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A Liveliness Analysis of a Distributed Constrained Coordination Strategy for Multi-Agent Linear Systems

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***Abstract*— In this paper we present a complete liveliness analysis for a novel distributed supervision strategy for multi-agent systems connected via data links and subject to coordination constraints. A geometrical *Constraints Qualification* (CQ) condition on the prescribed constraints is proposed whose fulfilment avoid deadlock situations and ensure viable solutions. Such a coordination paradigm, referred to as *coordination-by-constraint*, is accomplished by solving on-line in a distributed manner a convex optimization problem involving the modification of the prescribed nominal set-point of each agent. A sequential strategy, where only one agent at the time is allowed to manipulate its own setpoint, is discussed and it is shown that the solvability at each time instant of the underlying optimization problem, unlike the centralized case, can be ensured only if the constraints satisfy such a (CQ) condition, which can be easily checked from the outset via a numerical procedure provided in the paper. An algorithm to compute *Constraints Qualified* (CD) arbitrarily accurate inner approximations of the original constraint set is also presented and exemplified in final example.**

I. INTRODUCTION

The problem of interest here is a detailed investigation of a sort of *liveliness* properties of the sequential distributed supervision strategy proposed in the companion paper [1], based on *Feed-Forward* Command Governor (FFCG) ideas recently proposed in [2]. In the latter work, at the price of some additional conservativeness, a scheme able to accomplish the CG task in the absence of an explicit measure or estimate of the state has been presented.

The system paradigm of interest is depicted in Fig. 1. There, a data link connecting the agents exists which allows them to exchange relevant data. The agents are in charge to coordinate in a distributed way N locally regulated plants by acting on their nominal set-point r_i. In particular, whenever necessary, the agents are instructed to modify the prescribed set-point r_i in their admissible versions g_i, when the tracking of the nominal set-points would produce constraint violations and hence loss of coordination. In such a context, the supervision task can be expressed as the requirement of satisfying some tracking performance, viz. $y_i \approx r_i$, whereas the coordination task consists of enforcing some constraints $c_i \in \mathcal{C}_i$ and/or $f(c_1, c_2, ..., c_N) \in \mathcal{C}$ on each remote plant and/or on the overall network evolutions at each time instant.

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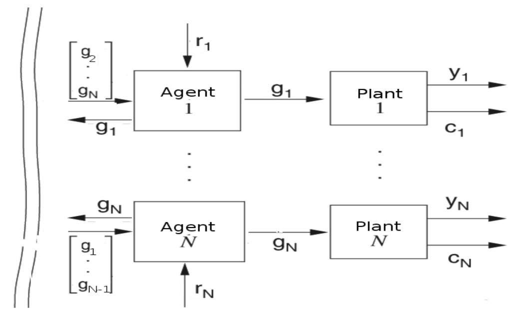

Fig. 1. Multi-agent architectures

The peculiarity of this sequential distributed scheme is that only one agent at the time is allowed to modify its own reference signal. Moreover, because resorting to a FFCG approach which doesn't exploit the current measure of the state, the scheme under investigation may be an attractive solution in distributed contexts because it alleviates the need to make the entire aggregate state, or substantially parts of it, known to all agents at each time instant.

This paper complements and integrates the companion paper [1] by making clear several theoretical aspects of this novel sequential distributed scheme, not discussed in [1], and proposing verifiable sufficient conditions on the constraints whose fulfilment ensure that the liveliness of the scheme is preserved. In fact, it is shown here that under specific constraint structures the proposed distributed supervising scheme may fail to find a solution because, unlike in centralized solutions, the agents are only allowed to update their commands one at the time. This restriction may lead to deadlock situations and prevent the agents from being able to modify their local set-points r_i altogether.

The geometrical characterization of the structure of the constraints set is exploited in order to establish if deadlock situation may occur. Such an analysis involves the verification of a particular geometrical property, here referred as *Constraints Qualification* (CQ), for all points belonging to the boundaries of the constrained region. To this end, a numerical procedure is also proposed. Finally, it is shown that it is possible to determine arbitrarily accurate inner approximations of the prescribed constrained region (via a multi-box approach) which, in the case where each agent manages a monodimensional g_i, result *Constraints Qualified* (CD) by construction and avoid deadlock situations.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a set of N subsystems $\mathcal{A} = \{1, \ldots, N\}$, each one being a LTI closed-loop dynamical system regulated by a local controller which ensures stability and good closed-loop properties when the constraints are not active (small-signal regimes when the coordination is effective). Let the i-th closed-loop subsystem be described by the following discrete-time model

$$\begin{cases} x_i(t+1) &= \Phi_{ii}x_i(t)+G_i^g g_i(t)+\sum_{j\in\mathcal{A}-\{i\}}\Phi_{ij}x_j(t) \\ y_i(t) &= H_i^y x_i(t) \\ c_i(t) &= H_i^c x(t)+L_i g(t) \end{cases} \quad (1)$$

where: $t \in \mathbb{Z}_+$, $x_i \in \mathbb{R}^{n_i}$ is the state vector (which includes the controller states under dynamic regulation), $g_i \in \mathbb{R}^{m_i}$ the manipulable reference vector which, if no constraints (and no CG) were present, would coincide with the desired reference $r_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^{m_i}$ is the output vector which is required to track r_i. Finally, $c_i \in \mathbb{R}^{n_i^c}$ represents the local constrained vector which has to fulfill the set-membership constraint

