $P_\Sigma(s)$ denote the *system matrix* of system (1), that is,

$$P_\Sigma(s) := \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}. \tag{3}$$

The values of $\bar{s} \in \mathbb{C}$ for which the complex valued matrix $P_\Sigma(\bar{s})$ has rank less than the rank of $P_\Sigma(s)$ as a polynomial matrix constitute the *invariant zeros* or *system zeros*, which include all the transmission zeros plus a subset of the *input decoupling zeros* (eigenvalues of the unreachable subsystem) and the *output decoupling zeros* (eigenvalues of the unobservable subsystem).

The following assumptions define the class of plant and exosystem models considered in this paper. For a square matrix A, $\Lambda(A)$ denotes the set of the eigenvalues of A.

Assumption 1: In (1), the pair (A, B) is stabilizable and the pair (A, C) is detectable.

Assumption 2: In (2), all eigenvalues of A_w are simple (i.e. their associated Jordan blocks have dimension 1) and lie on the imaginary axis.

The following assumption is a standard sufficient condition [6] for the solvability of the regulator problem (which becomes necessary if mild assumptions on parametric uncertainties affecting the plant matrices are considered).

Assumption 3: $\text{rank} P_\Sigma(\lambda) = n + q$, for all $\lambda \in \Lambda(A_w)$.

Remark 1: By Assumption 3, no invariant zero of (A, B, C, D) coincides with an eigenvalue of A_w. By Assumption 1, no eigenvalue of the unreachable/unobservable subsytems of (A, B, C, D) (and then no input or output decoupling zero of (A, B, C, D)) lie on the imaginary axis. Recalling that invariant zeros are the union of the transmission zeros and (a subset of) the decoupling zeros, it follows that under Assumption 1 and Assumption 2 it is possible to state Assumption 3 in the equivalent form that the set of transmission zeros of (A, B, C, D) is disjoint from the spectrum of A_w.

The standard regulator (or servomechanism) problem can be stated as follows.

Definition 1: Given plant (1) and exosystem (2), find, if possible, a compensator that suitably connected to the plant ensures that

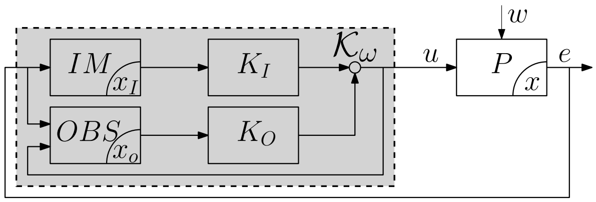

Fig. 1. A regulator solving the standard problem

(i) the closed loop is asymptotically stable;
(ii) $\lim_{t\to+\infty} e(t) = 0$, for any initial state of the exosystem (2), of the plant (1) and of the compensator.

A. A servocompensator design for a known exosystem

A solution to the above problem is given by a compensator $\mathcal{K}_\omega$ composed by three pieces (see Fig. 1):

- an *internal model* of the exogenous signals
$$\dot{x}_I = A_I x_I + B_I e$$
which is fed by the error signal (1a), with (A_I, B_I) reachable, A_I having the same eigenvalues of A_w and q Jordan blocks of dimension 1 for each eigenvalue;
- an *observer* for the undisturbed ($w \equiv 0$) plant (1a), namely
$$\dot{x}_o = Ax_o + Bu + L(Cx_o + Du - e)$$
where L makes $(A + LC)$ Hurwitz;
- a *stabilizing state feedback*
$$u = K \begin{bmatrix} x_I \\ x \end{bmatrix} = \begin{bmatrix} K_I & K_o \end{bmatrix} \begin{bmatrix} x_I \\ x \end{bmatrix}$$
for the cascade of the plant (with $w \equiv 0$) and the internal model
$$\begin{bmatrix} \dot{x}_I \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A_I & B_I C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_I \\ x \end{bmatrix} + \begin{bmatrix} B_I D \\ B \end{bmatrix} u.$$

