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Pole Structure Estimation from Laguerre Representations Using Hyperbolic Metrics on the Unit Disc

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***Abstract*— This paper proposes a new approach to identification of the poles in a linear system from frequency domain data. The discrete rational transfer function is represented in a rational Laguerre–basis, where the basis elements can be expressed by powers of the Blaschke–function. This function can be interpreted as a congruence transform on the Poincaré unit disc model of the hyperbolic geometry, leading to a nice geometric interpretation of the identification algorithm. Convergence results in hyperbolic metrics will be given. The full procedure is illustrated by simulation examples.**

I. INTRODUCTION

Determining the pole locations associated to signals and systems modeling is needed in many applications and several approaches are known to cope with this problem. Detection of spectral peaks in noisy signals – a frequent task in analyzing vibrating mechanical systems, electrical rotating machines, or industrial plants (such as nuclear power plants) – is in essence a pole-identification problem. The phase angle associated with the pole can be identified as the frequency, and its absolute value – corresponding to the attenuation represented by the pole – can be connected with the height and width of a spectral peak. The spectral peak-analysis can be performed by Fourier-analysis of the signals that can efficiently be realized by using the Fast-Fourier-Transform (FFT) algorithm. However the exact pole locations cannot be determined by this way, spectral peaks cannot be separated in many cases, as well as analyzing the height and width of the peaks does not result in unique solutions for determining the poles attenuations.

For use of parametric methods in time domain one can cite variations of Prony–methods [1] and that assuming linear signal- and system-models – and autoregressive (AR) or autoregressive moving-average (ARMA) model identifications and associated spectral analysis, identification of matrix partial fraction models, see [2], [3]. Disadvantage of these approaches is that associated to the parametrization problem, both the structure and the parameters have to be estimated, leading to, in many situation, unreliable results.

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Another approach is the use of rational orthogonal bases [4] that needs *a priori* knowledge upon the pole locations. Special attention paid on the problems of pole selection and validation [5], [6]. There exist methods to refine the pole locations starting from an approximate placement of poles [7], however the general identification problem has not been solved so far.

This paper proposes a new approach that is closely related to signal and system representations using discrete rational Laguerre–basis in the space H^2 upon the unit disc $\mathbb{D}$. The Laguerre–Fourier coefficients of the rational transfer function will be computed using frequency domain data obtained from non-uniformly distributed measurements defined by the Laguerre–basis parameter. These Fourier coefficients will be analyzed in terms of the basis parameter. Since the basis functions can be expressed by the Blaschke–function that have an interpretation in term of congruent transform on the Poincaré unit disc model of the hyperbolic geometry. This will allow a nice geometric interpretation of the resulting identification algorithm.

It will be shown that for a transfer function f analytic on the closed unit disc $\overline{\mathbb{D}}$, computing the quotients of consecutive conjugated Laguerre–Fourier coefficients as $q_n(b) = c_{n+1}(b)/c_n(b)$ the limit exists for almost all basis parameter b. The set of those points where this limit exists, can be described by perpendicular bisectors of hyperbolic geometry. Using the inverse Blaschke - maps, one of the poles of f can be computed. The procedure is continued until more new poles can be reconstructed.

The main result of the paper is formulated in Theorem 1 and this is directly used in the identification algorithm proposed. Numeric example for illustration of the theoretical results and providing geometric interpretation is given, too.

II. LAGUERRE SYSTEM REPRESENTATIONS, BLASCHKE – FUNCTIONS AND HYPERBOLIC GEOMETRY

A key concept in the $\mathrm{H}^2(\mathbb{D})$ system and signal representations is the Blaschke function based upon a parameter $b \in \mathbb{D}$, which can be considered as an *inverse pole* ($b = 1/\overline{p}$) of the function. The Blaschke function is defined as

$$B_{\mathfrak{b}}(z) := \epsilon \frac{z-b}{1-\overline{b}z} \quad (z \in \mathbb{C}, \mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}), \quad (1)$$

