2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Air Supply System of a PEM Fuel Cell Model: Passivity and Robust PI Control

Reine Talj, Romeo Ortega and Alessandro Astolfi

*Abstract*— **Fuel cells are widely regarded as potential future power sources, they convert the chemical energy of a gaseous fuel directly into electricity. In this paper, the study is concentrated on the control of the air subsystem that feeds the fuel cell cathode with oxygen—whose dynamics is described with a widely accepted nonlinear model. Due to the complexity of this model, the model–based controllers that have been proposed for this application are designed using its linear approximation at a given equilibrium point, which might lead to conservative stability margin estimates. On the other hand, practitioners propose the use of simple proportional or proportional–integral controllers around the compressor flow, which ensures good performance in most applications. Using some monotonicity characteristics of the system, in this paper we provide the theoretical justification to this scheme, proving that this output variable has the remarkable property that the linearization (around any admissible equilibrium) of the input-output map is *strictly passive*. Hence, the controllers used in applications yield (locally) asymptotically stable loops—for any desired equilibrium point and *all values of the controller gains*. Ensuring stability for all tuning gains overcomes the inherent conservativeness of linearized dynamics analysis, and assures robustness and high performance.**

## I. Introduction

Fuel cell systems offer a clean alternative to energy production and are currently under intensive development. We consider here a proton exchange membrane (PEM) fuel cell system composed of four main subsystems: the hydrogen and air subsystems, the humidifier and the cooler. Knowing that the degree of humidity and the temperature can not change rapidly, the control problem of these two subsystems can be decoupled from the rest of the system. On the other hand, the hydrogen subsystem is controlled by an electrical valve, while the air subsystem is controlled with a slower mechanical device (*e.g.*, an electrical motor and a compressor), suggesting another time–scale decomposition. For some physical reasons, our work is concentrated on the air supply subsystem of the fuel cell.

The fuel cell presents the problem of oxygen starvation when the load demand increases rapidly [1], [2]. To avoid the latter, the compressor should be accelerated to increase the amount of the supplied oxygen. Given that the compressor motor is electrically fed by the fuel cell itself, the increase in moto-compressor consumption could decrease the net power delivered to the load. Previous works [1] show that, for all operating points, the maximum net power delivered to the load is reached approximately for the same constant value of stoichiometry (ratio of the input oxygen flow over the reacted oxygen flow in the cathode) equal to two [2], [3]. The dynamic behavior of the air supply system is highly nonlinear and uncertain, which renders the analysis and design of suitable control laws very complicated [4].

In [1], [2] a 9-th order nonlinear model—derived from physical principles—to describe the dynamics of the fuel cell was proposed. Reduced, 4-th and 3-rd order models of the air supply system were proposed in [2] and [5], [6], respectively. In spite of its widespread acceptance by the scientific community, the complexity of these models stymies its application for controller design, which is almost invariably carried out using its linear approximation at a *given* equilibrium point [4], [3], [7]. This limits the validity of the analysis to a neighborhood of that particular point, even though the system has a wide operating region. Consequently, the resulting numerical designs are inherently conservative, and fail to provide tuning rules for high performance applications. Other control technics have been used for fuel cell applications, as the model predictive control (MPC) [8] and sliding mode control (SMC) [9]. On the other hand, practitioners propose the use of simple proportional or proportional–integral controllers around the compressor flow, or feedforward controllers, which ensures good performance in most applications.

Our objective in this paper is twofold: to overcome the aforementioned conservativeness problem and to provide the theoretical justification to the scheme used in practice. These objectives are attained invoking the property of passivity. It is well–known, that passive systems are "easy" to control, *e.g.*, with simple PI loops [10], [11], [12] whose gains, moreover, can take arbitrary positive values. Our main contribution is to prove that the map from the compressor voltage to the flow is such that, the linearization, around *any* admissible equilibrium, of the 3-rd order nonlinear model is *strictly passive* [10]. As a consequence, closing the loop with a PI controller yields—for *all tuning gains and all equilibrium points*—a robust (locally) asymptotically stable system. Instrumental to establish this result is the use of the monotonicity characteristics of the system, which stem from the basic physical laws.

The remaining of the paper is organized as follows. The mathematical model of the air supply system proposed in

R. Talj is with Heudiasyc, CNRS UMR 6599, University of Technology of Compiègne, Centre de Recherches de Royallieu, 60205 Compiègne, France

R. Ortega is with Laboratoire des Signaux et Systèmes, L2S, CNRS UMR 8506, Supélec, Univ Paris Sud-P11, 91192 Gif sur Yvette, France

A. Astolfi is with Dept. of Electrical and Electronic Eng., Imperial College, London. SW7 2AZ, UK and DISP, University of Roma, "Tor Vergata", Via del Politecnico 1, 00133 Rome, Italy

[5], [6], and the control problem formulation are presented in Section II. Some remarks on the limitations of numerical controller designs and the role of passivity are presented in Section III. The linearization of the model and some useful structural properties of it are then given in Section IV. Section V contains the main result of the paper, namely, the proof of passivity of the map. The stability analysis of a PI controller and some simulation results are presented in Sections VI and VII, respectively. We wrap up the paper with some concluding remarks and future work in Section VIII. A full version of the paper has been submitted to [13], and a summarized version to [14].

