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Performance and average sampling period of sub-optimal triggering event in event triggered state estimation

Lichun Li and Michael Lemmon

Abstract—**Event-triggered approaches to control and estimation have the sensor transmit processed information when a measure of information 'novelty' exceeds a threshold. Prior work has empirically demonstrated that event-triggered systems may have significantly longer average sampling intervals than comparably performing periodically triggered systems. There are, however, few results that analytically characterize the tradeoff that event-triggering introduces between average sampling period and system performance. This paper examines that tradeoff for a sub-optimal solution to the constrained state-estimation problem considered by Xu and Hespanha. The sub-optimal solution is comparable to that used by Cogill, and extends the earlier work to unstable systems. In particular, the paper derives a sub-optimal solution that guarantees the specified least average sampling period. The paper also derives upper and lower bounds on the event-triggered estimator performance. Simulation results are used to demonstrate the utility of these bounds.**

I. Introduction

Due to the digital nature of communication networks, the feedback flows in networked systems consist of discrete packets of information. Traditional networked control systems have used *periodic* feedback flows; where the time between consecutive packets is constant. There has, however, been recent interest in the impact that sporadic feedback flows have on the performance of such systems. A *sporadic* flow is one in which the time between consecutive packets is bounded but not necessarily constant. Such sporadic flows may arise due to variations in the reliability of the feedback channel or it may arise in an intentional manner as is the case in event-triggered systems.

An *event-triggered* system is one where sensor information is transmitted when a measure of information 'novelty' exceeds a threshold. On an intuitive level, event-triggering can be seen as only using the channel when there is something 'novel' to transmit. The hope is that such an approach to channel utilization will result in longer average sampling periods without significantly compromising the networked system performance. This fact was empirically demonstrated for event-triggered control systems using constant thresholds in [1], [2]. It has also been empirically demonstrated for state dependent thresholds for input-to-state stable (ISS) [3] or L2-stable [4] control systems. While this prior work suggests that event-triggering uses fewer communication resources than periodically triggered systems for comparable levels of performance, there has been little substantive work analytically investigating this tradeoff between performance and average sampling period. The purpose of this paper is to analytically examine the tradeoff between performance and average sampling period for event-triggered state estimation systems.

The *state estimation* problem considered in this paper was originally studied in [5]. That work considered a system in which a local sensor observing a discrete-time process transmits the observed state information to a remote observer. The problem is when to transmit the sensor information to the remote observer to minimize the mean square estimation error discounted by the cost of transmitting the data. The optimal decision logic was derived in [5], but computing the thresholds used in that decision was computationally complex. A simpler sub-optimal approach was proposed in [6] which was able to bound the difference in the performance achieved by the optimal and sub-optimal decision logics for stable systems. Since the proposed optimization metric is explicitly discounted by the cost of transmission, it implicitly considers the tradeoff between the performance and the sampling period. That tradeoff, however, was never made explicit in the earlier papers.

This paper re-examines the problem in [5] using a sub-optimal solution similar to that proposed in [6], and extends the earlier work in [6] to unstable systems. The paper's main results are the design of quadratic sub-optimal triggering events that guarantee the required least average sampling period and explicit lower and upper bounds on discounted mean square estimation error.

II. Problem statement

The event-triggering problem assumes that a sensor is observing an observable linear discrete-time process. The process state $x : \mathbb{Z}^+ \to \mathbb{R}^n$ ($\mathbb{Z}^+ = 0, 1, \cdots$) satisfies the difference equation

$$x(k) = Ax(k-1) + w(k-1)$$

for $k = 1, 2, \cdots$ where $A \in \mathbb{R}^{n\times n}$, $w : \mathbb{Z}^+ \to \mathbb{R}^n$ is a zero mean white Gaussian noise process with variance W. The initial state, x_0, is assumed to be a Gaussian random variable with mean μ_0 and variance Π_0. The sensor generates a measurement $y : \mathbb{Z}^+ \to \mathbb{R}^m$. The sensor measurement at time k is

$$y(k) = Cx(k) + v(k)$$

for $k \in \mathbb{Z}^+$ and where $v : \mathbb{Z}^+ \to \mathbb{R}^m$ is another zero mean white Gaussian noise process with variance V. We assume that the two noise processes w,v and the initial state x_0 are

Lichun Li and Michael Lemmon are with the department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA. (e-mail:lli3,lemmon@nd.edu). We acknowledge the partial financial support of the National Science Foundation (ECCS-0925229).