where $\mathbb{D}$ and $\mathbb{T}$ denotes the open unit disc and the unit circle, respectively. If $\mathfrak{b} \in \mathbb{B}$, then $B_{\mathfrak{b}}$ is an $1-1$ map on $\mathbb{T}$ and $\mathbb{D}$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or $\mathbb{T}$ with the operation $(B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2})(z) := B_{\mathfrak{b}_1}(B_{\mathfrak{b}_2}(z))$

form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2} = B_{\mathfrak{b}_1 \circ \mathfrak{b}_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B}\}, \circ)$. The neutral element of the group $(\mathbb{B}, \circ)$ is $\mathfrak{e} := (0, 1) \in \mathbb{B}$ and the inverse element of $\mathfrak{b} = (b, \epsilon) \in \mathbb{B}$ is $\mathfrak{b}^{-1} = (-b\epsilon, \overline{\epsilon})$.

It can be proved that the map

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \overline{z}_1\, z_2|} = |B_{z_1}(z_2)| \qquad (2)$$

$$(B_{z_1} := B_{(z_1,1)}, z_1, z_2 \in \mathbb{D})$$

is a metric on $\mathbb{D}$, called pseudohyperbolic metric (see [8], [9], [10]. Moreover the Blaschke functions B_b $(b \in \mathbb{D})$ are isometries with respect to this metric, i.e.

$$\rho(B_b(z_1), B_b(z_2)) = \rho(z_1, z_2) \quad (b \in \mathbb{D}, z_1, z_2 \in \mathbb{D}). \qquad (3)$$

The lines in this model are the sets

$$\mathcal{L}_{\mathfrak{b}} := \{B_{\mathfrak{b}}(r) : -1 < r < 1\} \quad (\mathfrak{b} \in \mathbb{B}),$$

i.e. circles crossing perpendicularly the unit circle. This model is known in the hyperbolic geometry as the unit disc Poincaré model.

The discrete Laguerre–functions are defined by

$$L_n^b(z) := \frac{\sqrt{1 - |b|^2}}{1 - \overline{b}z} \left(\frac{z - b}{1 - \overline{b}z} \right)^n \qquad (4)$$

$$(z \in \overline{\mathbb{D}}, b \in \mathbb{D}, n \in \mathbb{N}).$$

Using the function

$$F_{\mathfrak{b}}(z) := \frac{\sqrt{\epsilon(1 - |b|^2)}}{1 - \overline{b}z} \qquad (5)$$

$$(\mathfrak{b} := (b, \epsilon) \in \mathbb{B}, b \in \mathbb{D}, \epsilon \in \mathbb{T}, z \in \overline{\mathbb{D}}),$$

and the Blaschke maps according to (1), the discrete Laguerre–functions can be expressed in the form

$$L_n^b = F_{\mathfrak{b}} B_{\mathfrak{b}}^n \quad (\mathfrak{b} = (b, 1) \in \mathbb{B}, n \in \mathbb{N}). \qquad (6)$$

Denote $H = \mathrm{H}^2(\mathbb{T})$ the Hardy space with the usual scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \overline{g}(e^{it})\, dt \quad (f, g \in \mathrm{H}^2(\mathbb{T})). \qquad (7)$$

We introduce the collection of operators $U_{\mathfrak{b}} : \mathrm{H}^2(\mathbb{T}) \to \mathrm{H}^2(\mathbb{T})$ $(\mathfrak{b} \in \mathbb{B})$ defined by

$$U_{\mathfrak{b}} f := F_{\mathfrak{b}^{-1}}\, f \circ B_{\mathfrak{b}^{-1}} \quad (f \in \mathrm{H}^2(\mathbb{T}), \mathfrak{b} \in \mathbb{B}). \qquad (8)$$

It is known that $U_{\mathfrak{b}}$, $(\mathfrak{b} \in \mathbb{B})$ is a unitary representation of the Blaschke group $\mathbb{B}$ [11], i.e.

(i) $U_{\mathfrak{b}_1}(U_{\mathfrak{b}_2} f)) = U_{\mathfrak{b}_1 \circ \mathfrak{b}_2} f$,

(ii) $\langle U_{\mathfrak{b}} f, U_{\mathfrak{b}} g \rangle = \langle f, g \rangle$ $(f, g \in \mathrm{H}^2(\mathbb{T}))$.

