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Adaptive Super-Twist Control with Minimal Chattering Effect

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***Abstract*— The, so-called, adaptive super-twist controller is considered here. It contains the adaptive gain-parameter which adjusts the level of a scalar control action on-line based on direct measurements the "equivalent control" obtained by a first-order low-pass filter. It is shown that the problems of the finite time convergence and keeping a small chattering amplitude can be handled simultaneously if the gain magnitude is reduced to a minimal admissible level defined by the conditions for sliding mode to exist. The suggested methodology is compared numerically with another schemes of a gain-adaptation (including σ-adaptation) showing a high effectiveness under a significant level of uncertainties and external disturbances.**

I. INTRODUCTION

A. Brief survey

The basic idea of *Adaptive Control Approach* consists in designing the systems exhibiting the same dynamic properties under uncertainty conditions based on utilization of current information. It involves modifying the control law used by a controller to cope with the fact that the parameters of the system being controlled are slowly time-varying or uncertain. Even more, adaptive control implies improving dynamic characteristics while properties of a controlled plant or environment are varying [1], [20]. Without adaptation the original *Sliding Mode Control* (SMC) demonstrate *robustness* properties with respect to parameter variations and disturbances [22]. The first attempts to apply ideas of adaptation in SMC were made in the 60's [6], [7] and [8]: the control efficiency was improved by changing the position or equation of the discontinuity surfaces without any information on a plant parameters. The design idea might be formulated as follows: if sliding mode exists, then the coefficients of switching plane can be varied to improve the system dynamics. However those early publications did not take in to account the main obstacle for SMC application - the *chattering* phenomenon which is inherent in sliding motions (see, for example, [2], [3] and [4]). The phenomenon is well-known from literature on power converters and referred to as "*ripple*" [16]. Then the efforts of the researchers were oriented to the application of *adaptivity principles* to reduce the effect of chattering. Since the amplitude of chattering is proportional to discontinuity magnitude in control, one of possible adaptation methods is related to reducing this magnitude to the minimum admissible value dictated by the conditions for SM to exist. So, in [17] the control gain depended on the distance of system state to a discontinuity surface. The tracks of adaptivity can be found in the first publications about variable structure systems with SM (see [22], [23]) with the control gain proportional the system state. Similar ideas were developed later in [11] with different algorithms of tuning a control gain. There exist also adaptive SMC (ASMC) algorithms that allow adjusting dynamically the control gains without knowledge of uncertainties/perturbations bounds. In particular, several adaptive fuzzy SMC algorithms were proposed. However, they do not guarantee the tracking performance (see [15], [19] or overestimate the switching control gains as in [10]). Of course, another efficient tool to suppress chattering is the application of state observers [5], but for this method the plant parameters are assumed to be known.

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B. Motivation and the design idea

In [17] and [21] the adaptation process with the varying magnitude of the control gain terminates at the moment when the sliding mode starts. In [11] the authors tried to continue the adaptation process during sliding mode estimating equivalent control. However, none of the above algorithms resulted in minimum possible value of the discontinuous control. Finding the solution of this problem under uncertainty conditions is the *objective* of this paper. This leads to the minimization of chattering effect.

C. Primitive example

We start with a simple example. It is evident that for the first-order system

$$\begin{array}{c} \dot{x}(t) = a + u \\ u = -k\,\mathrm{sign}(x(t)),\ k > 0 \end{array} \quad (1)$$

with known range only $0 < |a| \leq a_+$ of a constant parameter a. If the value of a is unknown, the magnitude of control is selected such that sliding mode exists for the all values of unknown parameter $k > a_+$. However if parameter a is varying, the gain k can be decreased and as a result chattering amplitude can be reduced. The objective of adaptation is decreasing k to the minimal value preserving sliding mode, if parameter a is unknown.

