2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Distributed Robust Control of Spatially Interconnected Systems with Parameter Uncertainty

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*Abstract*— **This paper addresses the problem of robust control of spatially interconnected systems (SISs) subject to both exogenous disturbances and time-varying norm-bounded parameter uncertainties. A sufficient condition to the well-posedness, stability and attractiveness of an SIS with respect to all admissible uncertainties is presented in terms of a set of matrix inequalities. For the synthesis of distributed dynamic output feedback controllers, the nominal system is expanded in its output space so as to eliminate the uncertainties in the system matrices. In reference to the standard distributed scaled $H_\infty$ control, the synthesis of distributed robust $H_\infty$ controllers is proposed based on the system model in the expanded space. Numerical examples validate the effectiveness of the method.**

## I. INTRODUCTION

In recent years, large scale spatially interconnected systems (SISs) have received extensive research due to the wide range of applications in power systems, multi-agent cooperative systems, web-transport systems, and automated highway systems. To deal with this kind of large-scale systems, up till now, most literatures have been focused on decentralized control strategies[2], [5], and only in recent years, some scholars have begun to explore distributed control architecture from various perspectives [4], [7], [1].

Among other remarkable literatures to date, D'Andrea, Langbort and Bamieh, together with their colleagues, have played the leading role in the discussion over distributed control of SISs. Bamieh *et al.* focused mainly on distributed parameter systems that were modeled as linear infinite-dimensional systems[1]. Fourier transforms were introduced to diagonalize the relevant operators such that the transformed system was turned to a coupled family of standard finite-dimensional linear time-invariant systems. Later on, reference [12] considered a wider range of SISs that were spatially variant, and introduced spatially decaying operators in the cost function due to which the receding horizon controllers appeared to inherit spatial locality. D'Andrea *et al.*[4][8] modeled the SIS as a multidimensional system in both continuous time domain and spatial domain by the introduction of a spatial shift operator. The synthesis of the controllers is based on a set of convex linear matrix inequalities (LMIs) that are computational tractable and could be implemented in the LMI-Toolbox. Moreover, a more generic situation with subsystems that were heterogenous and were interconnected over an arbitrary network topology was discussed in [7] based on the dissipative theory. Most recently, an interesting work was reported in [13]. Based on the sequentially semi separable system model, the computational complexity for controllers synthesis was minimized.

Inspired by [7] and [3], this paper aims to introduce another type of robustness into the SIS with subsystems interconnected over an arbitrary network topology. we focus on the synthesis of distributed dynamic output feedback controllers such that the overall system is well-posed, exponentially stable and $H_\infty$ contractive to both exogenous disturbances and time-varying norm-bounded parameter uncertainties. In parallel to [15], we refer to this problem as the *distributed robust $H_\infty$ control*. Although a great efforts aforementioned have been made on the synthesis of distributed $H_\infty$ controllers, to our best knowledge, this problem has not yet been addressed in any of the related literatures. The results in this paper can be viewed as an extension of [7] and an analogy of the result in [15].

The rest of the paper is organized as follows: Section II presents some supporting results on SISs reported in the work [7] and several other fundamental inequalities. In Section III, sufficient conditions for well-posedness, stability and $H_\infty$ performance of the closed loop system over both directed and undirected interaction topologies are proposed in terms of LMIs. The synthesis of distributed controllers is discussed in Section IV. Section V presents numerical examples to illustrate the validity of the proposed method, and finally Section VI gives the conclusions.

## II. PRELIMINARY

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of positive real numbers respectively. The set of $m \times n$ real matrices is $\mathbb{R}^{m\times n}$ and the set of real vectors is $\mathbb{R}^m$. The concatenation of vectors $x_i \in \mathbb{R}^{m_i\times 1}$ is denoted by $\underset{i\in N}{\text{cat}}(x_i)$. Similarly, $\underset{i\in N}{\text{diag}}(a_i)$ where $a_i \in \mathbb{R}$ denotes a $N \times N$ diagonal matrix with $a_i$ being the $i$th diagonal entry. A set of real symmetric matrices is $\mathbb{R}_S^{n\times n}$, or abbreviated as $\mathbb{R}_S^n$. The set of positive(negative) definite real symmetric matrices is $\mathbb{R}_{S+}^{n\times n}(\mathbb{R}_{S-}^{n\times n})$, and the set of $n\times n$ skew-symmetric matrices is $\mathbb{R}_K^{n\times n}$. Let $E^n \in \mathbb{R}^{n\times n}$ be the identity matrix. When its dimension is clear from context, the superscript is omitted. Moreover, the $i$th column of $E$ is denoted by $E(i)$. For a matrix $A \in \mathbb{R}^{m\times n}$, its transpose is $A^T$ and we have the Hermitian operator acts on $A$: $\mathbf{He}(A) = A + A^T$.