The three pieces are combined in an overall compensator
$$\dot{x}_c = A_c x_c + B_c u_c, \tag{4a}$$
$$y_c = C_c x_c + D_c u_c, \tag{4b}$$
where $x_c = \begin{bmatrix} x_I' & x_o' \end{bmatrix}'$ and
$$A_c = \begin{bmatrix} A_I & 0 \\ (B + LD)K_I & (A + LC) + (B + LD)K_o \end{bmatrix}, \tag{5a}$$
$$B_c = \begin{bmatrix} B_I \\ -L \end{bmatrix}, \quad C_c = \begin{bmatrix} K_I & K_o \end{bmatrix}, \quad D_c = 0, \tag{5b}$$
to be connected to the plant according to the equation
$$u_c = e, \quad u = y_c. \tag{6}$$
For later use, the dependence of compensator $\mathcal{K}_\omega$ on ω is made explicit in the notation, where ω is the vector containing the frequencies of the signals generated by (2). The following classical result asserts the effectiveness of compensator (4), (5) in solving the problem in Definition 1.

Theorem 1: If Assumption 1, Assumption 2 and Assumption 3 hold, then the problem in Definition 1 is solvable and a solution is given by compensator (4), (5) with interconnection (6).

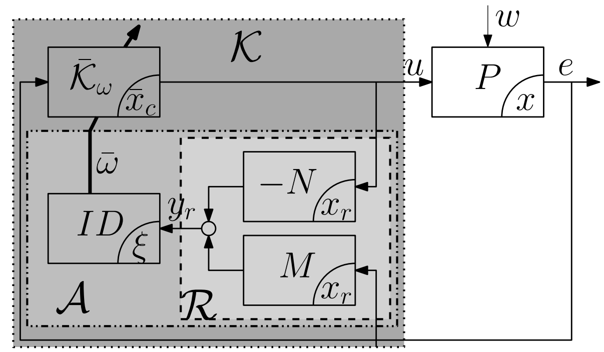

Fig. 2. The adaptive hybrid robust regulator solving the adaptive problem (note that N, M share the same state, *i.e.* are represented by two blocks but are implemented by a single system (7)).

In this paper, the focus is on the case when the exosystem in unknown (although it is known that it satisfies Assumption 2). The problem can be stated as follows.

Definition 2: Given a plant (1) with known A, B, C, D) and an exosystem (2) which satisfies Assumption 2 (but is otherwise unknown; in particular, matrices A_w, P, Q are unknown), find, if possible, a compensator that suitably connected to the plant ensures that

(i) the closed loop is asymptotically stable;
(ii) $\lim_{t\to+\infty} e(t) = 0$, for any initial state of the exosystem (2), of the plant (1) and of the compensator.

Since A_w is unknown in the problem in Definition 2, it is expected that Assumption 3 (which is a necessary condition for the solvability of the problem in Definition 1 under mild assumptions on the errors affecting the nominal values of the parameters of A) must be strengthened in order to guarantee that the required property holds for any admissible A_w. For this reason, Assumption 3 is replaced by the following Assumption 4.

Assumption 4: $\text{rank} P_\Sigma(\jmath\omega) = n + q$, $\forall \omega \in \mathbb{R}$.

III. Architecture of the Adaptive Hybrid Robust Regulator

The proposed controller $\mathcal{K}$ is composed by three subsystems (see Fig. 2):

- a *residual generator* $\mathcal{R}$, which provides a signal suitable for identification of the frequencies of the exogenous generator;
- the *frequency identifier* ID, which produces an estimate ω of the frequencies present in w, used for the internal model design;
- the *switching servo compensator* $\bar{\mathcal{K}}_\omega$, which provides both an internal model for the frequencies in ω and a stabilizing compensation.

A. The residual generator $\mathcal{R}$

Choose L_r such that $(A + L_r C)$ is Hurwitz. The *residual generator* is a LTI continuous time system described by
$$\dot{x}_r = (A + L_r C)x_r - (B + L_r D)u + L_r e, \tag{7a}$$
$$y_r = Cx_r - Du + e, \tag{7b}$$

corresponding to a matrix transfer function $\begin{bmatrix}-N(s) & M(s)\end{bmatrix}$ where $N(s) \in \mathcal{RH}_\infty$ and $M(s) \in \mathcal{RH}_\infty$ provide a right coprime factorization of the matrix transfer function of (1) from u to y, namely [18]

$$M^{-1}(s)N(s) = C(sI - A)^{-1}B + D. \qquad (8)$$

If the plant is (open loop) stable, the factorization (8) can be simply replaced by $M(s) = I$, $N(s) = C(sI - A)^{-1}B + D$.