If the condition $k > a_+$ holds, then sliding mode with $x(t) \equiv 0$ occurs and control in (1) should be replaced by the, so-called, *equivalent control* [22] u_{eq} for which the right-hand side in (1) is equal to zero, namely,

$$\dot{x}(t) = 0 = a + u_{eq} \quad (2)$$

that leads to

$$k\left[\mathrm{sign}(x(t))\right]_{eq} = a \quad (3)$$

If $k < a$, the set $x(t) \equiv 0$ is of zero measure in time and can be disregarded. The function $[\mathsf{sign}(x(t))]_{eq}$ is an average value, or a slow component of discontinuous function $\mathsf{sign}(x(t))$ switching at high frequency and can be easily obtained by a low pass filter filtering out the high frequency component [22]. Of course, the average value is in the range $(-1, 1)$. The design idea of adaptation looks transparent: *after sliding mode occurs the control parameter* $[\mathsf{sign}(x(t))]_{eq}$ *should be decreased until becomes close to 1*. Further decreasing will lead to ceasing sliding mode. As a result, the minimal possible value of discontinuity magnitude is found for the current value of parameter a to reduce the amplitude of chattering. For that purpose select the *adaptation algorithm* in the form

$$\begin{gathered}\dot{k}(t) = k(t)\mathsf{sign}(\delta(t)) - M[k(t) - k^+]_+ + M[\mu - k(t)]_+ \\ \delta(t) := \left|[\mathsf{sign}(x(t))]_{eq}\right| - \alpha,\ \alpha \in (0,1),\ M > k^+ > a^+ \\ [z]_+ := \begin{cases} z & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases}\end{gathered} \tag{4}$$

The gain k can vary in the range $[\mu, k^+]$, μ is a preselected minimal value of k. For the adaptation algorithm (4) sliding mode will occur after a finite time interval. Indeed, if it does not exist, then

$$\left|[\mathsf{sign}(x(t))]_{eq}\right| = 1$$

that leads to $\delta > 0$, and the increasing gain $k(t)$ will reach the value k^+ which is sufficient for enforcing sliding mode for any value of parameter a.

Show that in sliding mode the adaptation process (4) is over $\delta(t) = 0$ after a finite time t_f. To do that calculate the time derivative of the Lyapunov function $V(\delta) = \delta^2/2$. First, assume that during the adaptation process $k(t) \in [\mu, k^+]$ which means that $|a|/\alpha > \mu$. Then

$$\begin{gathered}\dot{V}(\delta(t)) = \delta(t)\dot{\delta}(t) = \delta(t)\frac{d}{dt}\left|[\mathsf{sign}(x(t))]_{eq}\right| = \\ \delta(t)\frac{d}{dt}(|a|/k) = -|a|\delta(t)k^{-2}\dot{k} = \\ -|a|\delta(t)k^{-1}\mathsf{sign}\left(\delta(t) - M[k(t) - k^+]_+ + M[\mu - k(t)]_+\right) \\ = -|a|\delta(t)k^{-1}\mathsf{sign}(\delta(t)) = -|a|k^{-1}|\delta(t)| \\ \leq -\sqrt{2}\frac{|a|}{k^+}\sqrt{V(\delta(t))}\end{gathered} \tag{5}$$

It is evident from the solution

$$0 \leq \sqrt{V(\delta(t))} \leq \sqrt{V(\delta(0))} - \frac{|a|}{\sqrt{2}k^+}t$$

of the differential inequality (5) that $\sqrt{V(\delta(t))} = 0$ at least after

$$t_f = \frac{k^+}{|a|}\sqrt{2V(\delta(0))} = \frac{k^+}{|a|}|\delta(0)|$$

and, as a result, $\delta(t)$ becomes equal to zero identically after the finite time t_f.

After the adaptation process is over ($t > t_f$) we have

$$\left|[\mathsf{sign}(x(t))]_{eq}\right| = \frac{|a|}{k} = \alpha$$

so, $k = |a|/\alpha$. If $|a|/\alpha < \mu$, then the gain k decreases until $k = \mu$ and then, as it follows from (4), it is maintained at this level.

Remark 1: The function $[\mathsf{sign}(x(t))]_{eq}$ is needed here for the implementation of the adaptation algorithm (4). As it was mentioned above, it can be derived by filtering out a high frequency component of the discontinuous function $\mathsf{sign}(x(t))$ by a low pass filter

$$\tau\dot{z} + z = \mathsf{sign}(x(t)),\ z(0) = 0$$

with a small time constant $\tau > 0$ and the output $z(t)$ which is, in fact, an estimate of $[\mathsf{sign}(x(t))]_{eq}$ satisfying

$$\left|z(t) - [\mathsf{sign}(x(t))]_{eq}\right| \leq O(\tau) \underset{\tau\to 0}{\to} 0$$

Then the convergence analysis of (4)-(5) with $\delta(t) = z(t) - \alpha$ is valid beyond the domain $|\delta(t)| \leq O(\tau)$. This inequality defines the accuracy of adaptation. The switching frequencies of the modern power converters are of order dozens of kHz, and very small time constant τ can be selected to get a high accuracy of adaptation.