The state-space model of the the $i$th subsystem takes the

This work was supported by the National Natural Science Foundation of China under grant No. 60736023 and grant No. 60704014

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form

$$\begin{bmatrix}\dot{x}(t)\\ w(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}A_{TT} & A_{TS} & B_{Td}\\ A_{ST} & A_{SS} & B_{Sd}\\ C_T & C_S & D\end{bmatrix}\begin{bmatrix}x(t)\\ v(t)\\ d(t)\end{bmatrix} \quad (1)$$

where subscript $i$ is omitted and vector $x_i(t) \in \mathbb{R}^{m_i}$. Vectors $v_i(t)$ and $w_i(t) \in \mathbb{R}^{l_i}$ are interconnected signals adjacent to subsystem $i$ and are further divided into $v_i = \underset{j\in V(i)}{\text{cat}}(v_{ij})$ and $w_i = \underset{j\in V(i)}{\text{cat}}(w_{ij})$ where $V(i)$ is the set of neighbors of subsystem $i$, $v_{ij} \in \mathbb{R}^{l_{ij}}$ and $w_{ij} \in \mathbb{R}^{l_{ij}}$ with $l_i = \sum_{j\in V(i)} l_{ij}$ being the input and output signals that flow between subsystems $i$ and $j$. An SIS that interconnected over an undirected graph $G$ satisfies $v_{ij} = w_{ji}$ Otherwise the SIS is a directed system.

All through this paper, discussions are carried out on each individual subsystem $i$. For the sake of simplicity, in most of cases, subscript $i$ is omitted.

In an SIS, well-posedness, stability and contractiveness are the three crucial indexes we use to evaluate the system performance. Langbort *et al.* in [7] have concluded a sufficient condition for those three indexes.

*Lemma 2.1:* The SIS (1) is well-posed, stable and contractive if, for each individual subsystem, inequality

$$\begin{bmatrix}E & 0 & 0\\ A_{TT} & A_{TS} & B_{Td}\\ A_{ST} & A_{SS} & B_{Sd}\\ 0 & E & 0\end{bmatrix}^*\begin{bmatrix}0 & X_T & 0 & 0\\ \star & 0 & 0 & 0\\ \star & \star & S_{11} & S_{12}\\ \star & \star & \star & S_{22}\end{bmatrix}\begin{bmatrix}E & 0 & 0\\ A_{TT} & A_{TS} & B_{Td}\\ A_{ST} & A_{SS} & B_{Sd}\\ 0 & E & 0\end{bmatrix} + \begin{bmatrix}C_T & C_S & D\\ 0 & 0 & E\end{bmatrix}^*\begin{bmatrix}E & 0\\ \star & -E\end{bmatrix}\begin{bmatrix}C_T & C_S & D\\ 0 & 0 & E\end{bmatrix} < 0 \quad (2)$$

is satisfied where $\star$ follows the symmetry of the matrix and the scalar matrices are

$$S_{11}^i = -\underset{j\in V(i)}{\text{diag}}(X_{11}^{ij}), \quad X_{11}^{ij} \in \begin{cases}\mathbb{R}_S^{l_{ij}\times l_{ij}}, & G \in \bar{G}\\ \mathbb{R}_{S-}^{l_{ij}\times l_{ij}}, & G \in \vec{G}\end{cases}$$

$$S_{12}^i = \begin{bmatrix}-\underset{j\in[1,i]\cap V(i)}{\text{diag}} X_{12}^{ij} & 0\\ 0 & \underset{j\in[i,N]\cap V(i)}{\text{diag}}(X_{12}^{ji})^*\end{bmatrix}$$

$$S_{22}^i = \underset{j\in V}{\text{diag}} X_{11}^{ji}, \quad X_T \in \mathbb{R}_{S+}^{m_i\times m_i}, \forall i, j \in V(i)$$

$$X_{12}^{ij} \in \begin{cases}\mathbb{R}^{l_{ij}\times l_{ij}}, & \forall i \geq j \in V(i) \text{ and } G \in \bar{G}\\ \mathbb{R}_K^{l_{ij}\times l_{ij}}, & \forall i = j \text{ and } G \in \bar{G}\\ 0, & G \in \vec{G}\end{cases} \quad (3)$$

It is highly recommended that readers referred to [7] for a detailed discussion of the three indexes we consider and the proof of Lemma 2.1.

The following two inequalities are well-known in the research of parameter uncertainties.

*Lemma 2.2 ([6]):* Let $D \in \mathbb{R}^{m\times n}$, $M \in \mathbb{R}^{m\times n}$ and $F \in \mathbb{R}^{m\times n}$. Then for an arbitrary $\varepsilon > 0$, inequality

$$\mathbf{He}(DFM) \leq \varepsilon^{-1}DD^* + \varepsilon M^*M \quad (4)$$

is satisfied if $F$ is of appropriate dimensions and satisfies $F^*F \leq E$.

*Lemma 2.3 ([6]):* Let $K \in \mathbb{R}^{m\times n}$ and $Z \in \mathbb{R}_{S+}^n$ then there exists $F \in \mathbb{R}^{m\times n}$ of appropriate dimension such that $F^*F \leq E$ and

$$K + M^*F^*D^*ZDFM \leq 0 \quad (5)$$

if and only if there exists a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix}K + \varepsilon^{-1}M^*M & 0\\ 0 & -Z^{-1} + \varepsilon DD^*\end{bmatrix} \leq 0 \quad (6)$$

## III. Analysis

In real-time applications, parameter uncertainties exist in the state-space model of an SIS, and have appeared in abundant of literatures in various forms. In one of the popular forms, the uncertainty is linear convex bounded[14]. In other words, the uncertainty is characterized by a polytopic model. A more general case is characterized by the norm-bounded uncertainties as seen in a lot of literatures [10], [11]. In this section, we focus on uncertainties under norm-bounded constraints.