$$c_i(t) \in \mathcal{C}_i,\ \forall t \in \mathbb{Z}_+, \quad (2)$$

$\mathcal{C}_i$ being a convex and compact polytopic set. It is worth pointing out that, in order to possibly characterize global (coupling) constraints amongst states of different subsystems, the vector c_i in (1) is allowed to depend on the aggregate state and manipulable reference vectors $x = [x_1^T,\ldots,x_N^T]^T \in \mathbb{R}^n$, with $n = \sum_{i=1}^N n_i$, and $g = [g_1^T,\ldots,g_N^T]^T \in \mathbb{R}^m$, with $m = \sum_{i=1}^N m_i$. Moreover, we denote by $r = [r_1^T,\ldots,r_N^T]^T \in \mathbb{R}^m$, $y = [y_1^T,\ldots,y_N^T]^T \in \mathbb{R}^m$ and $c = [c_1^T,\ldots,c_N^T]^T \in \mathbb{R}^{n^c}$, with $n^c = \sum_{i=1}^N n_i^c$, the other relevant aggregate vectors. The overall system arising by the composition of the above N subsystems can be described as

$$\begin{cases} x(t+1) &= \Phi x(t)+Gg(t) \\ y(t) &= H^y x(t) \\ c(t) &= H^c x(t)+Lg(t) \end{cases} \quad (3)$$

where

$$\Phi = \begin{pmatrix} \Phi_{11} & \ldots & \Phi_{1N} \\ \vdots & \ddots & \vdots \\ \Phi_{N1} & \ldots & \Phi_{NN} \end{pmatrix}, G = \begin{pmatrix} G_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & G_N \end{pmatrix}$$

$$H^y = \begin{pmatrix} H_1^y & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & H_N^y \end{pmatrix}, H^c = \begin{pmatrix} H_1^c \\ \ldots \\ H_N^c \end{pmatrix}, L = \begin{pmatrix} L_1 \\ \ldots \\ L_N \end{pmatrix}.$$

It is further assumed that

A1. The overall system (3) is asymptotically stable.

A2. System (3) is off-set free i.e. $H^y(I_n-\Phi)^{-1}G = I_m$.

Roughly speaking, the CG design problem we want to solve is that of locally determine, at each time step t and for each agent $i \in \mathcal{A}$, a suitable reference signal $g_i(t)$ which is the best approximation of $r_i(t)$ such that its application never produces constraints violation, i.e. $c_i(t) \in \mathcal{C}_i, \forall t \in \mathbb{Z}_+, i \in \mathcal{A}$

Classical centralized solutions of the above stated CG design problem (see [4], [3]) have been achieved by finding, at each time t, a CG action $g(t)$ as a function of the current reference $r(t)$ and measured state $x(t)$

$$g(t) := \underline{g}(r(t), x(t)) \quad (4)$$

such that $g(t)$ is the best approximation of $r(t)$ under the condition $c(t) \in \mathcal{C}$, where $\mathcal{C} \subseteq \{\mathcal{C}_1 \times \ldots \times \mathcal{C}_N\}$ is the global admissible region. In [2], the *Feed-Forward* CG (FF-CG) approach has been proposed, where a CG action having the following structure

$$g(t) \ = \ \underline{g}(r(t), g(t-\tau)) \quad (5)$$

was proved to have similar properties of the standard CG state-based approach when computed every τ steps and kept constant between two subsequent updating, without hinging upon on the explicit knowledge of the state vector.

III. The Feed-Forward CG approach

In this section we recall the basic ideas and notation of the FF-CG approach proposed in [2] which will be relevant for the the forthcoming discussion. To this end, consider, for a given $\delta > 0$, the sets:

$$\begin{array}{rcl} \mathcal{C}_\delta &:=& \mathcal{C} \sim \mathcal{B}_\delta \\ \mathcal{W}_\delta &:=& \{g \in \mathbb{R}^m :\ c_g \in \mathcal{C}^\delta\} \end{array} \quad (6)$$

where $\mathcal{B}_\delta$ is the ball of radius δ centered at the origin and $\mathcal{A} \sim \mathcal{E}$ is the Pontryagin set difference defined as $\{a\ :\ a+e \in \mathcal{A}, \forall e \in \mathcal{E}\}$. In particular, $\mathcal{W}^\delta$, which we assume non-empty, is the convex and closed set of all constant commands g whose corresponding equilibrium points $c_g := H^c(I_n-\Phi)^{-1}Gg + Lg$ satisfy the constraints with margin δ. Let introduce also the *virtual evolutions* of the c-variable

$$\hat{c}(k, x(t), g(t)) := H^c\left(\Phi^k x(t)+\sum_{i=0}^{k-1}\Phi^{k-i-1}Gg(t)\right)+Lg(t) \quad (7)$$

along the *virtual time* k, from the initial condition $x(t)$ at time $k=0$ under the application of a constant command $g(t)$, $\forall k$. The virtual c-variable evolution (7) can be rewritten as the sum of two amounts: a steady-state component represented by $c_{g(t)}$ and the transient evolution $H^c\Phi^k(x(t)-x_{g(t)})$:

$$\hat{c}(k, x(t), g(t)) = c_{g(t)} + H^c\Phi^k(x(t)-x_{g(t)}) \quad (8)$$

Because $g(t) \in \mathcal{W}_\delta$ and, in turn, $c_{g(t)} \in \mathcal{C}_\delta$ at each time t, then, a sufficient condition to ensure that the constraints are satisfied, although in a quite arbitrary and conservative way, is to ensure that the transient component is confined into a ball of radius $\rho_{g(t))}$

$$\|H^c\Phi^k(\hat{x}(t)-x_{g(t)})\| \le \rho_{g(t)}, \forall k \ge 0 \quad (9)$$

where $\rho_{g(t)}$ represents the minimum distance between $c_{g(t)}$ and the border of $\mathcal{C}$

$$\begin{array}{rcl} \rho_g &:=& \arg\max_\rho \rho \\ && \text{subject to } \mathcal{B}^\rho(c_g) \subseteq \mathcal{C} \end{array} \quad (10)$$

where $\mathcal{B}^\rho(c_g)$ represents the ball of radius ρ centered in c_g. Details on the computation of $\rho_{g(t)}$ can be found in [2].

Then, the FF-CG design problem translates into the problem of defining an algorithm that is able to select, at each time t, a reference value $g(t)$ such that (9) holds true for all $k \ge 0$. This has been achieved in [2] by selecting a suitable integer τ, referred to as a *Generalized Settling Time*, and a sequence of positive scalars $\rho(t)$ such that the following more strict condition than (9) is satisfied at each time t.

$$\|H^c\Phi^k(x(t)-x_{g(t)})\| \le \rho(t) \le \rho_{g(t)}, \forall k \ge 0 \quad (11)$$

Then, if condition (11) were holding true at time $t-\tau$ and a certain command $g(t-\tau)$ were constantly applied to the system, then the transient contribution from t onwards could be bounded as follows

$$\| H^c\Phi^k(\hat{x}(t)-x_{g(t-\tau)}) \| \leq \gamma\rho(t-\tau) \leq \gamma\rho_{g(t-\tau)}, \quad \forall k \geq 0 \tag{12}$$

with $\gamma < 1$ because of asymptotical stability. Then, if the FF-CG action were computed every τ sampling steps and kept constant between two successive updating, at time t our goal would be that of selecting a new command $g(t)$ such that

$$\| H^c\Phi^k(\hat{x}(t)-x_{g(t)}) \| \leq \rho(t) \leq \rho_{g(t)}, \forall k \geq 0 \tag{13}$$

is satisfied for some $\rho(t) > 0$. By introducing the τ-step incremental reference $\Delta g(t) = g(t) - g(t-\tau)$, and by observing that $x_{\Delta g(t)} = x_{g(t)} - x_{g(t-\tau)}$, a sufficient condition for (13) to hold is

$$\| H^c\Phi^k x_{\Delta g(t)} \| \leq \rho_{g(t-\tau)+\Delta g(t)} - \gamma\rho(t-\tau)\}, \forall k \geq 0. \tag{14}$$

Please note that the latter inequalities always hold true for Δg in a sufficiently small ball centered in $\Delta g = 0$. Finally, by taking the definition of $x_{\Delta g}$ into account, we can formulate the Feed-Forward CG selection algorithm as follows

The FF-CG Algorithm

REPEAT AT EACH TIME t

1.1 IF ($t == \kappa\tau, \kappa = 1, 2 \ldots$)

1.1.1 SOLVE

$$g(t) = \arg\min_g \| g - r(t) \|^2_\Psi \tag{15}$$

$$\text{SUBJECT TO}: \begin{cases} g \in \mathcal{W}_\delta \\ (g - g(t-\tau)) \in \Delta\mathcal{G}(g(t-\tau)) \end{cases} \tag{16}$$

1.2 ELSE $g(t) = g(t-1)$

2.1 APPLY $g(t)$

3.1 UPDATE $\rho(t) = \gamma\rho(t-\tau) + max_{k\geq 0} \| H_c\Phi^k(I-\Phi)^{-1}G\Delta g(t) \|$.

where $\Psi > 0$ is a weighting matrix and $\Delta\mathcal{G}(g)$ is the set of all possible τ-step incremental commands Δg which ensure (14) to hold true

$$\Delta\mathcal{G}(g) := \{\Delta g : \ \|H^c\Phi^k(I-\Phi)^{-1}G\Delta g\| \leq \rho_{g+\Delta g} - \gamma\rho_g, \forall k \geq 0\} \tag{17}$$

It is worth pointing out that the sets $\mathcal{W}_\delta$, $\Delta\mathcal{G}(g)$ and the generalized settling time τ can be computed from the outset. Finally the following properties can be proved [2]

Proposition *1:* - Let assumptions **A1-A2** be fulfilled. Consider system (3) along with the **FF-CG** selection rule and let an admissible command signal $g(0) \in \mathcal{W}_\delta$ be applied at $t = 0$ such that (9) holds true. Then:

1) the minimizer in (15), computed every τ steps, uniquely exists and can be obtained by solving a convex constrained optimization problem;
2) constraints are fulfilled for all $t \in \mathbb{Z}_+$;
3) the overall system is asymptotically stable and, whenever $r(t) \equiv r$, the sequence of $g(t)$ converges in finite time either to r or to its best steady-state admissible approximation: $g(t) \to \hat{r} := \arg\min_{g\in\mathcal{W}_\delta} \|g-r\|^2_\Psi$.□

IV. Distributed Sequential FFCG (S-FFCG)

Here we summarize the distributed CG scheme of [1]. See also [6] for more details. It is assumed that the agents are connected via a communication network modeled by an undirected graph $\mathcal{G} = (\mathcal{A}, \mathcal{B})$, $\mathcal{A}$ denoting the set of N agents and $\mathcal{B} \subset \mathcal{A}\times\mathcal{A}$ the set of edges representing a direct communication link amongst the agents. Moreover, let $\mathcal{G}$ be a Hamiltonian graph and assume, without loss of generality, that the sequence $\mathcal{H} = \{1, 2, ..., N-1, N\}$ is a Hamiltonian cycle. The idea behind the approach is that only one agent at decision time is allowed to manipulate its local command signal $g_i(t)$ while all others are instructed to keep applying their previous applied commands. After each decision, the "agent in charge" will update the global command received from the previous updating agent and will forward this new value to the next updating agent in the cycle. The resulting distributed FF-CG algorithm is as follows:

Sequential-FFCG Algorithm (S-FFCG) *- Agent i*

REPEAT AT EACH TIME t

1.1 IF($t == \kappa\tau, \kappa = 0, 1, \ldots$)&&($\kappa \mod N$) == i

1.1.1 RECEIVE $g(t-\tau)$ FROM THE PREVIOUS AGENT IN THE CYCLE $\mathcal{H}$

1.1.2 SOLVE

$$g_i(t) = \arg\min_{g_i} \| g_i - r_i(t) \|^2_{\Psi_i}$$

$$\text{SUBJECT TO}: \begin{cases} g(t) = [g_1^T(t-\tau), \ldots, g_i^T, \ldots, g_N^T(t-\tau)]^T \in \mathcal{W}_\delta \\ (g_i - g_i(t-\tau)) \in \Delta\mathcal{G}_i^0(g(t-\tau)) \end{cases} \tag{18}$$

1.1.3 APPLY $g_i(t)$

1.1.4 UPDATE $g(t) = [g_1^T(t-\tau), ..., g_i^T(t), ..., g_N^T(t-\tau)]^T$

1.1.5 TRANSMIT $g(t)$ TO THE NEXT AGENT IN $\mathcal{H}$

1.2 ELSE

1.2.1 APPLY $g_i(t) = g_i(t-1)$

where $\Psi_i > 0$ is a weighting matrix, $\kappa \mod N$ is the remainder of the integer division κ/N and

$$\Delta\mathcal{G}_i^0(g) := \{\Delta g_i : [0_{m_1}^T, 0_{m_2}^T, \ldots, \Delta g_i^T, \ldots, 0_{m_N}^T]^T \in \Delta\mathcal{G}(g)\} \tag{19}$$

is the set of all possible command variations for g_i in the case that the commands of all other agents are frozen. The following definitions are in order [1].

Definition (***Admissible direction***) - Let a convex set $\mathcal{S} \subset \mathbb{R}^m$ and a point $g \in \mathcal{S}$. The vector $v \in \mathbb{R}^m$ represents an admissible direction for $g \in \mathcal{S}$ if there exists a real $\bar{\lambda} > 0$ such that $(g + \lambda v) \in \mathcal{S}, \lambda \in [0, \bar{\lambda}]$. □

Definition (***Decision Set of agent i***) - The Decision Set $\mathcal{V}_i^{\mathcal{S}}(g)$ of the agent i at a point $g \in \mathcal{S}$ represents the set of all admissible directions belonging to $\mathbb{R}_i^m$ that such an agent could move along in updating its action when all other agents held their commands unvaried, viz. $\mathcal{V}_i^{\mathcal{S}}(g) := \{d \in \mathbb{R}^{m_i} : [0_1^T, \ldots, 0_{i-1}^T, d^T, 0_{i+1}^T, \ldots, 0_N^T]^T$ *is an admissible direction for* $g \in \mathcal{S}\}$. □

Definition (***Viability property***) - A point $g \in \mathcal{S}$ is said to be "*viable*" if, for any admissible direction $v = [v_1^T, ..., v_N^T]^T \in \mathbb{R}^m$, $v_i \in \mathbb{R}^{m_i}$ with $\sum_{i=1}^N m_i = m$, at least one subvector $v_i \neq 0$ there exists such that $v_i \in \mathcal{V}_i^{\mathcal{S}}(g)$. □

Definition ***(Pareto Optimal Solution)*** - Let vectors r_i, $i = 1, 2, ..., N$ be given. Consider the following multi-objective problem:

$$\begin{array}{l}\min_g [\| g_1 - r_1 \|^2_{\Psi_1}, \ldots, \| g_i - r_i \|^2_{\Psi_i}, \ldots, \| g_N - r_N \|^2_{\Psi_N}] \\ \text{subject to } g = [g_1^T, ..., g_i^T, ..., g_N^T]^T \in \mathcal{W}_\delta \end{array} \tag{20}$$