Remark 2: The adaptation algorithms

$$\dot{k}(t) = \rho k(t)\mathsf{sign}(\delta(t)) - M\left[k(t) - k^+\right]_+ + M[\mu - k(t)]_+$$

with $\rho > 0$ works as well, if the parameter a is time varying ($a = a(t)$) with bounded time derivative, namely, $\left|\frac{d}{dt}|a(t)|\right| \leq d_a = \mathsf{const}$. Indeed, again assuming that $k(t) \in [\mu, k^+]$ or, equivalently, $a/\alpha > \mu$, we have

$$\begin{gathered}\dot{V}(\delta(t)) = \delta(t)\dot{\delta}(t) = \delta(t)\frac{d}{dt}\left|[\mathsf{sign}(x(t))]_{eq}\right| = \\ \delta(t)\frac{d}{dt}(|a|/k) = -|a|\delta(t)k^{-2}\dot{k} + \delta(t)k^{-1}\frac{d}{dt}|a(t)| \leq \\ -|a|\rho\delta(t)k^{-1}\mathsf{sign}(\delta(t)) + |\delta(t)|k^{-1}d_a \leq \\ -|\delta(t)||a|\rho k^{-1} + |\delta(t)|d_a\mu^{-1} \leq -\rho|\delta(t)|\mu\alpha/k^+ \\ + |\delta(t)|d_a\mu^{-1} = -|\delta(t)|(\rho\mu\alpha/k^+ - d_a/\mu) = \\ -\rho_0\sqrt{2V(\delta(t))},\ \rho_0 = \rho\mu\alpha/k^+ - d_a/\mu\end{gathered}$$

If $\rho > d_a k^+/\left(\alpha\mu^2\right)$ it follows that $\rho_0 > 0$, and similarly to the analysis for $a = \mathsf{const}$ it can be shown that $\delta(t) \equiv 0$ after the finite time $t_f = \frac{\rho_0}{\sqrt{2}}|\delta(0)|$. If in the course of the motion the gain $k(t)$ decreases and becomes equal to μ, then it will be maintained at this level. Since the gain $a(t)$ is time varying its increase can result in $|a|/\mu = \alpha$ and $\delta = 0$ at some time t_μ. As it follows from the above analysis, for the further motion in the domain $k(t) \in (\mu, k^+]$ with the initial condition $\delta(t_\mu) = 0$ the time function $\delta(t)$ will be equal to zero identically with $|a|/k = \alpha$.

Certainly, we have described the idea only. The generalization of the adaptive procedure (4) for the, so-called, *super-twist controller* (with a high-order sliding mode) constitutes the main result of this paper.

II. Main properties of the standard Super Twist without adaptation

Consider the simple two dimensional nonlinear system containing discontinuous nolinearities in the right-hand sides of both components of the corresponding ODE:

$$\begin{cases} \dot{x}(t) = y(t) - \bar{\alpha}\sqrt{|x(t)|}\mathsf{sign}(x(t)) \\ \dot{y}(t) = \phi(t) + u(t) \\ u(t) := -\bar{\beta}\mathsf{sign}(x(t)) \end{cases} \tag{6}$$

referred below to as the "*super-twist" controller* [12], [13] and [14]. In (6) there is supposed that

$$\bar{\alpha} > 0 \text{ and } |\phi(t)| \leq \phi_0 < \bar{\beta} \tag{7}$$

a) The both state variable are sign-varying, therefore the initial conditions can be selected as follows

$$x(0) = -x_0,\, x_0 > 0,\, y(0) = 0$$

In our case $\dot{x}(0) > 0$ and, hence, $x(t)$ is increasing. Denote

$$t_1^* := \inf\{t > 0 : x(t_1^*) = 0,\, x(t) < 0 \text{ for } t \in [0, t_1^*)\}$$