## II. Mathematical Model and Control Problem Formulation

With the aim of facilitating the design of a suitable nonlinear model-based control, a reduced 3-rd order model, which suitably captures the behavior of the system, was proposed in [5], [6]. Its dynamical equations are[1]

$$\begin{aligned} \dot{x}_1 &= -b_1 x_1 + b_2 x_2 + b_3 - b_4 \xi, \\ \dot{x}_2 &= \psi(x_2)\left(\frac{h(x_2,x_3)}{c_{16}} + x_1 - x_2\right), \\ \dot{x}_3 &= -c_9 x_3 - \frac{c_{10}}{x_3}\varphi(x_2) h(x_2,x_3) + c_{13} u \end{aligned} \qquad (1)$$

with
$$\begin{aligned} \varphi(x_2) &= \left(\frac{x_2}{c_{11}}\right)^{c_{12}} - 1, \\ \psi(x_2) &= c_{14} c_{16}\left[1 + c_{15}\varphi(x_2)\right], \end{aligned}$$

where $x_1$, $x_2$ and $x_3$ are, respectively, the air pressure inside the cathode, the air pressure in the supply manifold (between the compressor and the fuel cell cathode input), and the moto-compressor angular speed. Furthermore, $\xi$ is the fuel cell current, which is a measurable disturbance input, and $u$ is the voltage applied on the compressor motor that is the control input. In view of the difference in time scales between the electrical and the mechanical dynamics, both dynamics can be decoupled, and $\xi$ is assumed *constant*.

All the constants $b_i$ and $c_i$—that are functions of the physical parameters of the system—are positive (see details in [5]). The static function of the compressor, $h(x_2, x_3)$, has the shape shown in Fig. 1. We draw the readers attention to the monotonic behavior of the graph, that is essential for the development of our results.

The vector of measurable outputs is

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} V(x_1,x_2) \\ x_2 \\ h(x_2,x_3) \end{bmatrix} \qquad (2)$$

where $y_1$ is the fuel cell voltage and $y_3$ is the compressor flow. See [1], [3] for the analytic expressions of the static functions $V(x_1,x_2)$ and $h(x_2,x_3)$. Note that the moto-compressor angular speed $x_3$ is not measurable. In fact, the driving motor and the compressor are both integrated in a closed volume that, due to mechanical constraints, does not include a speed sensor. Speed can, in principle, be recovered from the measurements $y_2$ and $y_3$, inverting the function $h(x_2,x_3) = y_3$. But this function is highly nonlinear in $x_3$, complicating its inversion.

[1]The interested reader is referred to [5], [6] for further details on the model.

The performance variables for the fuel–cell system are

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1\xi - c_{21}u(u - c_{22}x_3) \\ \frac{c_{23}}{c_{24}\xi}(x_2 - x_1) \end{bmatrix}, \qquad (3)$$

where $z_1$ is the net power delivered to the load and $z_2$ is the oxygen excess ratio, known also as stoichiometry coefficient. As already mentioned, the control objective is to design a controller that will regulate $z_2$ around a desired constant value, typically taken to be $z_2^\star = 2$.

The pressures $x_1$ and $x_2$ are positive. Furthermore, the moto–compressor is unidirectional, hence, the speed $x_3$, the flow $y_3$ and the feeding voltage $u$ are also positive. Finally, the fuel cell is an irreversible device, hence, its current $\xi$, voltage $y_1$ and the net power $z_1$ delivered by the fuel cell are always positive. Note that the stoichiometry $z_2$ represents a flow ratio and is also positive. Consequently, all the variables of the system belong to the positive orthant.

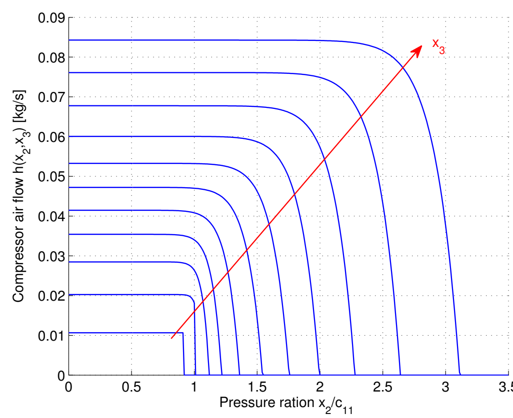


Fig. 1. Compressor map: the curves are parameterized by $x_3$, and grow when it increases

As indicated above, the control objective is to drive $z_2$ to a constant desired value. The proposition below shows that this objective can be recast in terms of *stabilization of an equilibrium point*, that we denote $x^\star \in \mathbb{R}^3_+$. Furthermore, it also proves that regulating $x_3$ to its desired value drives $x_1$ and $x_2$ to their equilibrium value.