To focus on the discussion of parameter uncertainties, in this paper we only consider parameter perturbations on matrices $A_\bullet$ and $B_\bullet$. We will discuss the more complete case in our later work.

With perturbations acting on matrices $A_\bullet$ and $B_\bullet$, the system model is then

$$\begin{bmatrix}\dot{x}(t)\\ w(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}A_{TT}+\Delta A_{TT} & A_{TS}+\Delta A_{TS} & B_{Td}+\Delta B_{Td}\\ A_{ST}+\Delta A_{ST} & A_{SS}+\Delta A_{SS} & B_{Sd}+\Delta B_{Sd}\\ C_T & C_S & D\end{bmatrix}\begin{bmatrix}x(t)\\ v(t)\\ d(t)\end{bmatrix} \quad (7)$$

where the uncertainties are structured by

$$\begin{bmatrix}\Delta A_{TT} & \Delta A_{TS} & \Delta B_{Td}\\ \Delta A_{ST} & \Delta A_{SS} & \Delta B_{Sd}\end{bmatrix} = \begin{bmatrix}H_T\\ H_S\end{bmatrix} F(t)\begin{bmatrix}P_T & P_S & P_d\end{bmatrix} \quad (8)$$

with $F(t)$ being the matrix-valued uncertainty satisfying

$$F(t)^*F(t) \leq E, \forall t \in \{0, \mathbb{R}_+\} \quad (9)$$

The matrices $H_\bullet, P_\bullet$ are known real matrices with appropriate dimensions characterizing how uncertainty $F(t)$ affects the nominal system. Uncertainties in (8) are convex in $F(t)$.

We have the following theorem that are inherited from the scaled $H_\infty$ control for nominal systems to evaluate an SIS.

*Theorem 3.1:* An SIS with interconnected subsystem (7) under norm-bounded perturbations (8) and (9) is well-posed, stable and contractive if, for each individual subsystem $i$, there exist symmetric matrices $X_T, S_{11}, S_{12}$ satisfying (3) and scalars $\gamma, \varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ such that the following LMI is satisfied:

$$\begin{bmatrix}M_0+N_0 & P^* & R_1 & R_2 & 0\\ \star & -\alpha^{-1}I & 0 & 0 & 0\\ \star & \star & -\varepsilon_1^{-1}I & 0 & 0\\ \star & \star & \star & -\varepsilon_2^{-1}I & 0\\ \star & \star & \star & \star & -(S_{11})^{-1}+\gamma H_S H_S^*\end{bmatrix} \leq 0 \quad (10)$$

with

$$R_1 = \begin{bmatrix}X_T H_T\\ 0\\ 0\end{bmatrix}, R_2 = \begin{bmatrix}A_{ST}^* S_{11}\\ A_{SS}^* S_{11} + S_{21}\\ B_S^* S_{11}\end{bmatrix} H_S,$$

$P = \begin{bmatrix} P_T & P_S & P_d \end{bmatrix}$ and $\alpha = \varepsilon_1^{-1} + \varepsilon_2^{-1} + \gamma^{-1}$. Matrices $M_0$ and $N_0$ designate the first and second term in the left-hand side of inequality (2), respectively.

*Proof:* When system matrices of a nominal SIS is perturbed by (8) and (9), the criterion matrix in (2) then contains the disturbance items in addition to the nominal ones as shown in (11) at the top of the next page.

According to Lemma 2.2, (11) is negative semi-definite if

$$M_0 + N_0 + (\varepsilon_1^{-1} + \varepsilon_2^{-1}) P^* P + P^* F^*(t) H_S^* S_{11} H_S F(t) P + \varepsilon_1 R_1 R_1^* + \varepsilon_2 R_2 R_2^* \leq 0 \tag{12}$$

where $\varepsilon_1, \varepsilon_2$ are arbitrary positive scalars. Under the assumption that $S_{11} \in \mathbb{R}^n_{S+}$ and Lemma 2.3, condition (12) is satisfied if and only if there exist $\alpha, \gamma > 0$ such that

$$\begin{bmatrix} M_0 + N_0 + \varepsilon_1 R_1 R_1^* + \varepsilon_2 R_2 R_2^* + \alpha P^* P & 0 \\ 0 & -(S_{11})^{-1} + \gamma H_S H_S^* \end{bmatrix} \leq 0 \tag{13}$$

Now the condition (10) can be obtained based on the Schur complement equivalence.

Thus condition (10) is sufficient to the wellposedness, stability and contractiveness of both the nominal system and the uncertain system. ■

The criterion (10) is convex in the scaling matrices. With (2) embedded, inequality (10) provides us an effective way to determine the stability and $H_\infty$ performance of an SIS subject to norm-bounded perturbations. Note that (13) is more restrictive than the condition given in [7]. The extra restrictions are caused by the parameter perturbations represented in the form $R_1, R_2, P$ and $H_S$.