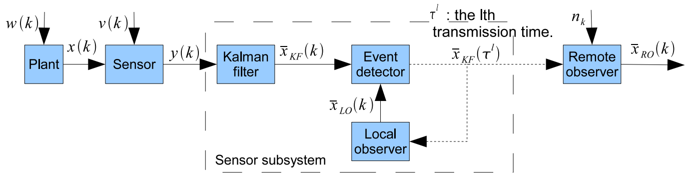

Fig. 1. Structure of event triggered networked state estimator

independent with each other. The process and sensor blocks are shown on the left hand side of Figure 1. In this figure, the output of the sensor feeds into a sensor subsystem that decides when to transmit information to a remote observer. The subsystem consists of three components: a *Kalman filter*, a *local observer* and an *event detector*.

Let $\mathcal{Y}(k) = \{y(0), y(1), \cdots, y(k)\}$ denote the measurement information available at step k. The *Kalman filter* generates a state estimate $\overline{x}_{KF} : \mathbb{Z}^+ \to \mathbb{R}^n$ that minimizes the weighted mean square estimation error (MSEE) $E\left[\|x(k) - \overline{x}_{KF}(k)\|_M^2 \,|\, \mathcal{Y}(k)\right]$ at each step conditioned on all of the sensor information received up to and including step k, where $M \geq 0$ is the weight matrix and $\|\theta\|_M^2 = \theta^T M \theta$. Let $M = P_M^T P_M$. For the process under study $\overline{x}_{KF}$ satisfies

$$\overline{x}_{KF}(k) = A\overline{x}_{KF}(k-1) + L\left(y(k) - CA\overline{x}_{KF}(k-1)\right),$$

where $L = XC^T(CXC^T + V)^{-1}$, and X satisfies the discrete linear Riccati equation

$$AXA^T - X - AXC^T(CXC^T + V)^{-1}CXA^T + W = 0.$$

The steady state estimation error $\overline{e}_{KF}(k) = x(k) - \overline{x}_{KF}(k)$ is a Gaussian random variable with zero mean and weighted variance $E(\overline{e}_{KF} M \overline{e}_{KF}^T) = Q = (I - LC)X$.

Let $\{\tau^\ell\}_{\ell=1}^{\infty}$ denote a sequence of increasing times ($\tau^\ell \in [0, +\infty]$) when information is transmitted from the sensor to the local and the remote observers. We require that τ^ℓ is forward progressing, i.e. for any $k \geq 0$, there always exists an ℓ such that $\tau^\ell \geq k$. Let $\overline{\mathcal{X}}(k) = \left\{\overline{x}_{KF}(\tau^1), \overline{x}_{KF}(\tau^2), \ldots, \overline{x}_{KF}(\tau^{\ell(k)})\right\}$ denote the filter estimates that are transmitted to the local and the remote observers by step k where $\ell(k) = \max\left\{\ell : \tau^\ell \leq k\right\}$. We can think of this as the 'information set' available to both the local observer and the remote observer at time k. The local observer generates a posteriori estimate $\overline{x}_{LO} : \mathbb{Z}^+ \to \mathbb{R}^n$ of the process state that minimizes the weighted MSEE, $E\left[\|x(k) - \overline{x}_{LO}(k)\|_M^2 \,|\, \overline{\mathcal{X}}_k\right]$, at time k conditioned on the information received up to and including time k. The a priori estimate of the local observer, $\overline{x}_{LO}^- : \mathbb{Z}^+ \to \mathbb{R}^n$, minimizes $E\left[\|x(k) - \overline{x}_{LO}^-(k)\|_M^2 \,|\, \overline{\mathcal{X}}_{k-1}\right]$, the weighted MSEE at time k conditioned on the information received up to and including step $k-1$. These estimates take the form