Remark 2: The main role of (7) is to guarantee, at least in the nominal parameters, that *the frequency identifier is virtually in open loop*, in the sense that y_r is independent from u in Fig. 2 (the corresponding transfer function is zero, as shown below). In fact, consider a complete right coprime factorization of (1) such that (8) is complemented with

$$M^{-1}(s)N_w(s) = C(sI - A)^{-1}P + Q; \qquad (9)$$

consider also zero initial conditions (otherwise, an additional exponentially converging term would also appear, which bears no consequence on the closed-loop stability analysis). It is then possible to write

$$\begin{aligned} y_r(s) &= M(s)e(s) - N(s)u(s) \\ &= M(s)M^{-1}(s)\begin{bmatrix}N(s) & N_w(s)\end{bmatrix}\begin{bmatrix}u(s)\\ w(s)\end{bmatrix} - N(s)u(s) \\ &= N_w(s)w(s), \end{aligned}$$

As a consequence, the combination of the plant, the residual generator and the frequency identifier (from u to the output of the frequency identifier) will essentially behave as an open loop stable filter on w, and if switching stability of the closed loop system in Fig. 2 without $\mathcal{A}$ is achieved, then stability of the overall adaptive closed loop system in Fig. 2 will also follow. A small gain reasoning can be used to show that the same conclusions still holds in the presence of sufficiently small mismatch between the real plant and the model (1).

Remark 3: One could be tempted to use only the error signal e instead of y_r for frequency detection. In order to see why this choice is not advisable, consider the case of an exogenous signal composed by a single sinusoid at frequency ω_1 for $t \in [t_0, t_1)$, to which a second sinusoid at frequency ω_2 is added for $t \in [t_1, +\infty)$. Assume also that at time $\tau_1 \in [t_0, t_1)$ the first sinusoid has been identified, and almost completely compensated in the error signal e. After time t_1, the frequency identifier would start detecting the sinuosoid at frequency ω_2 (but not the one at frequency ω_1, which is already compensated for) and then at the next controller switch at time τ_2 the internal model would be designed to compensate for the sinusoid at frequency ω_2 but not for the one at frequency ω_1. The same process would then happen again with the roles of ω_1 and ω_2 exchanged, thus leading to cycles without convergence of e to zero. It is easy to see that this kind of problems does not arise when the control input information is suitably elaborated, as is done by (7). The dynamics of (7) can also be used to introduce a suitable filtering action on high frequency noise possibly affecting the measurements of e.

B. The frequency identifier ID

We briefly recall the adaptive hybrid observer in [4] for the frequencies estimation of the signal

$$y_\omega(t) = \sum_{i=1}^{l} E_i \sin(\omega_i t + \phi_i), \qquad (10)$$

with unknown l, angular frequencies ω_i's, E_i's and Φ_i's. The hybrid estimator proposed in [4], tailored for numerical implementation, assumes the signal $y(t)$ to be sampled with a period of $T_s > 0$ seconds, called hardware-sampling time[1]. This first fact limits the angular frequencies that can be estimated to be smaller than π/T_s by Nyquist-Shannon's Theorem. Rewriting $y_\omega(t) = C_\omega x_\omega(t)$, $x_\omega \in \mathbb{R}^{2l}$, $C_\omega = [0, 1, 0, 1, 0, \dots, 1] \in \mathbb{R}^{1\times 2l}$, as the output of the linear time invariant system described by

$$\dot{x}_\omega = A_\omega x_\omega = \text{diag}\left\{\begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}\right\} x_\omega, \qquad (11)$$

for $i = 1, \dots, l$, with unknown initial condition $x_\omega(0) = x_\omega(t_0)$ and ω_i's. Define the measurements vector

$$Y_k := \begin{bmatrix} y_\omega(t_{k-2l}) \\ \cdots \\ y_\omega(t_{k-2}) \\ y_\omega(t_{k-1}) \end{bmatrix}, \qquad (12)$$

where shortly $t_k = kT$. The characteristic polynomial of $A_D := e^{A_\omega T}$ is

$$\begin{aligned} p_{A_D}(\lambda) &= \prod_{i=1}^{l}(\lambda^2 - 2\cos(\omega_i T)\lambda + 1) \\ &= \lambda^{2n} + a_{2l-1}\lambda^{2l-1} + \cdots + a_1\lambda + 1, \end{aligned} \qquad (13)$$