In Lemma 2.1, when considering directed graph $\bar{G}$, the scaling matrix $S_{11}$ is defined to be an arbitrary symmetric one. However for a parameter perturbed SIS, matrix $S_{11}$ is confined by $S_{11} > 0$ so as to admit to Lemma 2.3. This restriction may add to conservatism as can be clearly showed by setting $H_T = 0$ and $H_S = 0$ in (10). One the other hand, when subsystems in an SIS are communicated over a directed graph, this conservativeness appears simultaneously both in criterions for the nominal system and the perturbed system.

## IV. SYNTHESIS

In a closed-loop interconnected system, control signal $u(t)$ is added to each subsystem (1), and the nominal subsystem $i$ is cast into

$$\begin{bmatrix} \dot{x}(t) \\ w(t) \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A & B_d & B_u \\ C_z & D_{zd} & D_{zu} \\ C_y & D_{yd} & D_{yu} \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \\ \begin{bmatrix} d(t) \\ u(t) \end{bmatrix} \end{bmatrix} \tag{14}$$

In addition to perturbations (8) and (9), we have

$$\begin{bmatrix} \Delta B_{Tu} \\ \Delta B_{Su} \end{bmatrix} = \begin{bmatrix} H_T \\ H_S \end{bmatrix} F(t) P_u \tag{15}$$

The controllers are assumed to maintain exactly the same dynamic structures as their corresponding plants while share the unique network topology. By canceling the interconnected signals $y$ and $u$, The $i$th uncertain closed-loop system is captured by

$$\begin{bmatrix} \begin{bmatrix} \dot{x}^G(t) \\ \dot{x}^K(t) \\ w^C(t) \end{bmatrix} \\ z(t) \end{bmatrix} = \begin{bmatrix} \bar{A}^C & \bar{B}^C \\ \bar{C}^C & \bar{D}^C \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x^G(t) \\ x^K(t) \\ v^C(t) \end{bmatrix} \\ d(t) \end{bmatrix} \tag{16}$$

with

$$\begin{bmatrix} \bar{A}^C & \bar{B}^C \\ \bar{C}^C & \bar{D}^C \end{bmatrix} = U \begin{bmatrix} A + \Delta A & 0 & B_d + \Delta B_d \\ 0 & 0 & 0 \\ C_z & 0 & D_{zd} \end{bmatrix} U^* + U \begin{bmatrix} B_u + \Delta B_u & 0 \\ 0 & E \\ D_{zu} & 0 \end{bmatrix} V \Sigma V^* \begin{bmatrix} C_y & 0 & 0 \\ 0 & E & D_{yd} \end{bmatrix} U^*$$

$$\Sigma = \begin{bmatrix} A^K_{TT} & A^K_{TS} & B^K_T \\ A^K_{ST} & A^K_{SS} & B^K_S \\ C^K_T & C^K_S & D^K \end{bmatrix}, V = \begin{bmatrix} E(2) & E(1) \end{bmatrix}$$

$$U = \begin{bmatrix} E(1) & E(3) & E(2) & E(4) & E(5) \end{bmatrix}$$

and $l_i^C = l_i + l_i^K, m_i^C = m_i + m_i^K$.

Let

$$\bar{H}_T = \begin{bmatrix} H_T \\ 0 \end{bmatrix}, \bar{P}_T = \begin{bmatrix} P_T & 0 \end{bmatrix}, \bar{P}_u = \begin{bmatrix} 0 & 0 & P_u \end{bmatrix}$$

$$\bar{P}_S = \begin{bmatrix} P_S & 0 \end{bmatrix}, \bar{P}_d = P_d, \bar{H}_S = \begin{bmatrix} H_S \\ 0 \end{bmatrix} \tag{17}$$

*Proposition 4.1:* A closed-loop system with subsystems (16) is well-posed, stable and contractive if, for each individual subsystems, there exist symmetric matrices $X_T^C \in \mathbb{R}^n_{S+}$ , $S_{11}^C \in \mathbb{R}^m_{S+}$, $S_{12}^C \in \mathbb{R}^m$ and scalars $\lambda, \varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$. such that the following matrix inequality is satisfied.

$$\begin{bmatrix} M_0^C + N_0^C + \varepsilon_1 R_1^C (R_1^C)^* + \varepsilon_2 R_2^C (R_2^C)^* + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \lambda^{-1})(P^C)^* P^C & 0 \\ 0 & -(S_{11}^C)^{-1} + \lambda \bar{H}_S \bar{H}_S^* \end{bmatrix} \leq 0 \tag{18}$$

where

$$R_1^C = \begin{bmatrix} X_T^C \bar{H}_T \\ 0 \\ 0 \end{bmatrix}, R_2^C = \begin{bmatrix} (A_{ST}^C)^* S_{11}^C \\ (A_{SS}^C)^* S_{11}^C + (S_{12}^C)^* \\ (B_S^C)^* S_{11}^C \end{bmatrix} \bar{H}_S \tag{19}$$

and the scaling matrices in the closed-loop criterion matrix $M_0^C + N_0^C$ is

$$S^C = P \begin{bmatrix} S_{11}^P & S_{12}^P & S_{11}^{PK} & S_{12}^{PK} \\ \star & S_{22}^P & (S_{12}^{PK})^* & S_{22}^{PK} \\ \star & \star & S_{11}^K & S_{12}^K \\ * & * & * & S_{22}^K \end{bmatrix} P^* \tag{20}$$

$$P = \begin{bmatrix} E(1) & E(3) & E(2) & E(4) \end{bmatrix} \tag{21}$$

with $S_{11}^\bullet, S_{12}^\bullet$ and $S_{22}^\bullet$ admitting the same structures as the open loop ones shown in (3).