A solution $g^* \in \mathcal{W}_\delta$ is a Pareto Optimal solution of the optimization problem (20) if there not exist $g \in \mathcal{W}_\delta$, such that: $\| g_i - r_i \|^2_{\Psi_i} \leq \| g_i^* - r_i \|^2_{\Psi_i}$ $\forall i \in \{1, \ldots, N\}$ and $\| g_j - r_j \|^2_{\Psi_j} < \| g_i^* - r_i \|^2_{\Psi_i}, j \in \mathcal{A}$. □

The above definitions are instrumental to characterize situations of deadlocks that, unlike the centralized solution, may exist in this decentralized scheme even if the same feasibility set $\mathcal{W}_\delta$ of the centralized scheme is used. The rationale is that by acting one agent at the time certain viable paths existing in the centralized scheme, when the solutions for all subsystems are computed simultaneously, are precluded and the agents could get stuck indefinitely in a certain solution. In order to clarify the situation, next Fig. 2 depicts different viable and no-viable situations for points on the border of $\mathcal{W}_\delta$. In order

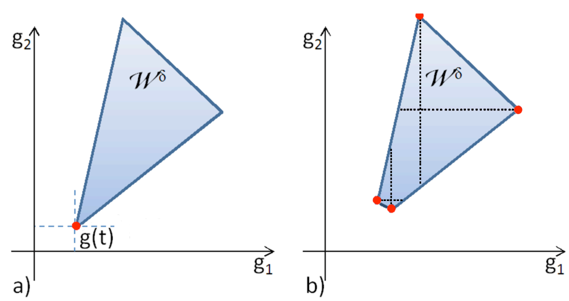

Fig. 2. Several two-agent cases. Each agent can select its command along a different axis. a) A no viable point: in this case the point $g(t)$ is not viable. In fact both agents cannot change their local command without violating boundaries; b)Viable points: in this case, for each vertex at least one of two agents can move inside $\mathcal{W}_\delta$. In fact, each admissible direction $v = [v_1, v_2]^T$ at each vertex is such that either $\lambda_1[v_1, 0]^T, \lambda_1 > 0$ or $\lambda_2[0, v_2]^T, \lambda_2 > 0$ is admissible.

to avoid this infeasible situations we have to introduce the following assumption for the points belonging to the border of $\mathcal{W}_\delta$

A3. Each point belonging to $\partial(\mathcal{W}_\delta)$ is viable, $\partial(\mathcal{W}_\delta)$ denoting the boundaries of $\mathcal{W}_\delta$.

In the next section, the characterization of viable points, a computable way of checking if the viability property **A3** is satisfied for the polyhedral set $\mathcal{W}_\delta$ at hands and a geometrical method allowing one to compute suitable inner approximations of $\mathcal{W}_\delta$ satisfying **A3** are presented, along with a proof that all internal points of $\mathcal{W}_\delta$ are viable. Finally, in [6] the following properties have been shown to hold under **A3** for the above stated S-FFCG scheme

Theorem *1:* Let assumptions **A1-A2-A3** be fulfilled. Consider system (3) as the composition of N subsystems in form (1) along with the distributed **S-FFCG** selection rule (18) and let an admissible aggregate command signal $g(0) = [g_1^T(0), \ldots, g_N^T(0)]^T \in \mathcal{W}_\delta$ be applied at $t = 0$ such that (9) holds true. Then

1) for each agent $i \in \mathcal{A}$, at each decision time $t = k\tau, k \in \mathbb{Z}_+$, the minimizer in (18) uniquely exists and can be obtained by locally solving a convex constrained optimization problem;
2) the overall system acted by the agents implementing the S-FFCG policy never violates the constraints, i.e. $c(t) \in \mathcal{C}$ for all $t \in \mathbb{Z}_+$;
3) whenever $r(t) \equiv [r_1^T, \ldots, r_N^T]^T, \forall t$, with r_i a constant set-point, the sequence of solutions $g(t) = [g_1^T(t), \ldots, g_N^T(t)]^T$ asymptotically converges to a Pareto-Optimal stationary (constant) solution of (20), which is given by r whenever $r \in \mathcal{W}_\delta$, or by any other Pareto-Optimal solution $\hat{r} \in \mathcal{W}_\delta$ otherwise. □

V. Constraints Qualification

In this section, some aspects of the viability property **A3** will be clarified and, in particular, a numerical procedure to check that this property is satisfied for all points of the boundaries of a given polyhedric set of constraints is presented. In the following analysis, we refer to a generic convex polyhedron $\mathcal{S} \subset \mathbb{R}^m$ expressed as a set of p linear inequalities $a_j^T g - b_j \leq 0, j \in \mathcal{J} := \{1, \ldots, p\}$. If all points on the boundaries of $\mathcal{S}$ are viable we say that the constraints are CD and no deadlocks can take place under the stated distributed S-FFCG policy. All proofs of the results presented in this section and in next one are omitted for space reasons, but they can be found in [6].