*Proposition 1:* Consider the system (1), (3) with $\xi$ fixed to a constant value.

(i) The equilibrium points $x_1^\star, x_2^\star$ are uniquely defined by $x_3^\star$.

(ii) If $x = x^\star$ then $z_2 = z_2^\star$. Furthermore, for all $z_2^\star$ there exists an $x^*$, uniquely defined.

*Proof:* To streamline the proof of the proposition the following observations are in order. First, note that, given an equilibrium $x^\star$, the (constant) control that assigns this equilibrium, say $u^\star$, is univocally defined by the third equation in (1) as

$$u^\star := \frac{1}{c_{13}}[c_9 x_3^\star + \frac{c_{10}}{x_3^\star}\varphi(x_2^\star)h_3(x_2^\star, x_3^\star)]. \tag{4}$$

Hence, our attention is concentrated on the first two of equations (1). Define the parameterized mapping $r^\xi : \mathbb{R}^3_+ \to \mathbb{R}^3$ as

$$r^\xi(x) := \begin{bmatrix} -b_1 x_1 + b_2 x_2 + b_3 - b_4\xi \\ h(x_2, x_3) + c_{16}(x_1 - x_2) \\ x_2 - x_1 - \frac{c_{24}}{c_{23}}\xi z_2^\star \end{bmatrix}. \tag{5}$$

It is clear that the set $\{x \in \mathbb{R}^3_+ \mid r^\xi(x) = 0\}$ identifies the equilibrium points $x^\star$ such that $z_2 = z_2^\star$. Now, $r^\xi(x) = 0$ is a set of three nonlinear algebraic equations in three unknowns. We use the first two equations to prove the first claim. Then, the third equation is solved to find a suitable $x_3^\star$.

(Proof of (i)) The first claim is established showing that the equilibria of the system admit the following parametrization

$$\begin{aligned} x_1^\star &= \frac{1}{b_1}\left[b_2\zeta(x_3^\star) + b_3 - b_4\xi\right], \\ x_2^\star &= \zeta(x_3^\star), \end{aligned} \tag{6}$$

where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function. Now, fix $x_3^\star \in \mathbb{R}_+$. Setting $r_1^\xi(x^\star) = r_2^\xi(x^\star) = 0$ get

$$\begin{aligned} x_1^\star &= \frac{1}{b_1}\left[b_2 x_2^\star + b_3 - b_4\xi\right] \\ x_2^\star - x_1^\star &= \frac{1}{c_{16}}h(x_2^\star, x_3^\star). \end{aligned}$$

Then $h(x_2^\star, x_3^\star) = \frac{c_{16}}{b_1}\left[(b_1 - b_2)x_2^\star - (b_3 - b_4\xi)\right].$

In [5] it is shown that the term $b_1 - b_2$ is positive—see Section IV. Hence, the equilibrium point corresponds to the intersection between the curve $h(x_2, x_3^\star)$ and a straight line with positive slope. Since $h(x_2, x_3)$ is strictly increasing in $x_3$, see Fig. 1, it is clear that there exists a strictly increasing function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$h(\zeta(x_3^\star), x_3^\star) = c_{16}\left(1 - \frac{b_2}{b_1}\right)\zeta(x_3^\star) - \frac{c_{16}}{b_1}(b_3 - b_4\xi), \tag{7}$$

completing the proof.

(Proof of (ii)) In view of the parametrization (6), to prove this claim it is enough to show that there exists $x_3^\star$ such that $x_3 = x_3^\star$ implies $z_2 = z_2^\star$. Setting $r_3^\xi(x^\star) = 0$ and using (6) we get

$$\zeta(x_3^\star) = \frac{1}{b_1 - b_2}\left(b_1\frac{c_{24}}{c_{23}}\xi z_2^\star + b_3 - b_4\xi\right).$$

Now, $\zeta(x_3)$ is a strictly increasing function therefore it is one–to–one and admits a left inverse, say $\zeta_L : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\zeta_L(\zeta(x_3)) = x_3$. The proof is completed selecting

$$x_3^\star = \zeta_L\left[\frac{1}{b_1 - b_2}\left(b_1\frac{c_{24}}{c_{23}}\xi z_2^\star + b_3 - b_4\xi\right)\right],$$

that, together with (6), defines the desired equilibrium. ■

## III. Limitations of Numerical Controller Designs and the Role of Passivity

In view of the complexity of the nonlinear model—even the reduced 3–rd order system (1)—most of the controllers for this application are designed based on its linear approximation [4], [3], [15]. Obviously, the *analytic* model of the linearized dynamics still preserves some of the structural properties of the physical system. Unfortunately, this is lost when *numerical values* are inserted to obtain the model for a *given* equilibrium point. This limits the validity of the analysis—in particular, the predicted stability margins of the controller—to a neighborhood of that particular point. Using numerical *ranges* and designing controllers that are robust to parameter uncertainty, as done in [5], partially palliates this problem.