$$\begin{aligned}
\overline{x}_{LO}^-(k) =& A\overline{x}_{LO}(k-1) \\
\overline{x}_{LO}(k) =& \begin{cases} \overline{x}_{LO}^-(k), & \text{if no transmission at step } k; \\ \overline{x}_{KF}(k), & \text{otherwise ,} \end{cases}
\end{aligned}$$

where $\overline{x}_{LO}^-(0) = \mu_0$.

Let $e_{KF,LO}^-(k) = \overline{x}_{KF}(k) - \overline{x}_{LO}^-(k)$ and $S(k) \subseteq \mathbb{R}^n$ be a *triggering set* at step k. The *event detector* detects the a priori gap $e_{KF,LO}^-(k)$ and compares the gap with the triggering set $S(k)$. If the gap is inside the triggering set $S(k)$, then no data is transmitted. Otherwise, the state estimate in Kalman filter $\overline{x}_{KF}(k)$ is sent to both the local and the remote observers.

The *remote observer* and the local observer have similar behavior. It produces an a priori state estimate $\overline{x}_{RO}^-(k)$ and an a posteriori state estimate $\overline{x}_{RO}(k)$ to minimize the weighted MSEE at step k based on the information received by step $k-1$ and by step k with weight matrix M, respectively. Because there is communication error, the remote observer receives the corrupted state estimate of the Kalman filter when transmission occurs. The dynamics of the state estimate $\overline{x}_{RO}^-(k)$ and $\overline{x}_{RO}(k)$ in the remote observer are

$$\begin{aligned}
\overline{x}_{RO}^-(k) =& A\overline{x}_{RO}(k-1) \\
\overline{x}_{RO}(k) =& \begin{cases} \overline{x}_{RO}^-(k), & \text{if no transmission at step } k; \\ \overline{x}_{KF}(k) + n(k), & \text{otherwise ,} \end{cases}
\end{aligned}$$

where $\overline{x}_{RO}^-(0) = \mu_0$, $n(k)$ is a zero mean white Gaussian noise with variance N and independent with w and v.

The *communication* between the sensor and the remote observer is limited in the sense that the communication channel can only reliably transmit a limited number of packets over the channel. This limitation on channel capacity means that the average interval between any consecutive packets has to be greater than a number $T_r \geq 1$. Formally, we express it as

$$\min\{t : E(e_{KF,LO}^-(t+\tau^\ell)) \notin S(t+\tau^\ell)\} \geq T_r, \forall \ell \in \mathbb{Z}^+. \quad (1)$$

Let $\mathcal{S}$ be the collection of all triggering sets. The average cost is

$$J(\{S(k)\}_{k=0}^{\infty}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\left(c(\overline{e}_{RO}^T(k), S(k))\right), \quad (2)$$

where $\overline{e}_{RO}(k) = x(k) - \overline{x}_{RO}(k)$ is the remote state estimation error, $\lambda \in \mathbb{R}^+$ is the communication price and the cost function $c : \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^+$ is defined as

$$c(\overline{e}_{RO}^T(k), S(k)) = \|\overline{e}_{RO}^T(k)\|_M^2 + \lambda 1_{e_{KF,LO}^-(k) \notin S(k)},$$

which is the weighted mean square estimation error discounted by the cost of transmitting data with $1_{\{\cdot\}}$ a characteristic function.

Our objective is to find the optimal triggering sets $\{S(k)\}_{k=0}^{\infty}$ to minimize the average cost $J\left(\{S(k)\}_{k=0}^{\infty}\right)$ subject to the communication requirement (1), and the optimal cost is denoted by J^*.