Next, compare two ODE's:

$$\dot{x}(t) = y(t) - \bar{\alpha}\frac{x(t)}{\sqrt{|x(t)|}},\, x(0) = -x_0 < 0 \tag{8}$$

where

$$y(t) = \int\limits_{\tau=0}^{t} \left[\bar{\beta} + \phi(\tau)\right] d\tau$$

satisfying

$$\begin{aligned} y(t) \geq \int\limits_{\tau=0}^{t} \left[\bar{\beta} - \phi_0\right] d\tau = m \cdot t,\ m := \bar{\beta} - \phi_0 > 0 \\ y(t) \leq \int\limits_{\tau=0}^{t} \left[\bar{\beta} + \phi_0\right] d\tau = Mt,\ M := \bar{\beta} + \phi_0 \end{aligned} \tag{9}$$

and

$$\dot{z}(t) = m \cdot t - \bar{\alpha}\frac{z(t)}{\sqrt{x_0}},\, z(0) = x(0) = -x_0 < 0 \tag{10}$$

Obviously that the ODE (8) is equivalent to (6). Since $|x(t)|$ is decreasing it follows that

$$\frac{1}{\sqrt{|x(t)|}} > \frac{1}{\sqrt{x_0}} := k_0 \text{ for } t > 0$$

For any $t \in [0, t_1^*)$ we have $x(t) < 0$ and $z(t) < 0$ which implies $\dot{x}(t) > \dot{z}(t)$ and, as the result, $x(t) > z(t)$. So that

$$t_1^* < t' := \inf\{t > 0 : z(t) = 0\}$$

The solution to (10) is

$$z(t) = \frac{m}{\bar{\alpha}k_0}\left(t - \frac{1}{\bar{\alpha}k_0}\right) + \left[\frac{m}{(\bar{\alpha}k_0)^2} - x_0\right] e^{-\bar{\alpha}k_0 t}$$

and, in view of this, one can conclude that

$$t' = (k_0)^{-1} O\left(\frac{1}{\bar{\alpha}}\right) = \sqrt{x_0} O\left(\frac{1}{\alpha\bar{\alpha}}\right) < t_1^*$$

and by (9) it follows that $y(t) \leq Mt$ and hence

$$y(t_1^*) \leq Mt_1^* \leq Mt' = M\sqrt{x_0} O\left(\frac{1}{\bar{\alpha}}\right)$$

b) For $t > t_1^*$ we already have that $x(t) > 0$ and $y(t)$ is a decaying function since

$$\dot{y}(t) = \phi(t) - \bar{\beta}\mathsf{sign}(x(t)) = \phi(t) - \bar{\beta} \leq 0$$

that implies

$$y(t) \leq y(t_1^*) - m \cdot t \tag{11}$$

For the instant t_1^{**}

$$t_1^{**} := \inf\{t > t_1^* : y(t) = 0\}$$

we have

$$\begin{aligned} t_1^{**} \leq t_1^* + y(t_1^*)/m = t_1^* + \frac{M}{m}\sqrt{x_0} O\left(\frac{1}{\bar{\alpha}}\right) \\ = \left(1 + \frac{M}{m}\right)\sqrt{x_0} O\left(\frac{1}{\bar{\alpha}}\right) \end{aligned} \tag{12}$$

So, by (6) and (11)

$$\begin{aligned} x(t_1^{**}) = \int\limits_{\tau=t_1^*}^{t_1^{**}} \left[y(\tau) - \bar{\alpha}\sqrt{|x(\tau)|}\mathsf{sign}(x(\tau))\right] d\tau \\ = \int\limits_{\tau=t_1^*}^{t_1^{**}} \left[y(\tau) - \bar{\alpha}\sqrt{|x(\tau)|}\right] d\tau \leq \int\limits_{\tau=t_1^*}^{t_1^{**}} y(t_1^*) d\tau = \\ y(t_1^*)\left(t_1^{**} - t_1^*\right) \leq y^2(t_1^*)/m \leq \gamma x_0 \end{aligned} \tag{13}$$

where

$$\gamma := \frac{M^2}{m}\left[O\left(\frac{1}{\bar{\alpha}}\right)\right]^2 \tag{14}$$