Eq. (18) is non-convex in the controller matrices and the scaling matrices. In [7] the sufficient and necessary condition for the existence of distributed controllers of the nominal

$$
\begin{aligned}
& M_0 + N_0 + \begin{bmatrix} 0 & (H_T F(t) P)^* \end{bmatrix} \begin{bmatrix} 0 & X_T \\ X_T & 0 \end{bmatrix} \begin{bmatrix} 0 \\ H_T F(t) P \end{bmatrix} + \begin{bmatrix} I & A_{TT}^* \\ 0 & A_{TS}^* \\ 0 & B_T^* \end{bmatrix} \begin{bmatrix} 0 & X_T \\ X_T & 0 \end{bmatrix} \begin{bmatrix} 0 \\ H_T F(t) P \end{bmatrix} \\
& + \begin{bmatrix} 0 & (H_T F(t) P)^* \end{bmatrix} \begin{bmatrix} 0 & X_T \\ X_T & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ A_{TT} & A_{TS} & B_T \end{bmatrix} + \begin{bmatrix} (H_S F(t) P)^* & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ \star & S_{22} \end{bmatrix} \begin{bmatrix} A_{ST} & A_{SS} & B_S \\ 0 & I & 0 \end{bmatrix} \\
& + \begin{bmatrix} A_{ST}^* & 0 \\ A_{SS}^* & I \\ B_S^* & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ \star & S_{22} \end{bmatrix} \begin{bmatrix} H_S F(t) P \\ 0 \end{bmatrix} + \begin{bmatrix} (H_S F(t) P)^* & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ \star & S_{22} \end{bmatrix} \begin{bmatrix} H_S F(t) P \\ 0 \end{bmatrix} \\
= \; & M_0 + N_0 + \mathbf{He}\left( \begin{bmatrix} X_T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} H_T \\ 0 \\ 0 \end{bmatrix} F(t) P \right) + P^* F^*(t) H_S^* S_{11} H_S F(t) P + \mathbf{He}\left( A_{SS}^* S_{11} H_S F(t) P + \begin{bmatrix} 0 \\ S_{21} H_S F(t) P \\ 0 \end{bmatrix} \right) \quad (11)
\end{aligned}
$$

closed-loop system is concluded based on the elimination lemma. In the following context, we will extend results therein to interconnected system subject to norm-bounded perturbations.

*Theorem 4.1:* A closed-loop SIS with subsystems (16) under norm-bounded parameter perturbations (15), (8) and (9) is well-posed, stable and contractive if, for each individual subsystems, there exist $X_T^C \in \mathbb{R}^n_{S+}$ , $S_{11}^C \in \mathbb{R}^m_{S+}$, $S_{12}^C \in \mathbb{R}^m$ and $\gamma, \varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ such that the following two properties holds:

i

$$
\begin{bmatrix} E & 0 & 0 \\ A_{TT}^C & A_{TS}^C & B_{Td}^C \\ \hline A_{ST}^C & A_{SS}^C & B_{Sd}^C \\ 0 & E & 0 \end{bmatrix}^* \left[\begin{array}{cc|cc} \bar{X}_T & X_T^C & 0 & 0 \\ X_T & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{S}_{11}^C & \bar{S}_{12}^C \\ 0 & 0 & \star & \bar{S}_{22}^C \end{array}\right] \begin{bmatrix} E & 0 & 0 \\ A_{TT}^C & A_{TS}^C & B_{Td}^C \\ \hline A_{ST}^C & A_{SS}^C & B_{Sd}^C \\ 0 & E & 0 \end{bmatrix} + \begin{bmatrix} C_T^C & C_S^C & D^C \\ K_T & K_S & K_d \\ 0 & 0 & E \end{bmatrix}^* \begin{bmatrix} E & 0 & 0 \\ 0 & \alpha E & 0 \\ 0 & 0 & -E \end{bmatrix} \begin{bmatrix} C_T^C & C_S^C & D^C \\ K_T & K_S & K_d \\ 0 & 0 & E \end{bmatrix} < 0 \quad (22)
$$

ii

$$
-(S_{11}^C)^{-1} + \gamma \bar{H}_S^* \bar{H}_S \leq 0 \quad (23)
$$

where $\bar{X}_T = \varepsilon_1 X_T^C \bar{H}_T \bar{H}_T^* X_T^C$,

$$
\begin{aligned}
& \begin{bmatrix} K_T & K_S & K_d \end{bmatrix} = \\
& \begin{bmatrix} \bar{P}_T & \bar{P}_S & \bar{P}_d \end{bmatrix} + \bar{P}_u \Sigma V^* \begin{bmatrix} C_y & 0 & 0 \\ 0 & E & D_{yd} \end{bmatrix} U^* \quad (24)
\end{aligned}
$$

and the scaling matrices

$$
\begin{bmatrix} \bar{S}_{11}^C & \bar{S}_{12}^C \\ \star & \bar{S}_{22}^C \end{bmatrix} = \begin{bmatrix} S_{11}^C & S_{12}^C \\ \star & S_{22}^C \end{bmatrix} + \varepsilon_2 \begin{bmatrix} S_{11}^C \\ (S_{12}^C)^* \end{bmatrix} \bar{H}_S (\bar{H}_S)^* \begin{bmatrix} S_{11}^C \\ (S_{12}^C)^* \end{bmatrix}^* \quad (25)
$$

with $S^C$ given in (20), and $\alpha = \gamma^{-1} + \varepsilon_1^{-1} + \varepsilon_2^{-1}$.