III. Main results

For the convenience of the rest of this paper, we define

$$\begin{aligned}
e_{KF,LO}^-(k) =& \overline{x}_{KF}(k) - \overline{x}_{LO}^-(k), \\
e_{KF,LO}(k) =& \overline{x}_{KF}(k) - \overline{x}_{LO}(k), \\
e_{LO,RO}^-(k) =& \overline{x}_{LO}^-(k) - \overline{x}_{RO}^-(k), \\
e_{LO,RO}(k) =& \overline{x}_{LO}(k) - \overline{x}_{RO}(k).
\end{aligned}$$

The variances of these random variables are denoted by $U^-_{KF,LO}(k)$, $U_{KF,LO}(k)$, $U^-_{LO,RO}(k)$ and $U_{LO,RO}(k)$, respectively. Note that $\overline{e}_{RO}(k) = (\overline{e}_{KF} + e_{KF,LO} + e_{LO,RO})(k)$. Since $\overline{e}_{KF}(k)$, $e_{KF,LO}(k)$ and $e_{LO,RO}(k)$ are uncorrelated with each other, it can be shown that

$$\begin{aligned} J_a(\{S(k)\}_{k=0}^{\infty}) =& J(\{S(k)\}_{k=0}^{\infty}) - tr(Q) \\ =& \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} E(c_a(e^-_{KF,LO}(k), S(k))), \end{aligned}$$

where

$$\begin{aligned} c_a(e^-_{KF,LO}(k), S(k)) = \left[\|e^-_{KF,LO}(k)\|^2_M + tr(MU^-_{LO,RO}(k))\right] \\ \cdot 1_{e^-_{KF,LO}(k)\in S(k)} + [\lambda + tr(MN)]\, 1_{e^-_{KF,LO}(k)\notin S(k)}. \end{aligned} \quad (3)$$

So finding $\{S(k)\}_{k=0}^{\infty}$ to minimize $J(\{S(k)\}_{k=0}^{\infty})$ in (2) subject to the communication requirement (1) is equivalent to finding $\{S(k)\}_{k=0}^{\infty}$ to minimize $J_a(\{S(k)\}_{k=0}^{\infty})$ with (1) satisfied, and the optimal cost of $J_a(\{S(k)\}_{k=0}^{\infty})$ is denoted by J_a^*. The problem stated above is an optimal average cost problem, and a method for solving it was given in [7].

A. The optimal cost and upper and lower bounds on it

This subsection states the optimal average cost and the corresponding optimal triggering sets in Lemma 3.1. Then, an upper bound on the cost of any triggering sets $\{S(k)\}_{k=0}^{\infty}$ is given in Lemma 3.4. Finally, Lemma 3.5 presents a lower bound on the optimal cost. The triggering sets discussed in this subsection can be any subsets of $\mathbb{R}^n$, and the next subsection will focus explicitly on quadratic ones.

Lemma 3.1: If there exist two sequences of bounded functions $\{J_k : \mathbb{R}^n \to \mathbb{R}\}$ and $\{h_k : \mathbb{R}^n \to \mathbb{R}\}$ such that

$$J_{k+1}(e^-_{KF,LO}(k)) + h_k(e^-_{KF,LO}(k)) = G\left(h_{k+1}(e^-_{KF,LO}(k))\right)$$

for all $k \in \mathbb{Z}^+$, where

$$\begin{aligned} G\left(h(\theta)\right) =& \min_{S(k)} \Big\{ E(h(e^-_{KF,LO}(k+1))|e_{KF,LO}(k) = \theta) \\ &+ c_a(\theta, S(k)) \Big\}, \end{aligned}$$

then the optimal cost is

$$J_a^* = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} E(J_{k+1}(e^-_{KF,LO}(k))), \quad (4)$$

and the optimal triggering set

$$\begin{aligned} S^*(k) =& \Big\{\theta : E(h_{k+1}(e^-_{KF,LO}(k+1))|e_{KF,LO}(k) = \theta) \\ &+ \|\theta\|^2_M + tr(MU^-_{LO,RO}(k)) \le \lambda + tr(MN) \\ &+ E(h_{k+1}(e^-_{KF,LO}(k+1))|e_{KF,LO}(k) = 0)\Big\}. \end{aligned} \quad (5)$$