with symmetric coefficients, *i.e.* such that $a_{2l-h} = a_h$, $h = 1, \dots, l-1$, and then the coefficient vector a can be expressed in compact form as

$$a := \begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \bar{S} \begin{bmatrix} 1 \\ a_c \end{bmatrix} \qquad (14)$$

where, denoting by S_i the $i-$th row of $S \in \mathbb{R}^{2n-1\times n}$, the matrices S, $\bar{S}$ and the compact coefficient vector a_c are defined according to

$$S_i := \begin{cases} 1 & \text{if } i = j \text{ or } 2n - i = j, \\ 0 & \text{otherwise}, \end{cases} \qquad (15a)$$

$$\bar{S} = \text{blockdiag}\{1, S\}, \qquad (15b)$$

$$a_{c,i} = a_i, \qquad i = 1, \dots, n. \qquad (15c)$$

From (13) we define $a_{c,i} = f_i(\omega)$, where $f_i(\omega) = \sum_{j=1}^{i}(-2)^i \mathcal{Q}_j^i(\omega)$ and $\mathcal{Q}_j^i(\omega)$ stands for the $j-$th combination (without repetition) of the vector $[\cos(\omega_1 T), \dots, \cos(\omega_l T)]'$ elements grouped by i. Then, the

[1] Of the Analog-to-Digital converter ADC.

signal $y_\omega(t)$ can be expressed as

$$y_\omega(t_k) = -{Y_k}'a = -{Y_k}' \begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 1 \\ a_c' \end{bmatrix}, \quad (16)$$

where

$$Y_{k+1} = \begin{bmatrix} y_\omega(t_{k-4l+1}) & \cdots & y_\omega(t_{k-2l}) \\ \vdots & & \vdots \\ y_\omega(t_{k-2l-1}) & \cdots & y_\omega(t_{k-2}) \\ y_\omega(t_{k-2l}) & \cdots & y_\omega(t_{k-1}) \end{bmatrix} a = -\bar{Y}_k a. \quad (17)$$

By the observability property of the pair (A_ω, C_ω), if $T < \pi/\omega_{\max}$ with $\omega_{max} = \max\{\omega_1, \dots, \omega_l\}$, then $\mathrm{rank}(\bar{Y}_k) = 2l$, and defining $\bar{Y}_k^c$ as

$$\bar{Y}_k = [Y_{k-2l-1}, Y_{k-2l}, \dots, Y_k] = \left[Y_{k-2l-1},\, \bar{Y}_k^c\right], \quad (18)$$

the compact coefficient vector can be obtained[2] as

$$a_c = -\left(S'\bar{Y}_k^{c\prime}\bar{Y}_k^c S\right)^{-1} S'\bar{Y}_k^{c\prime}\left(Y_{k+1} + Y_{k-2n-1}\right). \quad (19)$$

However, since we are interested in the characteristic polynomial of A_ω, we need to retrieve the ω's from a_c. This could be accomplished finding the roots of the $p_{A_D} = \lambda^{2l} + a_1\lambda^{2l-1} + \ldots + 1$ which can be easily determined by a line search over $\theta \in [0, 2\pi]$ by imposing $\lambda = e^{\jmath\theta}$, since they belong to the unit circle due to the symmetry of the coefficients.

On the other hand, an alternative implementation yielding an asymptotic estimate of the ω's can be obtained by resorting to the dynamical hybrid observer $\mathcal{H}$ in [4] defining

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad (20)$$

with $A_0 \in \mathbb{R}^{2l \times 2n}$ and $B_0 \in \mathbb{R}^{2l}$, and

$$\xi = \begin{bmatrix} \hat{\omega}' & \zeta' & \chi & \tau \end{bmatrix}' \in \mathcal{O}, \quad (21a)$$

where $\omega \in \mathbb{R}^l$, $\zeta \in \mathbb{R}^{2l}$, $\chi \in \mathbb{R}$, $\tau \in \mathbb{R}$, are

$$\left.\begin{aligned} \dot{\hat{\omega}} &= -\gamma \nabla f(\hat{\omega})'\hat{S}'\zeta e, \\ \dot{\zeta} &= 0, \\ \dot{\chi} &= 0, \\ \dot{\tau} &= 1, \end{aligned}\right\} \text{ if } \xi \in \mathcal{C}, \quad (21b)$$