Selecting α large enough we may conclude that $\gamma \in (0, 1)$ and

$$x(t_1^{**}) \leq \gamma x_0$$

Here $x(t_1^{**})$ is an initial value of (6) for the second interval $\Delta t_2 := t_2^{**} - t_1^{**}$ where

$$t_2^{**} := \inf\{t > t_1^{**} : y(t) = 0\}$$

Similarly to (13)

$$|x(t_2^{**})| \leq \gamma x(t_1^{**}) \leq \gamma^2 x_0 \tag{15}$$

c) Iterating this process we may conclude that

$$|x(t_i^{**})| \leq \gamma \left|x(t_{i-1}^{**})\right| \leq \cdots \leq \gamma^i x_0 \tag{16}$$

and

$$\begin{aligned} \Delta t_i := t_i^{**} - t_{i-1}^{**} \leq \sqrt{\left|x(t_{i-1}^{**})\right|}\left(1 + \frac{M}{m}\right) O\left(\frac{1}{\bar{\alpha}}\right) \\ \leq \gamma^{i/2}\sqrt{x_0}\left(1 + \frac{M}{m}\right) O\left(\frac{1}{\bar{\alpha}}\right) \end{aligned} \tag{17}$$

Last two inequalities permit to formulate the following result.

Proposition 1: If $|\phi(t)| \leq \phi_0 < \beta$, then for any initial value $x(0)$ from bounded domain there exist large enough $\bar{\alpha} > 0$, such that the super-twist procedure (6) has a finite time convergence or reaching time proceeding , the following properties holds:

1) for large enough $\bar{\alpha}$ we may guarantee that

$$\gamma := \frac{M^2}{m}\left[O\left(\frac{1}{\bar{\alpha}}\right)\right]^2 < 1$$

2) as the result,

$$|x(t)| \simeq O\left(\gamma^{t/2}\right) \underset{t\to\infty}{\to} 0$$

Proposition 2: The reaching time

$$t_{reach} := \inf_{\bar{t}\geq 0}\{\bar{t} : x(t) = 0 \text{ for all } t \geq \bar{t}\}$$

is estimated by

$$t_{reach} \leq \sum_{i=1}^{\infty}\Delta t_i \leq \sqrt{x_0}\left(1+\frac{M}{m}\right)O\left(\frac{1}{\bar{\alpha}}\right)\sum_{i=1}^{\infty}\gamma^{i/2} \leq \frac{\sqrt{\gamma}}{1-\sqrt{\gamma}}\sqrt{x_0}\left(1+\frac{M}{m}\right)O\left(\frac{1}{\bar{\alpha}}\right) \tag{18}$$

The important comments can be done:

- the reaching time tends to zero with gain $\bar{\alpha} \to \infty$;
- a finite-time convergence takes place for any small $m := \bar{\beta} - \phi_0 > 0$;
- the sufficient convergence conditions, derived in previous publications(see, for example, [13], [18]) led to upper estimate of admissible disturbance less than $0.5\bar{\beta}$. Note that the system is not even asymptotically stable for $L \geq \bar{\beta}$. As it follows from (6) in this case $y(t)$ is constant or diverging, if the disturbance ϕ is such that $|\phi(t)| \geq \bar{\beta}$, and has sign opposite to control $u(t)$.
- the upper bound (18) for the reaching time t_{reach} is proportional to the root of the initial state, namely, $\sqrt{|x_0|}$ and inverse-proportinal to the parameter $\bar{\alpha}$, i.e., $O\left(\frac{1}{\bar{\alpha}}\right)$.

A. Numerical simulations of the super-twist control without adaptation

The figure 1 illustrates the dynamics of the super-twist controller with the following parameters:

$$\bar{\alpha} = 4,\ \bar{\beta} = 1,\ \phi_0 = 0.1 \text{ and } x(0) = \begin{bmatrix} 0.1 & -0.2 \end{bmatrix}^{\intercal}$$

One can see a finite-time convergence to zero (approximately in 0.3 sec.) of the first state variable $x(t)$ and the corresponding discontinuous control of the amplitude $\bar{\beta} = 1$.