Lemma *1:* Let the polyhedral $\mathcal{S} \subset \mathbb{R}^m$ be given and expressed as $Ag \leq b, A \in \mathbb{R}^{|\mathcal{J}| \times m}, b \in \mathbb{R}^{|\mathcal{J}|}$. Consider also a generic point $g' \in \mathcal{S}$ and the set $\mathcal{J}' := \{j \in \mathcal{J} : a_j^T g' = b_j\}$. Then, g' is viable iff the following test fails.

Test - Find, if there exists, a vector $w = [w_1^T, ..., w_N^T]^T \in \mathbb{R}^m$ such that

$$\begin{cases} A(g' + w) \leq b \\ a_j^T [0_1^T, ..., w_i^T, ..., 0_N^T]^T > 0, & \text{for all } i \in \mathcal{A}, \text{ for at least one } j \in \mathcal{J}' \end{cases} \tag{21}$$

where a_j^T and b_j denote the rows of the matrix A and, respectively, vector b. □

Because the direct application of the above conditions to check the viability of all points of $\partial(\mathcal{S})$ would give rise to a cumbersome numerical procedure, hereafter several geometrical results are presented to shown that the viability of all points on the boundaries of $\mathcal{S}$, hereafter denoted as $\partial(\mathcal{S})$, can be established by only checking a finite number of points on $\partial(\mathcal{S})$. To this end, the usual notion of *face* of a polyhedron is recalled.

Definition (***Face of a polyhedron***) - Let a convex polyhedron $\mathcal{S} \subset \mathbb{R}^m$ expressed as a set of p linear inequalities $a_j^T g - b_j \leq 0, j \in \mathcal{J} := \{1, \ldots, p\}$ be given. Any region $\mathcal{P} := \{g \in \mathcal{S} : a_j^T g - b_j = 0, j \in \mathcal{J}' \subset \mathcal{J}\}$ is said to be a *face* of $\mathcal{S}$. Moreover, the quantity $m - |\mathcal{J}'|$ represents the *order* of $\mathcal{P}$. □

Based on the above definition, vertices are 0-order faces, facets are $(m-1)$-order faces, ridges are $(m-2)$-order faces and so on.

Lemma *2:* Let a convex polyhedron $\mathcal{S}$ and a $(m-k)$-order face $\mathcal{P}$ with $k \in \{1, ..., m-1\}$ be given. Then, $v \in \mathbb{R}^m$

is an admissible direction for all points of the interior of $\mathcal{P}$, say it $In(\mathcal{P})$, if it is an admissible direction for at a least one point g of $In(\mathcal{P})$. □

Based on the above Lemma, the following result which allows one to verify in an easy way the CD of $\mathcal{S}$ can be stated.

Lemma *3:* Let a convex polyhedron $\mathcal{S}$ be given and $\mathcal{P}_{\mathcal{S}}$ denote the set of all faces of $\mathcal{S}$. Then, each point of $\mathcal{S}$ is viable, that is the constraint set $\mathcal{S}$ is CD, if for each element $\mathcal{P} \in \mathcal{P}_{\mathcal{S}}$ there exists at least a viable point $g \in In(\mathcal{P})$. □

The above results allow one to introduce the following numerical procedure to check the viability all points belonging to the boundaries of a polyhedron $\mathcal{S}$.

Constraints Qualification test for polyhedrons $\mathcal{S}$

1.1 COMPUTE $\mathcal{P}_{\mathcal{S}}$
1.2 SET $Pl := \emptyset$
1.3 FOR EACH $\mathcal{P} \in \mathcal{P}_{\mathcal{S}}$
  1.3.1 SELECT $g \in In(\mathcal{P})$
  1.3.3 APPEND g TO Pl
1.4 SET $check := viable$
1.5 FOR EACH $g \in Pl$
  1.5.1 PERFORM ***Test***
  1.5.2 IF ***Test*** FAILS
    1.5.2.1 SET $check = notviable$
    1.5.2.1 BREAK
1.6 RETURN $check$

VI. Viable approximations

In this section we describe a method to find arbitrarily accurate viable multi-box inner-approximations of a no CD polyhedron in the case that all agents have mono-dimensional decision sets, viz. $m_i = 1, \forall i \in \mathcal{A}$ and $\mathcal{A} := \{1, ..., m\}$. To this end, the notion of *box* is recalled.

Definition (*Box in* $\mathbb{R}^m$) A box is a convex polytope with all the hyperplanes characterizing its boundaries parallel to the axes. More formally a box $\Omega(l, u)$ is defined as

$$\Omega(l, u) := x \in \mathbb{R}^m : l \le x \le u, \tag{22}$$

where l and u are real vectors of $\mathbb{R}^m$ and the inequalities hold componentwise. □

Consider, for a no viable polyhedron $\mathcal{S}$, a multi-box inner approximation $\mathcal{M}(\mathcal{S}) \subset \mathcal{S}$. That is, according to [5], a collection of full-dimensional boxes such that

1) the intersection between any two boxes is not full-dimensional;
2) the union of all boxes in $\mathcal{M}(\mathcal{S})$ is contained in $\mathcal{S}$;