As explained in Section I, to overcome the conservativeness problem we invoke the property of passivity. Our objective is to look for a passive output that should, additionally, be measurable and detectable. In this respect, it is interesting to recall the main result of [5], which proves that the nonlinear system (1) defines an *output strictly passive* map $u \to x_3$.[2] Unfortunately, the coordinate $x_3$ is not measurable nor it defines a detectable output. Moreover, this result does not even prove that the linearization of the system is passive. Indeed, the storage function in [5] is simply $x_3^2$, it does not have a minimum at the equilibrium.

The most common procedure to control the air subsystem [4], [5], [16] is to close the loop with a PI controller around the compressor air flow error $y_3 - h^\star$—recall that $y_3$, defined in (2), is measurable. Although this configuration ensures good performance in applications, to the best of our knowledge, no rigorous theoretical analysis has been carried out to prove it. A notable exception is [5] where, linearizing *only part* of the dynamics, a procedure to tune the gains of a cascaded controller configuration—ensuring stability of the system—is proposed. Unfortunately, consistent with the discussion of Section I, the admissible gain ranges predicted by the theory turned out to be extremely conservative, yielding below par performance [6]. In the following sections the remarkable property of passivity of the linearization of the map $u \to (y_3 - h^\star)$ is established.

## IV. Model Linearization and Some Useful Properties

*Proposition 2:* The linearization of system (1) around any equilibrium point $x^\star$ is given by

$$\dot\eta = A\eta + b\tilde u,$$

where $\tilde u = u - u^\star$,

$$\begin{aligned} A &= \begin{pmatrix} -b_1 & b_2 & 0 \\ \psi^\star & -\psi^\star\left(1 - \frac{\hbar_2}{c_{16}}\right) & \frac{\psi^\star\hbar_3}{c_{16}} \\ 0 & q_1 & q_2 \end{pmatrix} \\ b &= \begin{pmatrix} 0 & 0 & c_{13} \end{pmatrix}^\top, \end{aligned} \tag{8}$$

[2]In [5] the statement is made only for the dynamics of $x_3$, looking at $x_2$ as an arbitrary, external signal. From the proof it is clear that this is also true for the whole dynamics.

where
$$q_1 = -\frac{c_{10}}{x_3^\star}\left(\varphi^\star \hbar_2 + \varphi_1 h^\star\right),$$
$$q_2 = -\left[c_9 + c_{10}\frac{\varphi^\star}{x_3^\star}\left(\hbar_3 - \frac{h^\star}{x_3^\star}\right)\right],$$

and we have defined the constants

$$\varphi^\star := \varphi(x_2^\star), \quad \psi^\star := \psi(x_2^\star), \quad h^\star := h(x_2^\star, x_3^\star),$$
$$\varphi_1 := \frac{\partial \varphi}{\partial x_2}(x_2^\star),\ \hbar_2 := \frac{\partial h}{\partial x_2}(x_2^\star, x_3^\star),\ \hbar_3 := \frac{\partial h}{\partial x_3}(x_2^\star, x_3^\star)$$

Although the system (1) is a highly complicated set of nonlinear equations, there are several useful properties, which stem from the physical laws, that are instrumental to solve our task. It is shown in [5], that

$$b_1 > b_2 , \tag{9}$$
$$\varphi(x_2) > 0, \quad \forall\, x_2 \in \mathbb{R}_+ , \tag{10}$$
$$\psi(x_2) > 0, \quad \forall\, x_2 \in \mathbb{R}_+ . \tag{11}$$

Using these properties, and the monotonicity of some of the functions describing the dynamics, it is possible to identify some (sign) properties of the elements of the matrix $A$.

*Proposition 3:* The constants appearing in the matrix $A$ verify the following inequalities.

(C1) $\left[c_9 + c_{10}\frac{\varphi^\star}{x_3^\star}\left(\hbar_3 - \frac{h^\star}{x_3^\star}\right)\right] > 0.$
(C2) $\varphi^\star \hbar_2 + \varphi_1 h^\star > 0.$
(C3) $\hbar_2 < 0$ , $\hbar_3 > 0.$

*Proof:* In the operating domain of the fuel cell,
(C1) the function $f(x_3) := \frac{1}{x_3} h(x_2^\star, x_3)$ is strictly increasing for all $x_3$,
(C2) the function $f_2(x_2, x_3) := \varphi(x_2) h(x_2, x_3)$ is strictly increasing with respect to $x_2$,
(C3) the function $h(x_2, x_3)$ is strictly decreasing with $x_2$ and increasing with $x_3$ (Fig. 1). ■