$$\left.\begin{aligned} \hat{\omega}^+ &= \hat{\omega}, \\ \zeta^+ &= A_0\zeta + B_0\chi, \\ \chi^+ &= y, \\ \tau^+ &= 0, \end{aligned}\right\} \text{ if } \xi \in \mathcal{D}, \quad (21c)$$

where $\hat{S}' = [0, S'] \in \mathbb{R}^{n \times 2l}$, $\gamma > 0$ and $e = y(t_k) + \zeta'\hat{S}[1, f(\hat{\omega})']'$. The flow set $\mathcal{C}$ and the jump set $\mathcal{D}$ are defined as

$$\mathcal{C} \triangleq \{\xi \in \mathcal{O} : \tau \in [0, T]\}, \quad (21d)$$

$$\mathcal{D} \triangleq \{\xi \in \mathcal{O} : \tau \geq T\}. \quad (21e)$$

The vector ζ maintains the past $2l$ samples of the input y with sampling time T, *i.e.* $Y_k = \zeta(t, k)$, whereas $\chi(t, k) = y_\omega(t_k)$ for all $t \in [t_k, t_{k+1}]$ and it is the new measured sample to be fed into Y_k.

Under the assumption that $0 < \omega_i < \pi/T_s$, $\omega_i \neq \omega_j$ for all $(i, j) \in \{1, \dots, l\}$ (yielded by Assumption 2) it is then possible to prove exponential convergence of the estimate $\hat{\omega}(t, k)$ to ω as t goes to infinity. Note that the hybrid arc $\hat{\omega}(t, k)$ does not exhibits jumps ($\hat{\omega}^+ = \omega$) and by persistency of excitation arguments it can been proved that the convergence speed is proportional to the positive definiteness of the matrix $\bar{Y}_k\bar{Y}_k$. The estimated number of frequencies $\hat{l}$ is obtained by analyzing the rank of $\bar{Y}_k\bar{Y}_k$ by evaluating its minimum eigenvalue. In fact, if $0 < \omega_i < \pi/T$, the matrix $\bar{Y}_k\bar{Y}_k$ is non-singular iff $\hat{l} \geq l$ whereas it becomes singular iff $\hat{l} > l$. This property, that can be robustified in presence of measurement noise adding a greater number of samples in Y_k, is pursued in [4] to obtain $\hat{l} = l$. Moreover, the re-sampling time T that maximizes the minimum eigenvalue of $\bar{Y}_k\bar{Y}_k$ is selected to increase the exponential decaying rate of the estimation error that is exactly greater or equal than $\lambda_{\min}(\bar{Y}_k'\bar{Y}_k)$.

The maximum time to obtain l and T via the above algorithm depends on the time needed to collect the $4\hat{l}$ samples to evaluate $\bar{Y}_k$ with $T \leq \pi/\omega_{\max}$; then the maximum time required to obtain l and T is less or equal than $4l\pi/\omega_{\max}$. Thereafter, the flow and jump map in (21) allows to get the asymptotic estimate of the ω's.

The aforementioned adaptive observer is considered to retrieve the angular frequencies ω_i's of the signal w in (1), i.e. the non-zero eigenvalues on the imaginary axes of the matrix A_w (2). To this aim, we define

$$y_\omega(t_k) := C_r(y_r(t_k) - y_r(t_{k-1})),$$

with $C_r = [1, 1, \dots, 1] \in \mathbb{R}^q$ such that y_ω is scalar and it contains all the frequencies of w except the constant bias that could result by a zero eigenvalue of A_w. Certainly there might be other choices to get rid of this constant bias, for example pre-filtering the signal y_r sent to the frequency identifier. We do not estimate the constant bias of y_r, i.e. the zero eigenvalue of A_w, since we add by default the zero eigenvalue in A_I, which is in general a desirable feature since it improves the closed loop response at low frequency.

Once the correct value $2l = \bar{m}$ (where $\bar{m} = m$ if A_w has no zero eigenvalue, and $\bar{m} = m - 1$ otherwise) of eigenvalues related to the sinusoidal components is obtained, the estimate $\hat{\omega}$ is obtained by (21) with initial conditions $\hat{\omega}_i(0), i = 1, \dots, l$, evaluated as the roots of the characteristic polynomial (13) with coefficients yield by (18).