Proof: Given any $S(k)$,

$$\begin{aligned} & J_{k+1}(e^-_{KF,LO}(k)) + h_k(e^-_{KF,LO}(k)) \\ \le & E\left(h_{k+1}(e^-_{KF,LO}(k+1))|e^-_{KF,LO}(k)\right) + c_a(e^-_{KF,LO}(k)). \end{aligned}$$

Taking the expectation of both sides, we have

$$\begin{aligned} & E(J_{k+1}(e^-_{KF,LO}(k)) + E\left(h_k(e^-_{KF,LO}(k))\right) \\ \le & E(c_a(e^-_{KF,LO}(k))) + E\left(h_{k+1}(e^-_{KF,LO}(k+1))\right). \end{aligned}$$

Then adding the inequalities from step 0 to $N-1$ and taking the limit of N as it goes to infinity, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} E(J_{k+1}(e^-_{KF,LO}(k))) \le J_a(S(k)).$$

We know that the equality holds if $S(k) = S^*(k)$, so equation (4) holds and the optimal triggering set is (5). ∎

Remark 3.2: The sensor only has two choices: transmit information or not, and the optimal triggering sets reflect very simple logic which says the sensor should choose the one incurring less cost. If you pay close attention on the form of the optimal triggering set, you will find that the left hand side of the inequality in (5) is the cost introduced by the decision of not transmitting while the right hand side is the cost introduced by the decision of transmitting. If $e^-_{KF,LO}$ lies in the optimal triggering set $S_s^*(k)$, or in other words if the cost of not transmitting information is less, then the filtered state $\overline{x}_{KF}(k)$ should not be transmitted. Otherwise, transmission from sensor to remote observer occurs.

Remark 3.3: If the optimal strategy exists, it must be time varying, because the cost function c_a is a time varying function. If there is no communication noise, i.e. $N = 0$, the optimal strategy is the same as the strategy in [5] with time invariant function h_k and time invariant constant J_k.

It is difficult to calculate the optimal triggering sets $\{S^*(k)\}_{k=0}^{\infty}$ described in (5). There is, therefore, great interest in identifying computable approximations $\{S(k)\}_{k=0}^{\infty}$ of the optimal triggering sets . To characterize the performance of $\{S(k)\}_{k=0}^{\infty}$, an upper bound on the cost of $\{S(k)\}_{k=0}^{\infty}$ and the difference between the cost and the optimal cost should be derived. Lemma 3.4 and 3.5 derive an upper bound on the cost of $\{S(k)\}_{k=0}^{\infty}$ and a lower bound on the optimal cost, respectively. These two bounds can be used to characterize the performance of $\{S(k)\}_{k=0}^{\infty}$.

Lemma 3.4: Given the triggering set $\{S(k)\}_{k=0}^{\infty}$, if there exists a sequence of bounded function $\{f_k : \mathbb{R}^n \to \mathbb{R}\}$ and a sequence of finite constants $\{\overline{J}_k\}$ such that

$$\begin{aligned} & E\left(f_{k+1}(e^-_{KF,LO}(k+1))|e^-_{KF,LO}(k) = \theta, S(k)\right) \\ & + c_a\left(\theta, S(k)\right) \le \overline{J}_{k+1} + f_k(\theta), \forall k \in \mathbb{Z}^+ \end{aligned} \quad (6)$$

then

$$J_a(\{S(k)\}_{k=0}^{\infty}) \le \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \overline{J}_k < \infty \quad (7)$$

Lemma 3.5: If there exists a sequence of bounded function $\{g_k : \mathbb{R}^n \to \mathbb{R}\}$ and a sequence of nonnegative constants $\{\underline{J}_k\}$ such that

$$\underline{J}_{k+1} + g_k(e^-_{KF,LO}(k)) \le G\left(g_{k+1}(e^-_{KF,LO}(k))\right), \quad (8)$$

for any $k \in \mathbb{Z}^+$, then

$$J_a(\{S(k)\}_{k=0}^{\infty}) \geq J_a^* \geq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \underline{J}_k \geq 0$$