*Proof:*

$$
\begin{aligned}
(R_2^C)(R_2^C)^* &= \begin{bmatrix} (A_{ST}^C)^* S_{11}^C \\ (A_{SS}^C)^* S_{11}^C + (S_{12}^C)^* \\ B_S^* S_{11}^C \end{bmatrix} \bar{H}_S \bar{H}_S^* \begin{bmatrix} (A_{ST}^C)^* S_{11}^C \\ (A_{SS}^C)^* S_{11}^C + (S_{12}^C)^* \\ B_S^* S_{11}^C \end{bmatrix}^* \\
&= \begin{bmatrix} A_{ST}^C & A_{SS}^C & B_S^C \\ 0 & E & 0 \end{bmatrix}^* \begin{bmatrix} S_{11}^C \\ (S_{12}^C)^* \end{bmatrix} \bar{H}_S \bar{H}_S^* \begin{bmatrix} S_{11}^C \\ (S_{12}^C)^* \end{bmatrix}^* \begin{bmatrix} A_{ST}^C & A_{SS}^C & B_S^C \\ 0 & E & 0 \end{bmatrix} \quad (26)
\end{aligned}
$$

$$
\begin{aligned}
R_1^C (R_1^C)^* &= \begin{bmatrix} X_T^C \bar{H}_T \bar{H}^* X_T^C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} E & 0 & 0 \\ A_{TT}^C & A_{TS}^C & B_{Td}^C \end{bmatrix}^* \begin{bmatrix} X_T^C \bar{H}_T \bar{H}^* X_T^C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ A_{TT}^C & A_{TS}^C & B_{Td}^C \end{bmatrix} \quad (27)
\end{aligned}
$$

Recall the closed-loop system matrices we have

$$
\begin{aligned}
& \begin{bmatrix} \bar{A}_{TT}^C & \bar{A}_{TS}^C & \bar{B}_B^C \end{bmatrix} = \begin{bmatrix} A_{TT}^C & A_{TS}^C & B_T^C \end{bmatrix} \\
& + \bar{H}_T \bar{F} \left( \begin{bmatrix} \bar{P}_T & \bar{P}_S & \bar{P}_d \end{bmatrix} + \bar{P}_u \Sigma V^* \begin{bmatrix} C_y & 0 & 0 \\ 0 & E & D_{yd} \end{bmatrix} U^* \right) \quad (28)
\end{aligned}
$$

Thus the closed form of the perturbation matrix $E$ in (13) is recast into

$$
P^C = \begin{bmatrix} \bar{P}_T & \bar{P}_S & \bar{P}_d \end{bmatrix} + \bar{P}_u \Sigma V^* \begin{bmatrix} C_y & 0 & 0 \\ 0 & E & D_{yd} \end{bmatrix} U^* \quad (29)
$$

In accordance to (26), (27) and (29), block $(1,1)$ in (18) can then rewritten into the form of (22) with the substitution of the scaling matrices in (25), which finishes the proof. ■

The matrix inequality (22) in both the controller matrices and the scaling matrices is a non-convex one and is NP-hard. An effective way to solve this problem is introduced in [7] known as the Elimination Lemma. However when uncertainty (15), (8) and (9) enter the SIS, the criterion inequality (22) is no longer the standard form of the quadratic matrix inequality considered in [7]. Moreover, condition (22) is non-convex in $X_T^C$, $S_{11}^C$ and $S_{12}^C$ even when $\Sigma$ is known. In the following context, we will make adjustments upon (22) and propose an algorithm to solve this problem.

We define an expanded closed-loop matrix

$$
\begin{aligned}
\bar{\Sigma}_{cl} &= \begin{bmatrix} A_{TT}^C & A_{TS}^C & B_T^C \\ A_{ST}^C & A_{SS}^C & B_S^C \\ C_T^C & C_S^C & D^C \\ K_T & K_S & K_d \end{bmatrix} = \bar{U} \begin{bmatrix} A & 0 & B_d \\ 0 & 0 & 0 \\ C_z & 0 & D_{zd} \\ \begin{bmatrix} P_T & P_S \end{bmatrix} & 0 & P_d \end{bmatrix} \bar{U}^* \\
& + \bar{U} \begin{bmatrix} B_u & 0 \\ 0 & E \\ D_{zu} & 0 \\ u & 0 \end{bmatrix} V \Sigma V^* \begin{bmatrix} C_y & 0 & 0 \\ 0 & E & D_{yd} \end{bmatrix} U^* \quad (30)
\end{aligned}
$$

with $\bar{U} = \begin{bmatrix} U & E(6) \end{bmatrix}$ being the expanded matrix of $U$. If we set

$$\hat{C}_z = \begin{bmatrix} C_{Tz} & C_{Sz} \\ P_T & P_S \end{bmatrix}, \hat{D}_{zd} = \begin{bmatrix} D_{zd} \\ P_d \end{bmatrix}, \hat{D}_{zu} = \begin{bmatrix} D_{zu} \\ P_u \end{bmatrix} \quad (31)$$

we get a compact version of the system matrix $\bar{\Sigma}_{cl}$ that is an affine function of $\Sigma$. This allows us to bridge the gap between the robust $H_\infty$ control and scaled $H_\infty$ control by means of expansion in the output space.