III. Super-twist control with adaptation

Denoting

$$x_1 = x,\quad x_2 = y,\ k := \bar{\beta}$$

the system (6) can be represented as

$$\begin{cases} \dot{x}_1 = x_2 - \bar{\alpha}\sqrt{|x_1|}\mathsf{sign}(x_1) \\ \dot{x}_2 = \phi(t) + u \\ u := -k\mathsf{sign}(x_1) \end{cases}$$

or, in the vector format

$$\dot{x} = f(t,x) + b(t,x)u$$

with

$$f(t,x) := \begin{pmatrix} x_2 - \bar{\alpha}\sqrt{|x_1|}\mathsf{sign}(x_1) \\ \phi(t) \end{pmatrix},\ b(t,x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Taking

$$\sigma(x) = x_1$$

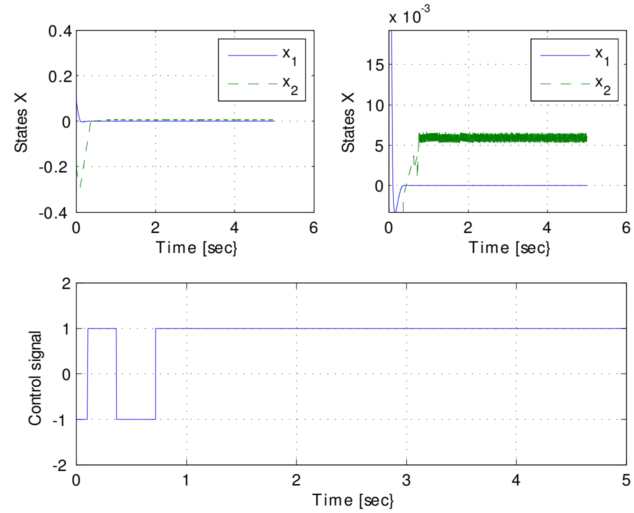

Fig. 1. The states and the control signal for the super-twist controller without adaptation of the gain parameter β.

and permitting for the gain parameter to be time-varying, i.e., $k(t) = \beta(t)$, we may apply the following adaptation procedure:

$$\dot{k}(t) = \begin{cases} [\gamma_0 + \gamma_1\|x\|]\,k(t)\mathsf{sign}(\delta(t)) \\ \quad \text{if } 0 < \mu \leq k(t) \leq k^+ \\ \\ 0 \text{ otherwise} \end{cases} \tag{19}$$

where[1]

$$\delta(t) := z(t)/k(t) - \alpha,\ \alpha \in (0,1),\lambda > 0$$

$$z(t) := \begin{cases} k(t)\left|[\mathsf{sign}(\sigma(x))]_{eq}\right| \\ \text{if } \left|k(t)\,[\mathsf{sign}(\sigma(x))]_{eq}\right| \geq \mu > 0 \\ \\ \mu \text{ otherwise} \end{cases} \tag{20}$$

A. The $\sigma-$adaptation

Here, following to [17], we apply the adaptation law given by

$$u(t) = -k(t)\mathsf{sign}(x_1(t))$$

$$\dot{k}(t) = \begin{cases} k(t)\,|\sigma(x(t))|\,\mathsf{sign}(|\sigma(x(t)| - \varepsilon) & \text{if } k(t) > \bar{\mu} \\ \bar{\mu} & \text{if } k(t) \leq \bar{\mu} \end{cases} \tag{21}$$

referred to as "$\sigma-$*adaptation*". In (21) $k(0) = 1$, $\varepsilon = 0.01$ and $\bar{\mu} = 0.001$. The specific feature of this procedure is that the adaptation process practically stops after the reaching time t_{reach} when $\sigma(x(t)) = x_1(t) = 0$ for any $t \geq t_{reach}$, and, as the result, the gain parameter $k(t) = \beta(t)$, defining the size of the discontinuous control (or a chattering amplitude) may be still too far from the disturbance level $|\phi(t)| \leq \phi_0$ which is minimal possible one guarantying the finite-time convergence. This effect is clearly seen in the figure 2: the reaching time $t_{reach} \simeq 0.2\,$sec., but the

[1]The condition $\dot{k}(t) = 0$ and $z(t) = \mu$ can be fulfilled by addition the discontinuous terms as in (4).

gain parameter (the chattering amplitude) remains around the initial level 1.0 which is too high comparing with $\phi_0 = 0.1$. So, the adaptation period is to short to decrease significantly the gain parameter $k(t) = \bar{\beta}(t)$, and in sliding mode regime there is no adaptation.