## V. Main Result: The Compressor Flow Error is a Passive Output

*Proposition 4:* Consider the system (1) and an equilibrium point $x^\star$ with the output

$$\tilde{y}_3 := h(x_2, x_3) - h(x_2^\star, x_3^\star) . \tag{12}$$

(P1) The linearization of (1), (12) is

$$\dot{\eta} = A\eta + b\tilde{u}, \quad \tilde{y}_3^\ell = c^\top \eta, \tag{13}$$

where $(A, b)$ are given in (8) and

$$c := \mathrm{col}(0, \hbar_2, \hbar_3). \tag{14}$$

(P2) The map $\tilde{u} \to \tilde{y}_3^\ell$ is strictly passive.

*Proof:* (P1) First, note that $\tilde{y}_3$ is zero at the equilibrium. Thus, the linearization of the output is

$$\tilde{y}_3^\ell = \frac{\partial h}{\partial x_2}(x_2^\star, x_3^\star)\eta_2 + \frac{\partial h}{\partial x_3}(x_2^\star, x_3^\star)\eta_3 = \hbar_2 \eta_2 + \hbar_3 \eta_3 . \tag{15}$$

(P2) The proof of strict passivity is established showing that the transfer function

$$H(s) = c^\top (sI_3 - A)^{-1} b , \tag{16}$$

is strictly positive real (SPR). As well–known, see *e.g.* Lemma 6.1 of [17], this is tantamount to verifying

(R1) $\Re\left[H(j\omega)\right] > 0, \quad \forall \omega \in \mathbb{R},$
(R2) $\lim_{\omega\to\infty}\left\{\omega^2 \Re\left[H(j\omega)\right]\right\} > 0.$

Given the matrices $A$, $b$ and $c$ in the equations (8) and (14), and after some calculation, the transfer function can be expressed as

$$H(s) = c_{13}\hbar_3 . \frac{s^2 + n_1 s + n_0}{s^3 + d_2 s^2 + d_1 s + d_0} , \tag{17}$$

Setting $s = j\omega$, the frequency response function becomes

$$H(j\omega) = c_{13}\hbar_3 . \frac{R_N + jI_N}{R_D + jI_D} ,$$

where $R_N$, $I_N$, $R_D$ and $I_D$ are the real and imaginary parts of the numerator and the denominator. Hence,

$$\Re\left[H(j\omega)\right] = c_{13}\hbar_3 . \frac{R_N R_D + I_N I_D}{R_D^2 + I_D^2} . \tag{18}$$

Since $c_{13}\hbar_3 > 0$ and $R_D^2 + I_D^2 > 0$, $\Re\left[H(j\omega)\right]$ is positive if and only if $R_N R_D + I_N I_D > 0$. Now,

$$R_N R_D + I_N I_D = \omega^4(d_2 - n_1) + \omega^2(n_1 d_1 - d_0 - n_0 d_2) + n_0 d_0 . \tag{19}$$

From the monotonicity characteristics of the system and the expressions of $n_i$ and $d_i$, one can prove that $(R_N R_D + I_N I_D)$ is positive for all $\omega \in \mathbb{R}$, proving (R1).
Furthermore, given that

$$R_D^2 + I_D^2 = \left(d_0 - d_2\omega^2\right)^2 + \left[\omega(d_1 - \omega^2)\right]^2 , = \omega^6 + (d_2^2 - 2d_1)\omega^4 + (d_1^2 - 2d_0 d_2)\omega^2 + d_0^2 , \tag{20}$$

and considering (19), we find that

$$lim_{\omega\to\infty}\left\{\omega^2 \Re\left[H(j\omega)\right]\right\} = c_{13}\hbar_3\ (d_2 - n_1) > 0$$

where the inequality follows from $c_{13}\hbar_3 > 0$ and $d_2 - n_1 > 0$. ■

## VI. An Asymptotically Stable PI Controller

In this section we prove that, in view of strict passivity of the linearized model, the nonlinear system in closed–loop with a PI controller has an asymptotically stable equilibrium at $x^\star$. Although the same result can be established for a simple proportional controller,[3] $\tilde{u} = -k_p \tilde{y}_3$, the addition of the integral action—besides the well–known constant disturbance rejection feature—has the advantage that the controller can be implemented without the knowledge of $u^\star$, which depends on uncertain model parameters.

*Proposition 5:* Given a fuel cell current $\xi$ and a desired value for the stoichiometry $z_2 = z_2^\star$, let $x^\star$ be the corresponding equilibrium. Consider the system (1) in closed–loop with the PI controller

$$\dot{x}_c = \tilde{y}_3 , \quad u = -k_p \tilde{y}_3 - k_i x_c , \tag{21}$$

[3] Actually, for any linear controller that ensures asymptotic stability of the closed–loop.

where $\tilde{y}_3$ is given in (12). The equilibrium $(x^\star, -\frac{u^\star}{k_i})$, with $u^\star$ given in (4), is *asymptotically stable*, for any $k_i, k_p \in \mathbb{R}_+$. Moreover, $lim_{t\to\infty} z_2(t) = z_2^\star$ .