*Algorithm 4.1 (Controller Construction):* Based on (22) and (23), the controllers construction for each subsystem $i$ proceeds as follows:

1) Get the expanded output matrices $\hat{C}_z$, $\hat{D}_{zd}$ and $\hat{D}_{zu}$ in (31).
2) Let $\bar{S}_{11}^C \in \mathbb{R}_{S+}^{n\times n}$, solve the standard controller synthesis problem:

$$\max_{\alpha>0} \alpha$$

s.t.

$$\begin{bmatrix} E & 0 & 0 \\ A_{TT}^C & A_{TS}^C & B_{Td}^C \\ \hdashline A_{ST}^C & A_{SS}^C & B_{Sd}^C \\ 0 & E & 0 \\ \hdashline \hat{C}_T^C & \hat{C}_S^C & \hat{D}^C \\ 0 & 0 & E \end{bmatrix}^* \begin{bmatrix} 0 & X_T^C & 0 & 0 & 0 & 0 \\ X_T & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & S_{11}^C & S_{12}^C & 0 & 0 \\ 0 & 0 & \star & S_{22}^C & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \bar{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & -E \end{bmatrix} \cdot \begin{bmatrix} E & 0 & 0 \\ A_{TT}^C & A_{TS}^C & B_{Td}^C \\ \hdashline A_{ST}^C & A_{SS}^C & B_{Sd}^C \\ 0 & E & 0 \\ \hdashline \hat{C}_T^C & \hat{C}_S^C & \hat{D}^C \\ 0 & 0 & E \end{bmatrix} < 0 \quad (32)$$

where

$$\bar{E} = \begin{bmatrix} E^{l_i^C} & 0 \\ 0 & \alpha E^{m_i^C} \end{bmatrix}$$

over the expanded closed-loop subsystems (30). The standard synthesis process was demonstrated in [7].
3) Let $\tilde{X} = S_{11}^C, \tilde{G} = \varepsilon_2 \bar{H}_S \bar{H}_S^*$ and $\tilde{H} = \bar{S}_{11}^C$, find the maximal $\varepsilon_2$ such that when $\tilde{G} \in \mathbb{R}_{S+}^{n\times n}$ and $\tilde{H} \in \mathbb{R}_{S+}^{n\times n}$ , the algebraic Riccati equation (ARE)

$$\tilde{X} + \tilde{X}\tilde{G}\tilde{X} - \tilde{H} = 0 \quad (33)$$

has solution $\tilde{X} \in \mathbb{R}_{S+}^{n\times n}$. Note that matrix "A" in the standard ARE is chosen to be a stable one $A = -\frac{1}{2}E$, which ensures the existence of $\tilde{X}$.
4) Find maximal $\gamma > 0$ such that condition (23) is satisfied. Condition $S_{11}^C > 0$ ensures the existence of $\gamma$.
5) Find a maximal scalar $\varepsilon_1 > 0$ such that

$$(32) + \begin{bmatrix} \varepsilon_1 X_T^C \bar{H}_T \bar{H}_T^* X_T^C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0 \quad (34)$$

6) If $\alpha > \gamma^{-1} + \varepsilon_1^{-1} + \varepsilon_2^{-1}$, set

$$\alpha = \gamma^{-1} + \varepsilon_1^{-1} + \varepsilon_2^{-1} \quad (35)$$

and condition (22) is still satisfied.

*Remark 4.1:* The scaling matrices we derived from step 2) is not the one for the nominal system. Thus we have to solve equation (25) so as to generate the scaling matrix $S_{11}^C$, and further applied it to condition (23). As we neglected the element $\bar{X}_T$ in step 2), we should validate the controllers we picked, which is carried out in step 5). When the optimal value of $\alpha$ in step 2) yields (35), the distributed controller derived during step 2) satisfies both the inequality (22) and (23).

*Remark 4.2:* This synthesis procedure may fail at step 6) if the inequality is not satisfied. In order to avoid this situation, when determining the parameters $\alpha$, $\gamma$, $\varepsilon_1$ and $\varepsilon_2$, we introduced the optimization operations. Although this dose not guarantee the success of the procedure, according to numerous examples we worked on, failure of the procedure is effectively avoided by the set of appropriate parameters $\alpha$, $\gamma$, $\varepsilon_1$ and $\varepsilon_2$.

Note that the controllers for the expanded system has the same dimension as that of the nominal system. Algorithm 1 guarantees well-posedness, stability and contractiveness of the nominal SIS as well.

The crucial steps in the algorithm are the expandation (Step 1)) and the synthesis (Step 2)). The synthesis techniques of robust controllers for expanded system is borrowed from [7] with minor differences in maximization of $\gamma$. Thanks to [7], the computational tractable method therein was transplantable.