The discrete Laguerre–functions L_n^b can be introduced as image of the power function $h_n(z) := z^n$ by the representation $U_{\mathfrak{b}}$:

$$L_n^b := U_{\mathfrak{b}}^{-1} h_n \quad (n \in \mathbb{N}, \mathfrak{b} = (b, 1) \in \mathbb{T}). \qquad (9)$$

Since $U_{\mathfrak{b}}$ is unitary, $U_{\mathfrak{b}}^* = U_{\mathfrak{b}}^{-1} = U_{\mathfrak{b}^{-1}}$ and consequently for any $m, n \in \mathbb{N}$

(i) $\langle L_n^b, L_m^b \rangle = \langle U_{\mathfrak{b}^{-1}} h_n, U_{\mathfrak{b}^{-1}} h_m \rangle = $
$= \langle h_n, h_m \rangle = \delta_{mn}$

(ii) $\langle f, L_n^b \rangle = \langle f, U_{\mathfrak{b}^{-1}} h_n \rangle = \langle U_{\mathfrak{b}} f, h_n \rangle$.

Thus the discrete Laguerre–Fourier coefficients of f are equal to the trigonometric Fourier coefficients of the function $U_{\mathfrak{b}} f$. This relation can be used to compute the discrete Laguerre–Fourier coefficients.

The representation of any function $f \in \mathrm{H}^2(\mathbb{D})$ in the Laguerre–system can be expressed as

$$f(z) = \sum_{n=0}^{\infty} c_{n,f} L_n^b(z), \qquad (10)$$

where $c_{n,f}$ coefficients – i.e. the so called Laguerre–Fourier coefficients belonging to function f – can be computed by the scalar products $\langle f, L_n^b \rangle$ $(n \in \mathbb{N})$.

III. ANALYZING THE LAGUERRE-FOURIER COEFFICIENTS

Let $\mathfrak{R}$ denote the set of rational functions with poles falling outside the closed unit disc. It is obvious that functions

$$r_{n,a}(z) := \frac{z^n}{(1 - \overline{a}z)^{n+1}} \quad (a \in \mathbb{D}, z \in \overline{\mathbb{D}}, n \in \mathbb{N}) \qquad (11)$$

belong to $\mathfrak{R}$. $a^* := 1/\overline{a}$ is a pole of multiplicity $(n + 1)$ of the function $r_{n,a}$, which is the inverse map of a with respect to the unit circle. a will be referred as the "inverse pole" of the function $r_{n,a}$ in the subsequent part of the paper. It is well known that the functions of form (11) generate the function class $\mathfrak{R}$, i.e. any $f \in \mathfrak{R}$ can be expressed in the form

$$f := \sum_{k=1}^{P} \Lambda_{a_k} \quad \Lambda_{a_k} := \sum_{i=0}^{m_k - 1} \lambda_{ki} r_{i,a_k}, \qquad (12)$$

where $a_k \in \mathbb{D}$ $(k = 1, \ldots, P)$ denote the inverse poles of the function f with their multiplicity m_k.

The following lemma will be used in computing the Laguerre–Fourier coefficients of f.

Lemma 1: For every function $g \in \mathfrak{R}$

$$\langle g, r_{n,a} \rangle = \frac{g^{(n)}(a)}{n!} \quad (n \in \mathbb{N}, a \in \mathbb{D}). \qquad (13)$$

Proof: By definition

$$\langle g, r_{n,a} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(e^{it}) e^{-int}}{(1 - ae^{-it})^{n+1}}\, dt = $$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(e^{it}) e^{it}}{(e^{it} - a)^{n+1}}\, dt = $$
$$= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{g(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta.$$