Proof: We can follow the same steps in the proof of Lemma 3.1 to get the two lemmas above. ■

Lemma 3.4 and 3.5 are very similar to Lemma 4 and 5 in [6]. One of the differences is that we allow time varying functions f_k, g_k and time varying constants $\overline{J}_k$ and $\underline{J}_k$. If there is no communication noise, then the cost function c_a becomes time invariant and all the time varying parameters f_k, $\overline{J}_k$, g_k and $\underline{J}_k$ should be time invariant, too. The other difference is that the functions f_k and g_k need to be bounded from both below and above while [6] only required that f_k was bounded from below and g_k was bounded from above. It seems that we have a more strict assumption, but in fact the property of being bounded from both below and above is an intrinsic property of the cost function, $c_a \in [0, \lambda + tr(MU_{LO,RO}^-)]$. Since f_k and g_k are approximations of the cost function, they should be bounded, too.

Lemma 3.4 and 3.5 can be used for any triggering sets $\{S(k)\}_{k=0}^{\infty}$. The next subsection makes use of these two lemmas, and considers the case where these sets are defined by quadratic forms. We are interested in quadratic sets, because they are easy to compute.

B. Quadratic sets, their average period and performance

The quadratic sets are analyzed in this subsection. Theorem 3.6 first states how to design quadratic sets such that the communication requirement (1) is satisfied, and then give the upper bounds of the cost of the quadratic sets and its difference from the optimal cost. Please see Section VI for the proof.

Theorem 3.6: Given a quadratic triggering set

$$S(k) = \{e_{KF,LO}^- : \|e_{KF,LO}^-\|_H^2 \leq \lambda + tr(MN) - \zeta(k)\}, \tag{9}$$

where matrix $H \geq 0 \in \mathbb{R}^{n\times n}$ satisfies the Lyapunov inequality

$$\frac{A^T H A}{1+\delta^2} - H + \frac{M}{1+\delta^2} \leq 0, \tag{10}$$

for some $\delta^2 \geq 0$, and

$$\zeta(k) = \frac{\delta^2(\lambda + tr(MN)) + tr(MU_{LO,RO}^-(k)) + tr(HR)}{1+\delta^2} \tag{11}$$

where $R = L(CAQA^TC^T + CWC^T + V)L^T$ and

$$\lambda \geq \max_{t=1,\cdots,T_r-1} \left[(1+\delta^2)\sum_{i=1}^{t} tr(HA^{t-i}R(A^T)^{t-i}) - tr(MN) + tr(MA^tR(A^T)^t) + tr(HR)\right], \tag{12}$$

the statements below are true.

1) the communication requirement (1) is guaranteed;

2) $J_a(\{S(k)\}_{k=0}^{\infty})$ is bounded above by

$$\begin{aligned}&\overline{J}_a(\{S(k)\}_{k=0}^{\infty}) \qquad (13)\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} E(f_k(e_{KF,LO}^-(k))|e_{KF,LO}(k-1)=0)\\ &\leq \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\min\{tr(HR)+\zeta(k), \lambda + tr(MN)\}\end{aligned}$$

where $f_k(\theta) = \min\{\|\theta\|_H^2 + \zeta(k), \lambda + tr(MN)\}$;

3) The difference between the cost of $\{S(k)\}_{k=0}^{\infty}$ and the optimal cost $\overline{J}_a(\{S(k)\}_{k=0}^{\infty}) - J_a^*$ is bounded above by

$$\overline{D} = \min\{tr(YR) + D_1, \lambda + tr(MN) - D_2\},$$

where

$$D_1 = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\max\{\zeta(k) - tr(MU_{LO,RO}^-(k)), 0\},$$

with $Y \geq 0$, the matrix which has the smallest trace such that $Y \geq H - M$,

$$D_2 = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\min\{tr(MU_{LO,RO}^-(k)), \lambda + tr(MN)\}.$$

Remark 3.7: For any A and $M > 0$, there always exists an $H \geq 0$ and a $\delta^2 \geq 0$ such that the Lyapunov inequality (10) holds. We should notice that the greatest singular value of A, $\overline{\sigma}(A)$, is always greater than or equal to the absolute value of any eigenvalue of A. So if we set δ^2 to be the value such that $\overline{\sigma}(A) \leq \sqrt{1+\delta^2}$, $A/\sqrt{1+\delta^2}$ is always stable, and there always exists $H \geq 0$ such that (10) holds for any semi-positive definite matrix M.