The piecewise constant signal $\bar{\omega}(t)$ sent to the servocompensator is the output of a sample and hold device with input $[\hat{\omega}(t)' \ \ 0]'$ (as mentioned above, the additional null eigenvalue is added by default) and sampling time $T_r := -\ln(\beta)/\lambda_{\min}(\bar{Y}_k'\bar{Y}_k)$, where $0 < \beta < 1$ is a selectable parameter and corresponds to the desired shrinking factor of the estimation error between each sampling time T_r. With this choice, the smaller the β the smaller the number of switches of the servocompensator.

At each new sample of y_ω the algorithm in [4] checks if

[2] A Least SQuare (LSQ) estimate of a_c can be obtained by extending the vectors Y_k with further measurements and using the pseudo-inverse.

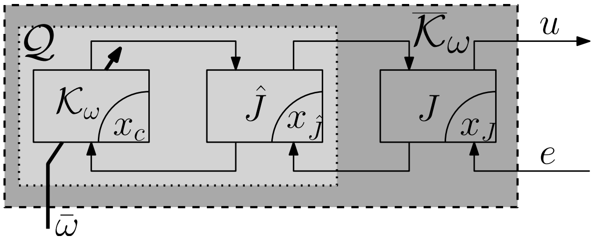

Fig. 3. The Youla-Kucera based realization (23), (24) of the switching controller.

$|e| = |y_\omega(t_k) + \zeta'\hat{S}[1, f(\hat{\omega})']'| > S_e||\zeta|| = S_e||Y_k||$ holds to identify a change in the frequency number or values and, if so, the past data collected in Y_k are discarded and the over all procedure of retrieving l and T is repeated.

To conclude, even with time varying $m(t)$ and $\omega(t)$, Theorem 2 in [4] can be invoked to show that if $m(t)$ and $\omega(t)$ are constant for $t \in [\bar{t}, \rho]$, with $\bar{t}+4m(\bar{t})T < \rho$, then for $t \in (-(\bar{t}+4m(\bar{t})T), \rho)$ the estimated number of frequencies is correct ($\hat{l} = m$) and

$$||\omega(\rho) - \hat{\omega}(\rho, k)|| \leq e^{-\sigma(\rho-\bar{t}-4m(\bar{t})T)}||\omega(\bar{t}) - \hat{\omega}(\bar{t}, \bar{k})||, \quad (22)$$

for some $\bar{k}$ and k and $\sigma = \lambda_{\min}(\bar{Y}_k'\bar{Y}_k)$ with the selected T. Hence, a similar exponential bound clearly holds for $\bar{\omega}(t)$ too.

C. The switching servo compensator $\bar{\mathcal{K}}_\omega$

In order to ensure switching stability, the *switching servo compensator* is implemented using a fixed part (composed by the subsystem J in Fig. 3) and a servocompensator $\mathcal{K}_\omega$ as in (4), (5) which is redesigned each time the signal $\bar{\omega}$ changes. By reasoning as in [9], [11], if a suitable realization of this changing part is adopted, closed loop stability can be guaranteed for arbitrary switchings due to the changes in $\bar{\omega}$.

Denote by $\mathcal{F}_L(M, R)$ the lower fractional transformation

$$\mathcal{F}_L(M, R) := M_{11} + M_{12}R(I - M_{22}R)^{-1}M_{21}.$$

Let K_k and L_k be such that $A + L_kC$ and $A + BK_k$ are Hurwitz. The subsystems J and $\hat{J}$ are characterized, respectively, by the matrices:

$$\left[\begin{array}{c|c} A_J & B_J \\ \hline C_J & D_J \end{array}\right] := \left[\begin{array}{c|cc} A + BK_k + L_kC + L_kDK_k & -L_k & B + L_kD \\ \hline K_k & 0 & I \\ -(C + DK_k) & I & -D \end{array}\right], \quad (23a)$$

$$\left[\begin{array}{c|c} A_{\hat{J}} & B_{\hat{J}} \\ \hline C_{\hat{J}} & D_{\hat{J}} \end{array}\right] := \left[\begin{array}{c|cc} A & -L_k & B \\ \hline -K_k & 0 & I \\ C & I & D \end{array}\right]. \quad (23b)$$