Hence by Cauchy's integral formula we get (13). ■

In the case if $m_k = 1$ the associated term will be r_{0,a_k}, and the conjugate of the Laguerre–Fourier coefficients belonging to it are directly given by (13) as

$$\langle L_n^b, r_{0,a_k} \rangle = L_n^b(a_k), \qquad (14)$$

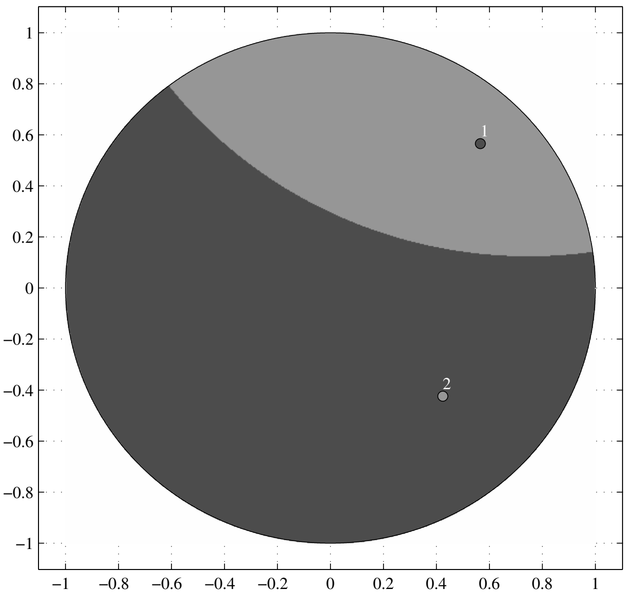

Fig. 1. Domains D_i belonging to a pair of poles.

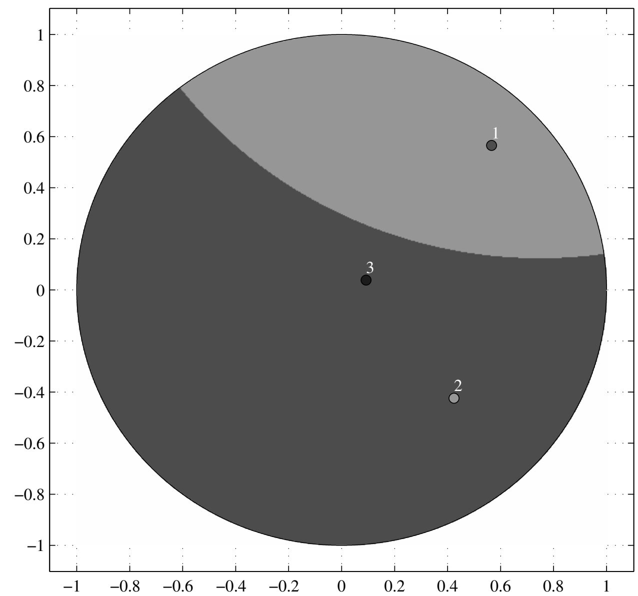

Fig. 2. Domains D_i belonging to three poles, one of them isolated.

that is equal to coefficient $c_{n,f}$. To indicate that $c_{n,f}$ coefficients belong to the parameter b used in the Laguerre representation, let us denote them as $c_{n,f}(b)$.

Suppose that the system under consideration contains only a single pole of multiplicity 1, in this case the conjugated Laguerre–Fourier coefficients are given as $c_{n,f}(b) = L_n^b(a)$, and the quotients

$$q_n(b) = \frac{c_{n+1,f}(b)}{c_{n,f}(b)} = B_b(a) \quad (n \in \mathbb{N}), \tag{15}$$

form a constant sequence and its elements equal to a Blaschke function applied to a. This fact can be used to identify the position of inverse pole a,

$$a = B_{b^{-1}}\left(\frac{c_{n+1,f}(b)}{c_{n,f}(b)}\right), \tag{16}$$

where $B_{b^{-1}}$ is the inverse of B_b, i.e. a is given by applying a hyperbolic transform corresponding to the inverse group element belonging to b.

This concept can be extended to multiple poles, it will be shown that in the case of multiple poles there exist a region of $\mathbb{D}$ where the sequence of the quotients generated by the conjugated Laguerre–Fourier coefficients converge.