*Proof:* The proof is established invoking Lyapunov's indirect method. That is, we prove that the linearization of the closed–loop system, around the equilibrium $x^\star$, is asymptotically stable, which implies that $x^\star$ is a (locally) asymptotically stable equilibrium of the nonlinear system. The proof that $lim_{t\to\infty} z_2(t) = z_2^\star$ then follows from claim (ii) in Proposition 1.

The linearization of the closed–loop is given by (13) together with

$$\dot{\tilde{x}}_c = \tilde{y}_3^\ell\ , \quad \tilde{u} = -k_p\tilde{y}_3^\ell - k_i\tilde{x}_c\ , \tag{22}$$

where we have defined $\tilde{x}_c := x_c - x_c^\star$, and used $x_c^\star = -\frac{u^\star}{k_i}$. The linearized closed–loop system then becomes

$$\begin{bmatrix} \dot{\eta} \\ \dot{\tilde{x}}_c \end{bmatrix} = \mathcal{A} \begin{bmatrix} \eta \\ \tilde{x}_c \end{bmatrix},\ \mathcal{A} := \begin{bmatrix} A - k_p b c^\top & -k_i b \\ c^\top & 0 \end{bmatrix}.$$

We show that $\mathcal{A}$ is Hurwitz proving that it satisfies an algebraic Lyapunov equation of the form

$$\mathcal{P}\mathcal{A} + \mathcal{A}^\top\mathcal{P} = -\mathcal{C}^\top\mathcal{C}, \tag{23}$$

where $\mathcal{P} > 0$ and the pair $(\mathcal{A}, \mathcal{C})$ is observable.

Since the transfer function $H(s)$ is strictly positive real, Kalman–Yakubovich–Popov's Lemma yields the existence of positive definite matrices $P \in \mathbb{R}^{3\times3}$ and $Q \in \mathbb{R}^{3\times3}$ such that

$$PA + A^\top P = -Q, \quad Pb = c. \tag{24}$$

Consider the positive definite matrix $\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & k_i \end{bmatrix}$, which yields

$$\mathcal{P}\mathcal{A} + \mathcal{A}^\top\mathcal{P} = -\begin{bmatrix} Q + 2k_p Pbb^\top P & 0 \\ 0 & 0 \end{bmatrix},$$

where (24) has been used. The matrix $Q + 2k_p Pbb^\top P$ is positive definite, therefore it admits a factorization of the form

$$Q + 2k_p Pbb^\top P = K^\top K,$$

where $K \in \mathbb{R}^{3\times3}$ is *nonsingular*. It is clear that (23) is satisfied with

$$\mathcal{C} = \begin{bmatrix} K & \vdots & 0 \end{bmatrix} \in \mathbb{R}^{3\times4}.$$

The observability claim follows from Popov–Belevitch–Hautus test, *e.g.*, showing that there is no eigenvector of $\mathcal{A}$ in the kernel of $\mathcal{C}$. Indeed, for any vector $v \in \mathbb{C}^4$,

$$\mathcal{C}v = 0 \Leftrightarrow K\mathrm{col}(v_1, v_2, v_3) = 0 \Leftrightarrow v = \mathrm{col}(0, 0, 0, v_4),$$

for some $v_4 \in \mathbb{C}$. Hence, $\mathcal{A}v = \begin{bmatrix} -k_i v_4 b\ ,\ 0 \end{bmatrix}^\top$, and, clearly, there is no (eigenvalue) $\lambda \in \mathbb{C}$ such that $\mathcal{A}v = \lambda v$. Completing the proof.

## VII. Simulation Results

To compute the output for the PI controller (21) we write

$$\begin{aligned} x_1^\star &= \frac{z_2^\star \frac{c_{24}}{c_{23}} b_2 - b_4}{b_1 - b_2}\xi + \frac{b_3}{b_1 - b_2} \\ x_2^\star &= \frac{z_2^\star \frac{c_{24}}{c_{23}} b_1 - b_4}{b_1 - b_2}\xi + \frac{b_3}{b_1 - b_2}, \end{aligned}$$

which are obtained setting $r_1^\xi(x^\star) = r_3^\xi(x^\star) = 0$ in (5). The computation of $h(x_2^\star, x_3^\star)$ is carried out from $r_2^\xi(x^\star) = 0$, with $x_1^\star$ and $x_2^\star$ being replaced by the expressions above, yielding

$$h(x_2^\star, x_3^\star) = \frac{1}{c_{16}}(x_2^\star - x_1^\star) = z_2^\star \frac{c_{24}}{c_{23}}\xi.$$

Hence, $\tilde{y}_3 = h(x_2, x_3) - h(x_2^\star, x_3^\star)$ is known, given that the compressor airflow $y_3 = h(x_2, x_3)$ is measured.