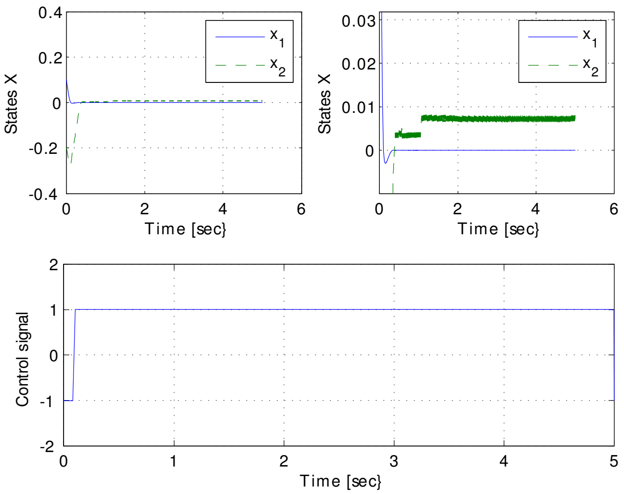

Fig. 2. The states and the control signal for the super-twist controller with σ-adaptation of the gain parameter β.

B. Adaptation based on the "equivalent control"

The adaptation procedure (19)-(20) suggested here is applied to minimize the magnitude of discontinuous input $\bar{\beta}\text{sign}(x(t))$ in (6). In sliding mode $y(t) \equiv 0$ therefore

$$[\text{sign}(\sigma(x(t)))]_{eq} = \phi(t)/\bar{\beta}(t) = \phi(t)/k(t)$$

and the algorithm (19)-(20) with $\lambda = \gamma_2 = 0$ can be used directly for this case if time derivative of $|\phi(t)|$ is bounded, namely, if $\frac{d}{dt}|\phi(t)| \leq L$. Indeed, let us fulfill the following steps *a)-e)*.

*Step a)*Following to (20)

$$\delta := \left(\chi \left|[\text{sign}(\sigma(x))]_{eq}\right| + (1-\chi)\mu/k\right) - \alpha \qquad (22)$$

and hence for $V(\delta) = \delta^2/2$ we have

$$\dot{V}(\delta(t)) = \delta(t)\dot{\delta}(t) = \delta(t)\left[\chi\frac{d}{dt}\left(\left|[\text{sign}(\sigma(x))]_{eq}\right|\right) + (1\text{-}\chi)\frac{d}{dt}(\mu/k)\right]$$

Here, according to (20),

$$\chi = \begin{cases} k(t)\left|[\text{sign}(\sigma(x))]_{eq}\right| = |\phi| & \text{if} \quad |\phi| \geq \mu > 0 \\ \mu & \text{if} \quad |\phi| < \mu \end{cases}$$

*Step b)*If $|\phi(t)|$ is differentiable then

$$\frac{d}{dt}\left(\left|[\text{sign}(\sigma(x))]_{eq}\right|\right) = \frac{1}{k}\frac{d}{dt}|\phi| - \frac{|\phi|}{k^2}\dot{k}$$

Step c).

$$\dot{V}(\delta(t)) = \delta(t)\left[\chi\left(\frac{1}{k}\frac{d}{dt}|\phi| - \frac{|\phi|}{k^2}\dot{k}\right) - (1-\chi)\frac{\mu}{k^2}\dot{k}\right] = \delta(t)\left[\chi\frac{1}{k}\frac{d}{dt}|\phi| - \frac{|\phi|\chi + (1-\chi)\mu}{k^2}\dot{k}\right] \qquad (23)$$

Step d).

$$\dot{k} = k(\gamma_0 + \gamma_1\|x\|)\,\text{sign}(\delta(t)) \qquad (24)$$

*Step e)*If $\left|\frac{d}{dt}|\phi|\right| \leq L$, then substitution (24) in (23) and using the estimate

$$|\phi|\chi + (1-\chi)\mu \geq \mu$$

imply

$$\dot{V}(\delta(t)) = \delta(t)\chi\frac{1}{k}\frac{d}{dt}|\phi| - \delta(t)\frac{|\phi|\chi + (1-\chi)\mu}{k^2}k(\gamma_0 + \gamma_1\|x\|)\,\text{sign}(\delta(t)) \leq |\delta(t)|\chi\frac{\phi_0}{k} - |\delta(t)|\frac{\mu}{k}(\gamma_0 + \gamma_1\|x\|) \leq |\delta(t)|(L - \mu\gamma_0)/k$$