The numerical method described in [5] can be used to find multi-box inner approximations $\mathcal{M}(\mathcal{S})$ of $\mathcal{S}$. We will show that for such a kind of approximation the convex hull of $\mathcal{M}(\mathcal{S})$, say it $\mathcal{S}' := \text{co}\{\mathcal{M}(\mathcal{S})\}$, is always CD. It is clear from the above discussion, that each vertex of $\mathcal{S}'$ is a vertex of a box contained in $\mathcal{M}(\mathcal{S})$. By exploiting this fact, the following preliminary results can be stated

Lemma *4:* Let $g' \in \mathbb{R}^m$ a point of $\partial(\mathcal{S}')$ such that $g' \in \partial(\mathcal{M}(\mathcal{S}))$. Then, m scalars $v_i \in \{-1, 1\}, i = 1, ..., m$, there exist such that, for any $\bar{\lambda} > 0$, $g' + [0_1, \ldots, \lambda v_i, \ldots, 0_m]^T \in \mathcal{S}', \forall \lambda \in [0, \bar{\lambda}], \forall i \in \mathcal{A}$. □

Lemma *5:* Let $g' \in \mathbb{R}^m$ and $g'' \in \mathbb{R}^m$ be two points of $\mathcal{S}'$ such that $g' + [0_1, \ldots, \lambda v_i', \ldots, 0_m]^T \in \mathcal{S}'$ and $g'' + [0_1, \ldots, \lambda v_i'', \ldots, 0_m]^T \in \mathcal{S}', \lambda \in [0, \bar{\lambda}_i'], v_i \in \{-1, 1\}, \forall i \in \mathcal{A}$. Then, for each point belonging to the convex combination $g = \gamma g' + (1 - \gamma) g''$ there exists, for each $i \in \mathcal{A}$, a pair $(\hat{\lambda}_i, v_i)$, with $\hat{\lambda}_i > 0$ and $v_i \in \{-1, 1\}$, such that $g + [0, \ldots, \lambda v_i, \ldots, 0]^T \in \mathcal{S}', \forall \lambda \in [0, \hat{\lambda}_i], \forall i \in \mathcal{A}$. □

Lemma *6:* For each point g of the border of $\mathcal{S}'$ there exist m admissible directions aligned to the axes, i.e. $g + [0_1, \ldots, \lambda v_i, \ldots, 0_m]^T \in \mathcal{S}', \lambda \in [0, \bar{\lambda}], v_i \in \{-1, 1\}, \forall i \in \mathcal{A}, \forall g \in \partial(\mathcal{S}')$. □

Lemma *7:* Let $\mathcal{S}'$ be expressed as the intersection of $|\mathcal{J}|$ inequalities $Ag \le b$. Then, for all $g \in \partial(\mathcal{S}')$ and for all w such that $A(g + w) \le b$ the following condition is satisfied

$$A(g + [0_1, ..., w_i, ..., 0_m]^T) \le b, \tag{23}$$

for at least an index $i \in \mathcal{A}$. □

Finally the viability property of each point $g \in \partial(\mathcal{S}')$ is ensured by the next lemma.

Lemma *8:* For a given $g \in \partial(\mathcal{S}')$, let $\mathcal{V}_i^{\mathcal{S}'}(g)$ be the decision set for agent i acting at g. Then, g is viable. □

In conclusion, a no CD polyhedron $\mathcal{W}_\delta$ can be always approximated with a CD polyhedron $\mathcal{W}_\delta'$ and the S-FFCG problem recast as follows

$$\begin{array}{l} g_i(t) = \arg\min_{g_i} \| g_i - r_i(t) \|^2_{\Psi_i} \\ \text{subject to :} \\ \begin{cases} g(t) = [g_1^T(t-\tau), \ldots, g_i^T, \ldots, g_N^T(t-\tau)]^T \in \mathcal{W}_\delta' \\ (g_i - g_i(t-\tau)) \in \Delta\mathcal{G}_i^0(g(t-\tau)) \end{cases} \end{array} \tag{24}$$

where the set $\mathcal{W}_\delta'$ is used in the place of $\mathcal{W}_\delta$. Each set $\Delta\mathcal{G}_i^0$ is not subject to modification because it represents a local constraint in the optimization problem (24). Then, its fulfillment does not depend on the global command vector g.

Remark - 1 The Constraints Qualification of multi-box approximations $\mathcal{S}'$ has been proved only for mono-dimensional decision set cases, the multi-box inner approximation method here proposed may represent a heuristic able to compute a CD polyhedron also in the general multi-dimensional decision set case. Actually, no formal proofs are available but in all performed numerical experiments the generated multi-box inner-approximations of no CD polyhedrons always gave rise CD approximations. Then, we conjecture that above approximation method can be extended to the general case without modification. The investigation is ongoing. □

VII. Illustrative Example

In this section a short example is presented in order to show the effectiveness of proposed method. The two-dimensional polytopic constraint set $\mathcal{S}$ of Figure 3 is considered. It is characterized by the following five inequalities

$$Ag \le b$$

where

$$A = \begin{pmatrix} -0.2693 & 0.9630 \\ 0.3288 & -0.9444 \\ -0.9874 & 0.1584 \\ 0.9877 & -0.1563 \\ 0.9837 & 0.1797 \end{pmatrix}, b = \begin{pmatrix} 6.1218 \\ -0.0681 \\ -1.7645 \\ 8.4683 \\ 10.7936 \end{pmatrix}$$