Figures (3) and (2) present the simulation results of the system (1) in closed—loop with the PI controller (21). The gains of the PI are chosen to be $k_p = 3000, \quad k_i = 18000$, so that the poles of the closed-loop system are given by $p_1 = -1.209, p_2 = -5.663, p_3 = -27.822, p_4 = -54.338$, when the fuel cell current is at the nominal value of 191A [2]. The simulation is performed with the current profile shown in Fig.2-(a). Fig.3 shows that the three states $x_1$, $x_2$ and $x_3$ converge to the equilibrium point corresponding to the desired output $z_2 = z_2^\star$. Fig.2-(b) shows the control input, that is the voltage feeding the motor of the compressor. The convergence of the compressor airflow to its equilibrium point is shown in Fig.2-(c). Finally, Fig.2-(d) shows the convergence of the controlled output $z_2$ to its desired reference $z_2^\star$, chosen equal to two.

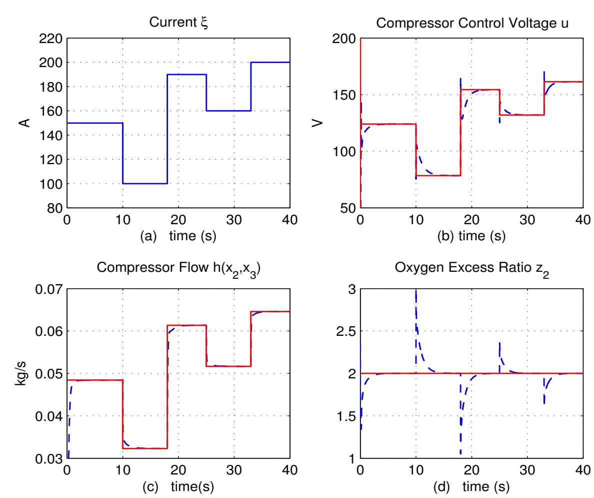


Fig. 2. Simulation response of the closed-loop system with the PI controller

## VIII. Concluding Remarks and Future Work

In this paper a theoretical justification has been given to the common practice of regulating the air supply system of a PEM fuel cell with a simple PI controller around the

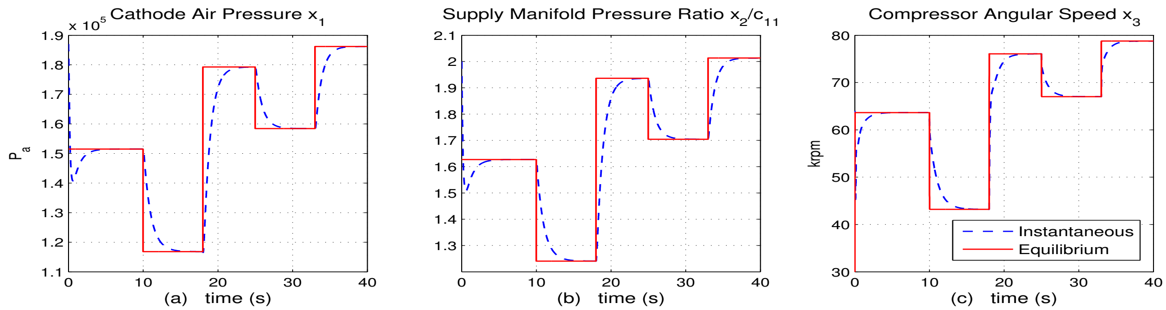


Fig. 3. Convergence of the three states to the equilibrium point

compressor flow. Exploiting the monotonic characteristics of some static functions of the fuel cell, it has been verified that the linearized incremental model of the system—at any equilibrium point—is strictly passive. Based on the latter property the stability of the closed-loop system, with a simple PI controller, is proved around any equilibrium point in the operating domain for all $k_p > 0$ and $k_i \geq 0$.

Unfortunately, as shown in [18], passivity of the linearized system does not imply (local) passivity of the original nonlinear map. Therefore, no conclusions can be drawn regarding the behavior of the system in closed–loop with a nonlinear controller. However, as proven in [10], $\mathcal{L}_2$–stability of the linearized system implies, local, $\mathcal{L}_2$–stability of the nonlinear system. Since the passivity property that we established is strict, the linearized system is $\mathcal{L}_2$–stable and small gain theorem arguments can be used to analyze nonlinear controllers. Current research is under way in this direction. It should be pointed out that the passivity property has been established for the linearization at the equilibrium points and not at *any* point in the state space. Some preliminary calculations suggest that the latter, stronger, property is unfortunately not true.