Taking $\gamma_0 > L/\mu$ and denoting $\varkappa := \mu\gamma_0 - L$ from the last inequality we get

$$\dot{V}(\delta(t)) \leq -|\delta(t)|\varkappa/k^+ = -\varkappa/k^+\sqrt{2V(\delta(t))}$$

which proofs the finite convergence before $t_f = |\delta(0)|k^+/\varkappa$. So, the following statement can be formulated.

Theorem 1 (on adaptive super-twist): The system (6) with disturbances $\phi(t)$ having a bounded derivative $\frac{d}{dt}|\phi(t)| \leq L$ and with the parameter $\bar{\beta}(t) = k(t)$ adapted on-line according to the adaptation law

$$\dot{k}(t) = k(t)(\gamma_0 + \gamma_1\|x(t)\|)\,\text{sign}(\delta(t))$$
$$0 < \mu \leq k(t) \leq k^+, \ \gamma_0 > L/\mu$$
$$\delta(t) \ \text{ is defined by (22)}$$

converges in the finite time $t_f = |\delta(0)|k^+/(\mu\gamma_0 - L)$ to the sliding mode regime $\sigma(x) = x_1 = 0$ maintaining within the relation

$$\phi(t)/k(t) = \alpha = 1 - \varepsilon$$

for small enough $\varepsilon > 0$.

To demonstrate the properties of the adaptation procedure (19)-(20), simulation was performed for the case $\sigma(x) = x_1$ with the following parameters:

$$\gamma_0 = 1, \ \gamma_1 = 1, \ \gamma_2 = 1, \ \lambda = 2$$
$$\mu = 0.001, \ \alpha = 0.95, \ k^+ = 10, \ k(0) = 1$$

we obtain the following dynamics (see the figure 3):

Here is clearly seen from Fig. 3 that gain parameter $k(t) = \beta(t)$, defining the chattering amplitude in the sliding mode (after the reaching time $t_{reach} \simeq 0.2\,\text{sec}$.), continues to decrease attaining after $2.5\,\text{sec}$.the minimal possible level 0.11 which is 9 times less than under the "$\sigma-$*adaptation*" (see Fig. 2) and 10 times less than without any adaptation (see Fig. 1).

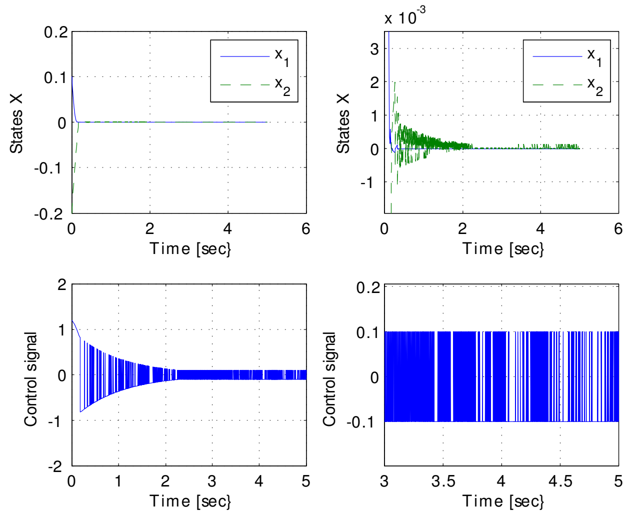

Fig. 3. The states and the control signal (with the zoom) for the super-twist controller with adaptation of the gain parameter β based on the "equivalent control" signal.

C. Conclusions

In this paper an adaptation methodology is developed to find the control gain of a sliding-mode control providing a minimum value of discontinuity resulting in minimization of the chattering effect. The application of this methodology to the super - twist control enables reducing of the control action magnitude to minimum possible value along with a finite-time convergence. The numerical examples clearly illustrate the positive effect of the gain coefficient adaptation being applied to the high-order sliding mode controllers (in particularly, to the super-twist controller).

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