Let the inverse poles $a_1, a_2 \ldots, a_P \in \mathbb{D}$ of function f be fixed. Applying the hyperbolic distance as defined by (2) let us introduce subsets of $\mathbb{D}$ as follows:

$$\begin{aligned} D_i &: = \{b \in \mathbb{D} : \rho(b, a_i) > \max_{1 \le j \le P, i \ne j} \rho(b, a_j)\}, \\ D &: = \bigcup_{j=1}^{P} D_i \quad (i = 1, 2, \ldots, P). \end{aligned} \tag{17}$$

Concerning these sets a rather informative interpretation can be given: the set

$$\mathcal{L}_{ij} := \{b \in \mathbb{D} : \rho(a_i, b) = \rho(a_j, b)\} \tag{18}$$

can be considered as the hyperbolic perpendicular bisector of the points a_i, a_j that divides $\mathbb{D}$ in two hyperbolic half-planes. Let the following notations be introduced:

$$\begin{aligned} D_{ij} &:= \{b \in \mathbb{D} : \rho(a_i, b) > \rho(a_j, b)\} \\ D_{ji} &:= \{b \in \mathbb{D} : \rho(a_i, b) < \rho(a_j, b)\}. \end{aligned} \tag{19}$$

The sets D_i can be generated as an intersection of the hyperbolic half-planes, i.e. according to the definitions in (17) and (19)

$$D_i = \bigcap_{k=1, k \ne i}^{P} D_{ik} \quad (i = 1, 2, \cdots, P). \tag{20}$$

As a consequence the sets D_i are hyperbolically convex regions, i.e. any hyperbolic line segment connecting two points belonging to any D_i is located as a whole in the same region.

An example illustrating the placement of regions D_i belonging to 2 poles can be seen in Figure 1. The limit between the two regions is a hyperbolic line satisfying the condition (18). In Figure 2 an additional pole has been taken (pole no. 3), however it cannot generate a nonempty region, i.e. region D_3 cannot be seen.

It will be shown that in any point of D the limit

$$(\mathcal{Q}f)(b) := \lim_{n \to \infty} \frac{c_{n+1,f}(b)}{c_{n,f}(b)} \quad (f \in \mathfrak{R}) \tag{21}$$

does exist, and it can be used to reconstruct the poles of function f. It should be mentioned that operator $\mathcal{Q}$ defined on domain $\mathfrak{R}$ is nonlinear.

Theorem 1: For any rational function f of form (12) in any point b of D the limit (21) exists, and

$$(\mathcal{Q}f)(b) = B_b(a_i), \quad b \in D_i \quad (i = 1, 2, \ldots, P). \tag{22}$$

In the case of poles of multiplicity 1 for the speed of convergence the estimation

$$\left|\frac{c_{n+1,f}(b)}{c_{n,f}(b)} - B_b(a_i)\right| = O(q_i^n) \ (n \in \mathbb{N}, b \in D_i, q_i < 1)$$

can be given.

For simplicity only a proof for the case of poles with multiplicity 1 is presented in the current paper. The general case has been discussed in [12].

Proof: Suppose that $b \in D_i$, and introduce the quotients as follow:

$$q_{ik}(b) := \frac{|B_b(a_k)|}{|B_b(a_i)|} = \frac{\rho(b, a_k)}{\rho(b, a_i)} \quad (k = 1, 2, \ldots, P).$$

In this case

$$q_i := \max_{1 \leq k \leq P, i \neq k} q_{ik} < 1 \quad (i = 1, 2, \ldots, P).$$

It is assumed that all the poles of the function $f \in \mathfrak{R}$ are of multiplicity 1. In this case according to (12) and (13), by applying the notation $\lambda_k = \lambda_{k0}$,

$$c_{n,f}(b) = \sum_{k=1}^{P} \overline{\lambda}_k L_n^b(a_k) = \tag{23}$$

$$= B_b^n(a_i)\Big(\overline{\lambda}_i F_b(a_i) + \sum_{k=1, k\neq i}^{P} \overline{\lambda}_k F_b(a_k) \frac{B_b^n(a_k)}{B_b^n(a_i)}\Big) =$$

$$= B_b^n(a_i)\overline{\lambda}_i F_b(a_i)(1 + \epsilon_n),$$

by using the notation

$$\epsilon_n := \sum_{k=1, k\neq i}^{P} \frac{\overline{\lambda}_k}{\overline{\lambda}_i} \frac{F_b(a_k)}{F_b(a_i)} \frac{B_b^n(a_k)}{B_b^n(a_i)}.$$