A limitation of the local asymptotic stability result of Proposition 5 is that we are unable to estimate the domain of attraction of the stable equilibrium. This requires a Lyapunov analysis of the nonlinear system. Another interesting open question is the potential advantage of using other passive outputs. For instance, it can be shown that $h(x_2, x_3) - h(x_2, x_3^\star) = y_3 - h(y_2, x_3^\star)$, which is measurable and detectable, is also a passive output.

Another interesting and important point to be studied is the effect of saturation of the input voltage in fast transients, and its influence on the stability analysis. Saturation is a first–third quadrant static nonlinearity that defines a passive operator.

The result presented in this paper provides a first, modest, step towards the development of physically–based controller design methodologies, where the mass and energy balance properties of the system are explicitly exploited. This is the essence and final objective of the research area generically known as passivity–based control.

## REFERENCES


[1] J.T. Pukrushpan, A.G. Stefanopoulou, and H. Peng. *Control of fuel cell power systems: principles, modeling, analysis and feedback design*. Springer, 2004.

[2] K.W. Suh. *Modeling, analysis and control of fuel cell hybrid power systems*. PhD thesis, University of Michigan, 2006.

[3] M. Grujicic, K.M. Chittajallu, E.H. Law, and J.T. Pukrushpan. Model-based control strategies in the dynamic interaction of air supply and fuel cell. *Proceedings of the Institution of Mechanical Engineers, Part A: Journal of Power and Energy*, 218(7):487–499, 2004.

[4] J.T. Pukrushpan, A.G. Stefanopoulou, and H. Peng. Control of fuel cell breathing. *IEEE Control Systems Magazine*, 24(2):30–46, 2004.

[5] R. Talj, R. Ortega, and M. Hilairet. A controller tuning methodology for the air supply system of a PEM fuel cell system with guaranteed stability properties. *International Journal of Control*, 82(9):1706–1719, 2009.

[6] R. Talj, D. Hissel, R. Ortega, M. Becherif, and M. Hilairet. Experimental validation of a PEM fuel cell reduced order model and a moto-compressor higher order sliding mode control. *IEEE Transactions on Industrial Electronics*, 57(6):1906–1913, 2010.

[7] J.T. Pukrushpan, H. Peng, and A.G. Stefanopoulou. Control-oriented modeling and analysis for automotive fuel cell systems. *Journal of Dynamic Systems, Measurement, and Control*, 126:14, 2004.

[8] C. Bordons, A. Arce, and AJ del Real. Constrained predictive control strategies for PEM fuel cells. In *American Control Conference, 2006*, page 6, 2006.

[9] C. Kunusch, PF Puleston, MA Mayosky, and J. Riera. Sliding Mode Strategy for PEM Fuel Cells Stacks Breathing Control Using a Super-Twisting Algorithm. *IEEE Transactions on Control Systems Technology*, 17(1):167–174, 2009.

[10] A. Van der Schaft. *$L_2$-gain and passivity techniques in nonlinear control*. Springer, $2^{nd}$ edition, 2000.

[11] F. Castaños, B. Jayawardhana, R. Ortega, and E. García-Canseco. Proportional plus integral control for set point regulation of a class of nonlinear RLC circuits. In *Circuits, Systems and Signal Processing*, volume 28, pages 609–615, 2009.

[12] M. Hernandez-Gomez, R. Ortega, G. Escobar, and F. Lamnabhi-Lagarrigue. Adaptive PI stabilization of switched power converters. *IEEE Transactions on Control Systems Technology*, 18(3):688–698, 2010.

[13] R. Talj, R. Ortega, and A. Astolfi. Passivity and Robust PI Control of the Air Supply System of a PEM Fuel Cell Model. *Automatica*, 2011. accepted.

[14] R. Talj and R. Ortega. Passivity of the air supply system of a PEM fuel cell model and robust PI control. In *Fundamentals and Developments of Fuel Cells (FDFC), Grenoble, France*, 19 – 21 January, 2011.

[15] N. Romani, E. Godoy, D. Beauvois, and V. Le Lay. Modélisation et commande du système d'alimentation en air d'un véhicule pile à combustible avec reformage embarqué. In *Conférence Internationale Francophone dAutomatique, IEEE CIFA'06*, 2006.

[16] K.W. Suh and A.G. Stefanopoulou. Coordination of converter and fuel cell controllers. *International Journal of Energy Research*, 29(12):1167–1189, 2005.

[17] H.K. Khalil. *Nonlinear systems, Third edition*. Prentice hall Pearson Education, Upper Saddle River, New Jersey, 2000.

[18] H. Nijmeijer, R. Ortega, A.C. Ruiz, and A. Van der Schaft. On passive systems: From linearity to nonlinearity. In *Proceedings of $2^{nd}$ IFAC Symposium on Nonlinear Control Systems, Bordeaux, France*, pages 214–219, 1992.