The lower bound on communication price λ in (12) is calculated numerically. Basically, the value in the bracket of (12) is calculated from step 1 to step $T_r - 1$, and λ is chosen to be the greatest one. But as T_r grows, one may want to find a more efficient way to calculate λ. Corollary 3.8 gives an explicit description of λ.

Proposition 3.8: Let $P_H^T P_H = H$, $P_R P_R^T = R$, $P_M^T P_M = M$ and $P_N P_N^T = N$. If

$$\begin{aligned}\lambda &= tr(HR) - tr(MN)\\ &+ \begin{cases}(1+\delta^2)n\overline{\sigma}^2(P_H)\overline{\sigma}^2(P_R)\frac{1-\overline{\sigma}^{2(P_r-1)}(A)}{1-\overline{\sigma}^2(A)}, & \text{if } \overline{\sigma}(A) \neq 1;\\ (1+\delta^2)n\overline{\sigma}^2(P_H)\overline{\sigma}^2(P_R)(P_r-1)\overline{\sigma}^2(A) & \text{otherwise,}\end{cases}\\ &+ \begin{cases}n\overline{\sigma}(P_M)\overline{\sigma}(P_N), & \text{if } \overline{\sigma}(A) \leq 1;\\ n\overline{\sigma}(P_M)\overline{\sigma}(P_N)\overline{\sigma}^{2(T_r-1)}(A), & \text{otherwise,}\end{cases}\end{aligned} \tag{14}$$

then the inequality (12) holds.

Proof: Equation (14) can be derived by finding the upper bound of the right side of inequality (12). From the fact that $tr(HA^iR(A^T)^i) = tr(P_HA^iP_R(A^T)^iP_H^T) \leq n\overline{\sigma}^2(P_HA^iP_R) \leq n\overline{\sigma}^2(P_H)\overline{\sigma}^2(P_R)\overline{\sigma}^{2i}(A)$, it can be shown that (14) is an upper bound on the right hand side of (12), and Corollary 3.8 is true. ■

IV. Simulation Results

An example is used to demonstrate that the proposed quadratic triggering sets can guarantee the communication requirement (1) and that the average cost triggered by the quadratic sets is bounded by the upper and lower bound derived in Theorem 3.6. We then compare the average cost of the quadratic sets in this paper against the average cost of the quadratic sets used in [6].

Let's consider the system with A to be $\begin{bmatrix} 0.95 & 1 \\ 0 & 1.01 \end{bmatrix}$, and C to be $\begin{bmatrix} 0.1 & 1 \end{bmatrix}$. The variances of the system noises are $W = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $V = 0.3$, and $N = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}$. The weight matrix M is chosen to be an identity matrix.

Given δ to be the greatest singular value of A, we calculate the quadratic triggering sets, and run the state estimation system with the quadratic triggering sets. In Figure 2, T_{sim} and J_{sim} are the average sampling period and the average cost of quadratic triggering set. J_{up} and D_{up} are upper bounds of the average cost and difference from the optimal cost, respectively. In the top plot of Figure 2, the x-axis is the required least average sampling period T_r, and the y-axis is the average sampling period in our experiment. We can find that the average sampling period T_{sim} (solid line) is always greater than or equal to the required least average sampling period T_r (dashed line). In the bottom plot of Figure 2, the x-axis denotes the required least average sampling period, and y-axis denotes the average cost. It shows that the average cost J_{sim} (solid line) is always bounded from below by $J_{up} - D_{up}$ (dotted line), and bounded from above by J_{up} (dot-dashed line). The simulation results confirm the statements in Theorem 3.6.