Recall Theorem 3.1, for an SIS with subsystems interconnected over directed networks, Algorithm 1 differs in step 2) and step 5) by setting $S_{12}^C = 0$ and $l_i^K = l_i$, compared to $l_i^K = 3l_i$ for undirected case. The other steps are retained. This fact indicates that the proposed synthesis process of uncertainty SIS accommodate to the scaled $H_\infty$ synthesis[7] in an effective manner.

## V. Illustrative Example

In this section, we validate the effectiveness of Algorithm 1 for subsystems interconnected over an undirected network.

Consider an undirected SIS consists of three interconnected subsystems over network topology of

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad (36)$$

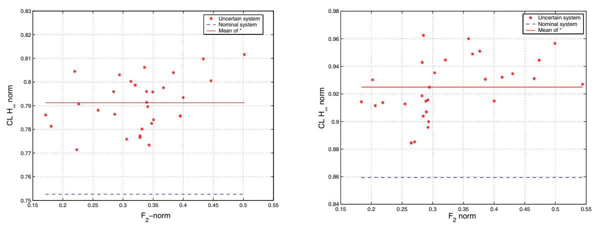
such that $v_{ij} = m_{ij} w_{ji}$.

The exogenous disturbance satisfy $d_2 = d_3 = 0$ and $d_1$ is the unit-pulse signal with a width of 0.2*sec*.

Following the controllers construction process in Section IV, the scalars were set to be $\varepsilon_1^1 = 57.76$, $\varepsilon_1^2 = 44.82$, $\varepsilon_1^3 = 55.42$, $\varepsilon_2^i = 1, i = 1,2,3$ and $\gamma = 0.6, i = 1,2,3$. We carried out 20 times experiments with randomly generalized uncertainty matrix $F^i(k) : \|F^i(k)\|_2 \le 1, \forall k \in [1,20], i \in [1,3]$ on each subsystem in addition to exogenous unit-pulse signal

TABLE I: Perturbations and the corresponding $H_\infty$ gain

| $k$ | 2 | 5 | 8 | 11 | 14 | 17 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $\|F\|_2$ | 0.4462 | 0.3392 | 0.2263 | 0.3493 | 0.1714 | 0.5016 | $\cdots$ |
| $\|\bar{T}_{zd}\|_\infty$ | 0.8005 | 0.7914 | 0.7907 | 0.7958 | 0.7861 | 0.8116 | $\cdots$ |




(a) $H_\infty$ performance (b) Output dynamics

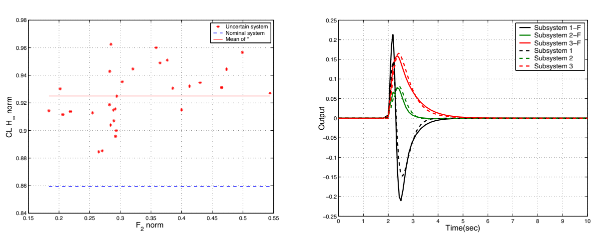
Fig. 1: System performance of the closed-loop uncertain system over undirected networks

on subsystem 1. Under the invariant distributed controllers, the resulting $H_\infty$ gain of the closed-loop SIS with respect to the mean perturbations $\|F(k)\|_2 = \sum_{i\in[1,3]} 1/3\|F^i(k)\|_2$ is shown in Table I and Fig. 1(a). In Fig. 1(a), the solid line with $y \equiv 0.7913$ denotes the mean value of $\|\bar{T}_{zd}\|_\infty$ while the dashed line with $y \equiv 0.7526$ represents the value of $\|T_{zd}\|_\infty$ of the nominal system . The distributed controllers we designed restricted the $H_\infty$ gain under parameter uncertainties within a scale of 7.8% of the nominal one.

The system response of both the nominal and the uncertain closed-loop SIS is shown in Fig. 1(b), where the dashed lines represent the nominal system response while the solid lines denote the system response against norm-bounded uncertainty. With the distributed robust controllers, the outputs of the system with or without uncertainty almost coincide with each other, indicating that the effects of parameter uncertainty are diminished into an satisfactory scale under the distributed controllers we developed.

## VI. Conclusion

In this paper, based on results from [7], sufficient conditions have been developed for the stability, wellposedness and contractiveness of SISs with norm-bounded parameter uncertainty in addition to exogenous disturbance. For the synthesis of distributed controllers, using techniques from linear matrix inequalities, we bridged the gap between distributed robust $H_\infty$ control and scaled $H_\infty$ control by expanding the output space of the state-space equation of the nominal system into higher dimensions that helps to eliminate the uncertainty matrices. Based on the expanded state-space model, the synthesis process was divided into several sequential steps including standard controllers synthesis process and the regain of the nominal scaling matrices. We showed that the robust $H_\infty$ control problem accommodate to scaled $H_\infty$ control in an effective manner. Numerical examples illustrated the efficiency of the proposed algorithm. Extension of this method to SISs consists of discrete-time subsystems will be considered in the future research.




(a) $H_\infty$ performance (b) Output dynamics

Fig. 2: System performance of the closed-loop uncertain system over directed networks


## Acknowledgement

The authors would like to thank Professor Cédric Langbort for the useful discussions on the technical details of the paper.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