From this

$$|\epsilon_n| \leq \kappa_i q_i^n, \quad \kappa_i := \sum_{k=1, k\neq i}^{P} \frac{|\lambda_k|}{|\lambda_i|} \frac{|F_b(a_k)|}{|F_b(a_i)|}.$$

follows. By using the obtained form in (21)

$$\frac{c_{n+1,f}}{c_{n,f}} = B_b(a_i) \frac{1 + \epsilon_{n+1}}{1 + \epsilon_n} \to B_b(a_i) \quad (n \to \infty)$$

is given that proves the first assertion of the theorem. For the speed of convergence

$$\left|\frac{c_{n+1,f}}{c_{n,f}} - B_b(a_i)\right| = \left|1 - \frac{1 + \epsilon_{n+1}}{1 + \epsilon_n}\right| \leq O(q_i^n)$$
$$(n \in \mathbb{N}, i = 1, 2, \cdots, P)$$

is given. ■

According to Theorem 1

$$B_b^{-1}((\mathcal{Q}f)(b)) = a_i \quad (b \in D_i, i = 1, 2, \cdots, P) \tag{24}$$

which can be used to reconstruct all the poles with region $D_i \neq \varnothing$ belonging to them.

IV. THE ALGORITHM FOR IDENTIFYING POLES

On the basis of Theorem 1 and its corollary (24) a practically realizable method can be constructed for the reconstruction of system poles by using frequency-domain signal measurements. Two problems has to be solved within the procedure:

1) The design of the frequency points to obtain data and the estimation of Laguerre–Fourier coefficients.
2) Reconstruction the poles from the Laguerre–Fourier coefficients.

It is assumed that the frequency points where the measurements are to be performed can be assigned arbitrarily. The concept of estimating the Laguerre–Fourier coefficients is based on the unitary representation of the Blaschke group, that implies that the Laguerre–Fourier coefficients of a function f are equal to the Fourier coefficients of the function $U_b f$, see (8). This means that the coefficients can be computed by the evaluation of Fourier-integrals, in discrete case by Fast Fourier Transform (FFT). This results in a non-uniformly spaced sampling scheme in the frequency scale. This non-uniform scale depends on the b parameter of the Laguerre–system and can be constructed from the inverse of the argument-function associated with the Blaschke function B_b. Let the argument function of B_b be denoted by $\beta_b(t)$ and denote the value of β_b by s. The inverse function $t = \beta_b^{-1}(s)$ can be expressed in the form

$$t = \varphi + 2\arctan\Big(\frac{1-r}{1+r}\tan\frac{s-\gamma}{2}\Big) \tag{25}$$

where $b = re^{i\phi}$ and γ is a parameter chosen such a way that

$$\beta_b : [-\pi, \pi] \to [-\pi, \pi].$$

An adequate selection for γ is

$$\gamma = 2\arctan\Big(\frac{1-r}{1+r}\tan\frac{\varphi}{2}\Big),$$

see for details in [13]. The interval $[-\pi, \pi]$ of parameter t corresponds to the physical frequency band $[-f_N, f_N]$, where f_N denotes the Nyquist frequency associated with the sampling rule applied in the time-domain signals.

Hence the estimation step can be performed by executing the following steps:

1) Derive a nonuniform sampling scheme associated to parameter b in the frequency scale and obtain N frequency measurement points.
2) Compute the values of the unitary representation U_b.
3) Compute the Laguerre–Fourier coefficients by applying FFT.

The pole reconstruction procedure – by assuming that the number of poles is P – consist of the following steps:

1) using N frequency measurements, estimate the Laguerre–Fourier coefficients $\{c_n(b)\}$ for the basis parameter b.
2) Compute the quotient sequence $\{q_n(b) = \frac{c_{n+1}(b)}{c_n(b)}\}$ $(n = 0, 1, \ldots, N-1)$.

3) Estimate the limit $Q(b)$ according to (23) and (24) of the sequence $\{q_n\}$ – a practical method that ensures some tolerance to noise – can be the estimation of the mean value of the tail of sequence $\{q_n\}$ from starting from an adequately large index.
4) Apply hyperbolic transform by using the inverse group element of B_b. This step provides one out of the system poles, i.e. $B_{b^{-1}}(Q(b)) = a_i$ $(1 \le i \le P)$.