By [18, Theorem 12.8], all stabilizing controllers for (1) are parameterized by the LFT

$$K = \mathcal{F}_L(J, Q), \ Q \in \mathcal{RH}_\infty, \ \det(I + DQ(\infty)) \neq 0; \quad (24a)$$

conversely [18, Remark 12.9], the parameter Q yielding the particular stabilizing controller K can be expressed as

$$Q = \mathcal{F}_L(\hat{J}, K). \quad (24b)$$

However, in general it is not enough to use $K = \mathcal{K}_\omega$ for the current value of $\bar{\omega}$ in order to achieve stability for any possible switching sequence; on the other hand, as shown in [9], [11], such property holds if the realization of Q in (24b) is suitably chosen. Although other approaches could be used to guarantee stability (taking into account, for example, that each updates of $\bar{\omega}$ requires a certain amount of time, and then a known dwell time is always guaranteed between two switchings[3]), this one is adopted here due to the ensuing simplified analysis (no additional special care has to be taken to ensure stability).

For simplicity of implementation, it is useful to fix a maximum state space dimension for Q. Let $\bar{n}_Q := n_{exo} + n$, where n is state dimension of (1) and n_{exo} is the maximum state dimension of the exosystem (2). When $\bar{\omega}$ changes, the following algorithm is performed:

- design a servocompensator $\mathcal{K}_\omega$ as in Section II-A;
- find a minimal realization $(\hat{A}_Q, \hat{B}_Q, \hat{C}_Q, \hat{D}_Q)$ of $\hat{Q} := \mathcal{F}_L(\hat{J}, \mathcal{K}_\omega)$ with $\hat{J}$ given in (23b);
- let n_Q be the size of $\hat{A}_Q$ and define $\tilde{n}_{Qi} = \bar{n}_Q - n_Q$.
- let $T_Q = X^{\frac{1}{2}}$ where X solves the Lyapunov equation:
$$\hat{A}_Q^T X + X\hat{A}_Q = -I; \quad (25)$$
- given $\tilde{\alpha} > 1$, define the realization of Q as
$$\left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] = \left[\begin{array}{cc|c} T_Q\hat{A}_QT_Q^{-1} & 0 & T_Q\hat{B}_Q \\ 0 & -\tilde{\alpha}I_{\tilde{n}_Q} & 0 \\ \hline \hat{C}_QT_Q^{-1} & 0 & \hat{D}_Q \end{array}\right],$$
where the blocks having at least one dimension equal to $\tilde{n}_Q$ are absent if $\tilde{n}_Q = 0$.

System Q in Fig. 3 is then implemented as the switching system (with the dependence on ω of the state space matrices is evidenced)

$$\dot{x}_Q = A_{Q(\omega)}x_Q + B_{Q(\omega)}u_Q, \quad (26a)$$

$$y_Q = C_{Q(\omega)}x_Q + D_{Q(\omega)}u_Q, \quad (26b)$$

where $x_Q(\cdot) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^{\bar{n}_Q}$ has fixed dimension and is continuous at the switching times of $\bar{\omega}$, that is $x_Q(t) = x_Q(t^-)$ for all t even if $\omega(t) \neq \omega(t^-)$. With the above choices, all considered Q's have the same state dimension, and share $V(x_Q) = x_Q'x_Q$ as a common Lyapunov function; moreover, such function satisfies $\dot{V}(x_Q) \leq -V(x_Q)$ when $u_Q = 0$.

IV. Properties of the Adaptive Hybrid Robust Regulator

Recall $\mathcal{K}$ in Fig. 2 is the hybrid compensator obtained by the interconnection of the residual generator $\mathcal{R}$ in (7), the frequency identifier ID in Section III-B, and the switching controller $\bar{\mathcal{K}}_\omega$ obtained as detailed in Section III-C.

Theorem 2: Under Assumption 1, Assumption 2 and Assumption 4, the hybrid compensator $\mathcal{K}$ in Fig. 2 solves the problem in Definition 2.