The CQ test presented in section V has been performed on it and the answer was achieved in 0.04 seconds by means of the

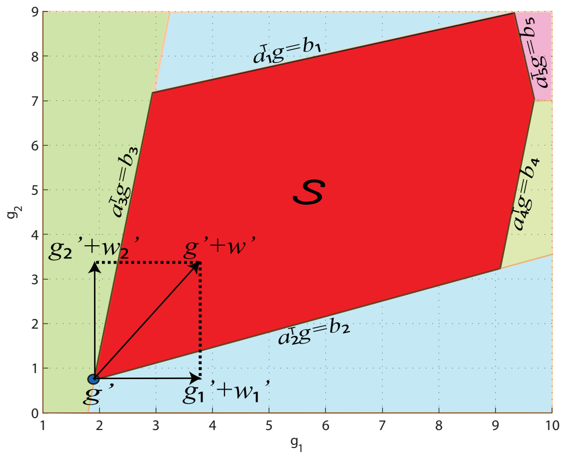

Fig. 3. The polytope $\mathcal{S}$. At g' $\mathcal{J}' = \{2, 3\} \subset \{1, 2, ..., 5\}$. Notice that the point g' is not viable because the admissible vector w' is such that $a_3^T [w_1', 0]^T > 0$ and $a_2^T [0, w_2']^T > 0$.

Multi-Parametric Toolbox (MPT) (please see [7] for details) with MATLAB 2009b® installed on a Intel Core$^{\text{TM}}$2® Quad machine. The polytope $\mathcal{S}$ resulted no CD because of the presence of the vertex $g' = [1.908, 0.7355]^T$, which is not viable as illustrated in Figure 3. Then, $\mathcal{S}$ has been inner-approximated by 91 boxes (Figure 4) by using the algorithm presented in [5] which, on the same machine, took 37 seconds to terminate its execution. The resulting CD polytope $\mathcal{S}'$, that doesn't contain g' in its convex hull, is depicted in Figures 5-6.

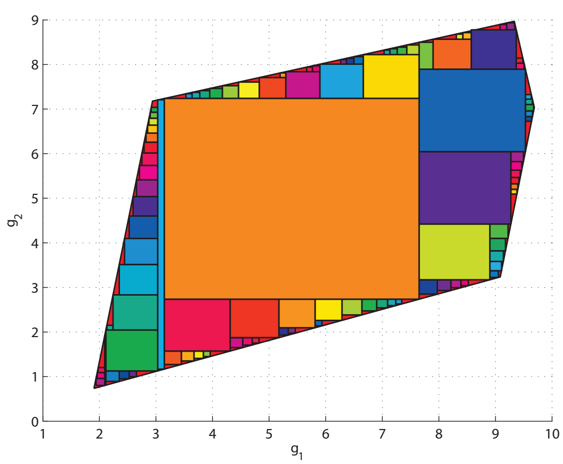

Fig. 4. Multi-inner box approximation $\mathcal{M}(\mathcal{S}) = \{\Omega_k\}_{k=1}^{91}$ of $\mathcal{S}$.

VIII. Conclusions

In this paper the liveliness properties of the distributed sequential supervisory scheme presented in [1] has been analyzed. First, the sequential distributed strategy has been recalled and a complete characterization of all possible deadlock situations discussed. Then, a numerical test on the constraint set has been proposed for ensuring the liveliness

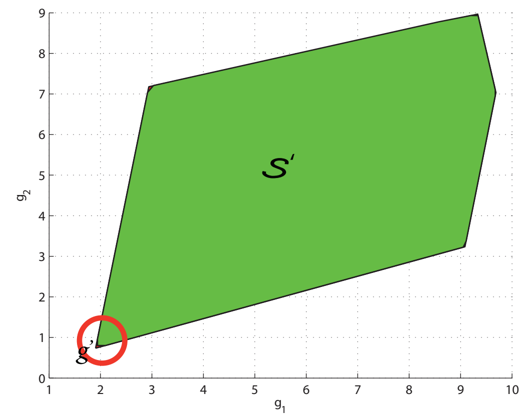

Fig. 5. Convex Hull of $\mathcal{M}(\mathcal{S})$.

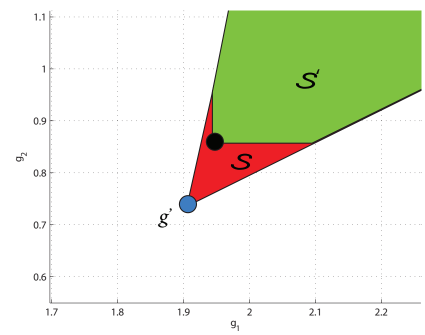

Fig. 6. Zoom on the no viable point g': the point g' (blue) is not inside $\mathcal{S}'$, having been substituted by a new viable vertex (black spot).

of the strategy. Finally, a procedure to achieve *Constraints Qualified* arbitrarily accurate inner approximations of the original constraint set has been presented for the case of mono-dimensional decision sets. The generalization of such a procedure to the multi-dimensional case represents a future step of this research work.

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