The above procedure identifies one of the poles. Since in a practical identification there is no a priori knowledge on the pole locations, to find all the poles a sequence of parameters b should be selected and the procedure should be repeated. The selection of b parameters can be arbitrary, or one can use any practically suitable scheme.

The poles that has no nonempty regions D_i cannot directly be found. An obvious method to identify them can be the removal of the already identified poles. This can be done by filtering the input signal data, or to put the known poles into an orthogonal rational basis.

There are various options to implement the above procedure numerically, but these are not detailed here. The complete identification and pole reconstruction procedure has been realized for test purposes in Matlab® environment, however the complexity of computations required allow realizations either in embedded platforms.

V. NUMERIC EXAMPLES

In the current phase of the research some simulation examples are provide with the purpose of conceptual testing of methods. The procedure is the following.

- Fix a set of system poles a_i and the associated coefficients λ_i of the fractional terms in (12).
- Compute the function values according to the form (12) in the specific scale used in the analysis, i.e. the nonuniform scale generated by b (this step substitutes the real measurement).
- Perform the pole identification process as described in Section IV.
- Test the result qualitatively, and compute the errors in the function reconstruction.

The system to be identified is specified by 3 poles, one real pole with inverse pole position $a_1 = 0.8$ and residue $\lambda_1 = 1.5$, as well as a conjugated complex pair of poles in position $a_{2,3} = 0.8 * e^{\pm i\frac{\pi}{4}}$, with associated residues $\lambda_{2,3} = 1$.

Figure 3 and 5 presents a visualization of the iteration processes for finding specific poles. The poles that belong to a given a_i and the Laguerre representation are drawn by grayscale shaded and white circles, respectively. The elements of the quotient sequence generated by (15) are transformed by the hyperbolic transform $B_{b^{-1}}$ to locate them at the same region where the poles are located, and these are drawn by white points in the Figures. Furthermore, the D_i regions, that are given in (17), and belong to several poles a_i are visualized in the figures with shades identical with those of poles.

The pole to be identified in Example 1 is a_1, hence the parameter b is selected to lie in the region D_2, i.e. $b = 0.2e^{i\frac{15\pi}{16}}$. Similarly, in Example 2 $b = 0.7e^{-i\frac{\pi}{2}}$ has been selected in the region D_2 with the purpose to find a_2.

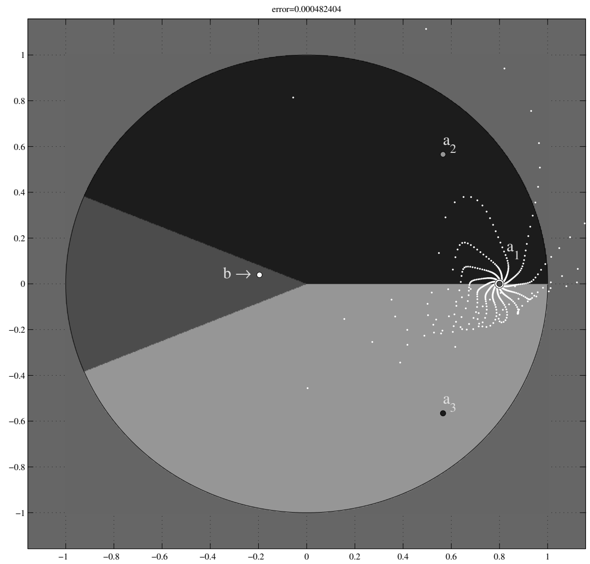

Fig. 3. Example1: – Finding pole a_1.

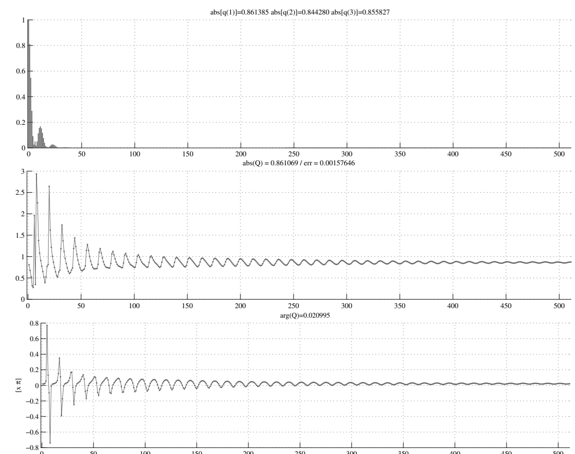

Fig. 4. Example1 – Abs of L-F coefficients, Abs–Phase of sequence q_n.