Remark 4: In the Assumption 2 the number of frequencies m and ω are assumed to be constant. However, thanks to the properties of the frequency estimator, if there exists at time $\bar{t}$ such that $m(t)$ and $\omega(t)$ are constant for $t \in [\bar{t}, \rho]$, with

[3]This hold true in our case $\bar{\omega}$ does not change before a minimum time equal to $4mT_s$ elapses.

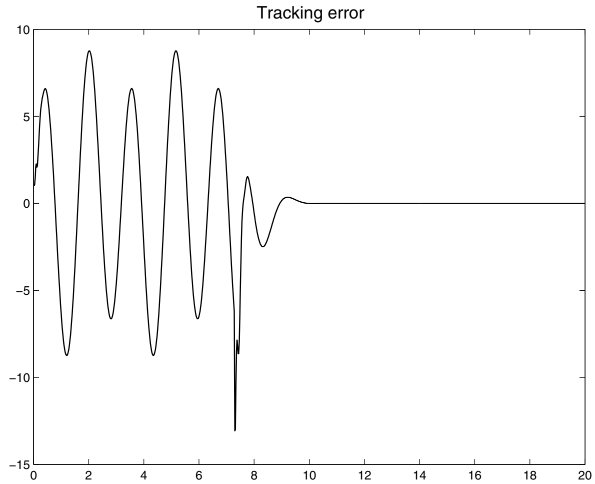

Fig. 4. The tracking error in case of two frequencies.

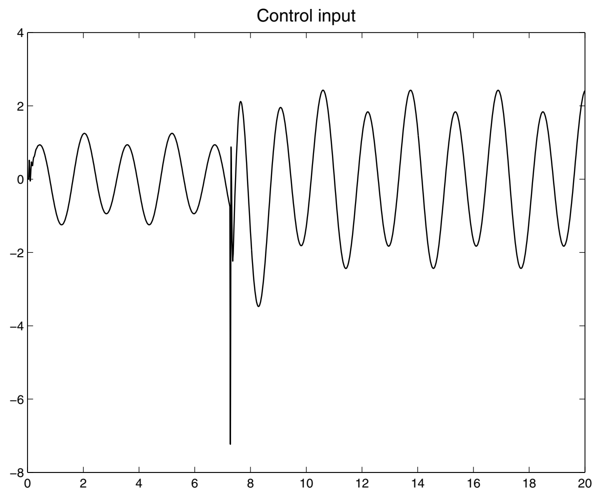

Fig. 5. The control input in case of two frequencies

$\bar{t} + 4m(\bar{t})T < \rho$, then for $t \in (-(\bar{t} + 4m(\bar{t})T), \rho)$ it holds $||e(t)|| \leq ||e(\bar{t})||e^{-\sigma(\rho - \bar{t} - 4m(\bar{t})T)}$ with $\sigma = \lambda_{\min}(\bar{Y}_k' \bar{Y}_k)$ and T selected by the ID.

Remark 5: By using reasonings similar to the proof of Theorem 2 and the additional robustness properties of the frequency identifier in Section III-B provided in the companion paper [4], it is also possible to show that the proposed compensator can achieve practical (instead of asymptotic) regulation, in the sense that for any bound $\varepsilon > 0$ on the norm of the acceptable error it is possible to find a bound $\delta > 0$ such that if the measurement noise is smaller than δ than also the error will become smaller that ε after a finite time. While the detailed description of such results is not provided here, it is mentioned here in order to substantiate further the practical interest of the proposed method.

V. An example

Consider the plant (1) with data

$$A = \begin{bmatrix} -7.5 & 31 \\ -31 & 7.5 \end{bmatrix}, \quad B = \begin{bmatrix} 36 \\ 68.5 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 36 \\ 0 & 68.5 \end{bmatrix} \quad (27a)$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0 \qquad Q = \begin{bmatrix} -1 & 1 \end{bmatrix} \quad (27b)$$

sampled with $T_H = 0.02$, subject to $w(t) = \begin{bmatrix} \sin(2t) & \cos(2t) & 1.5\sin(4t) & 1.5\cos(4t) \end{bmatrix}'$, modeling the reference signal as $\sin(2t)$ and the disturbance as $1.5\sin(4t)$. All the initial conditions of the servocompensator are set to zero. The resulting tracking error and control input are shown in Fig. 4 and Fig. 5, respectively. After the number of frequency is identified, approximatively at time 7.3, it is possible to see the fast settling to zero of the tracking error.

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