It can be observed that the transformed sequences of the quotients converge to the specific poles in both examples. The convergence can be checked on the lower two diagrams in Figures 4 and 6 for both examples respectively. The absolute value and the phase of the quotient sequences has been plotted against the indices. The upper diagram in these figures depicts the absolute value of the Laguerre-Fourier coefficients belonging to the specific representation.

The reconstruction error – defined as a root-mean-square difference – is rather small in both examples, typically it falls in the magnitude $10^{-5} \ldots 10^{-7}$.

The third pole a_3 can be identified analogously to a_2, due to the symmetry of the current pole locations. The identified poles represent the function f with high accuracy, f can be reconstructed by them with root-means-quare error in the magnitude less than 10^{-5}.

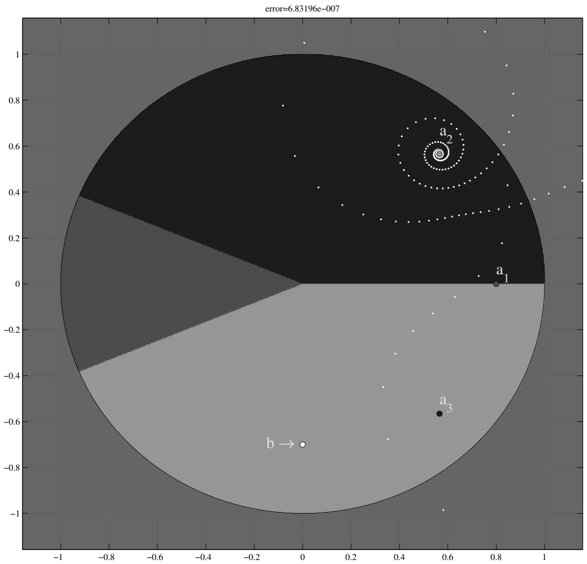

Fig. 5. Example2 – Finding pole a_2.

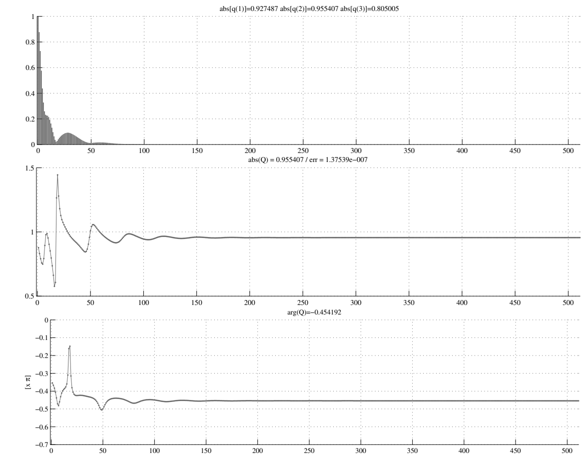

Fig. 6. Example2 – Abs of L-F coefficients, Abs–Phase of sequence q_n.

VI. CONCLUSIONS

In this paper a new method has been proposed that can efficiently be used to identify the poles in a linear system from frequency domain data. The discrete rational transfer function is represented in a rational Laguerre–basis, where the basis elements can be expressed by powers of the Blaschke–function. This function can be interpreted as a congruence transform on the Poincaré unit disc model of the hyperbolic geometry, leading to a nice geometric interpretation of the identification algorithm. The identification of poles can be done on the basis of Theorem 1; the reconstruction of a pole is given as a hyperbolic transform of the limit of a sequence formed of quotients of the Laguerre–Fourier coefficients belonging to the function. The Laguerre-Fourier coefficients can be estimated from frequency domain data by using an efficient FFT-based algorithm Convergence results in hyperbolic metrics has been given. The full procedure has been illustrated by simulation examples.

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