Comparing against the result in [6] is of interest since they also approximated the optimal triggering set with a quadratic form. Since their results can only be applied to stable systems without communication noise, we let A be $\begin{bmatrix} 0.95 & 1 \\ 0 & 0.95 \end{bmatrix}$, and N be 0. Because there is no result in [6] showing how λ is related with communication requirement T_r, the comparison is made given the same communication price, λ. To derive our quadratic triggering set, we let $\delta^2 = 1.5$. Figure 3 shows the average costs of both quadratic triggering sets in this paper and [6]. The x-axis is the communication price λ, and the y-axis is the average cost. The average cost of the quadratic triggering set in this paper is indicated by the solid line, and the average cost of the quadratic triggering set in [6] is indicated by the dashed line. Figure 3 shows that these two costs are almost the same. Our quadratic triggering set, however, can be applied to unstable linear time invariant systems while the results in [6] can not.

V. Conclusion

This paper explicitly states the relationship between the performance of the state estimation system and the least average sampling period when communication is triggered by quadratic events. Upper and lower bounds on the cost of the quadratic triggering sets are derived. The simulation results agree on the theoretic results, and indicate that the quadratic set in this paper is comparable with the quadratic set used in [6] for stable systems while our triggering sets can be applied to unstable systems.

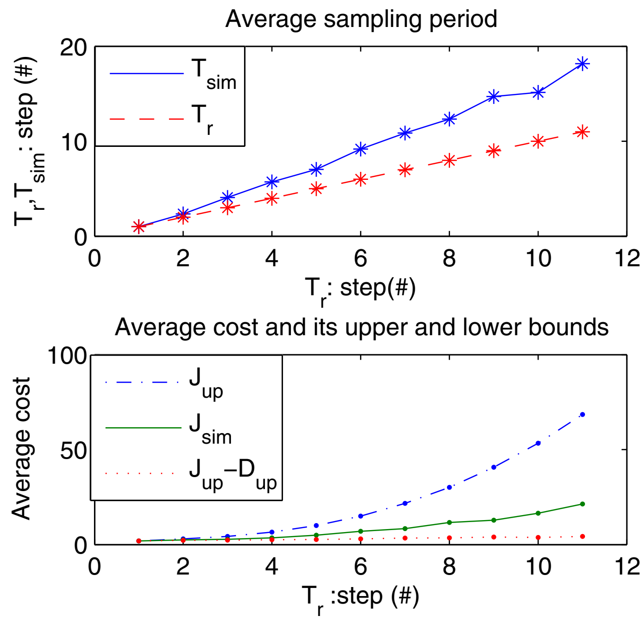

Fig. 2. The average sampling period of the quadratic sets and the upper and lower bounds of the average cost.

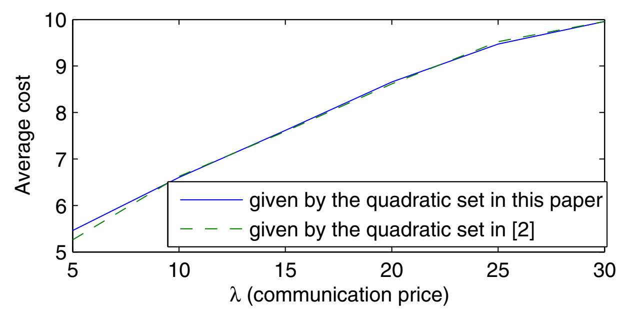

Fig. 3. Comparison of the average costs of the triggering sets in this paper and [2]

VI. Proof of Theorem 3.6

A. Proof of part 1)

To prove the first part, the communication requirement (1) needs to be rewritten as

$$E(\|e^{-}_{KF,LO}(k)\|_H^2) \le \lambda + tr(MN) - \zeta(k),$$

for any $k = \tau^\ell + 1, \cdots, \tau^\ell + T_r - 1$ and any $\ell \in \mathbb{Z}^+$. By applying $\zeta(k)$ into the inequality above, we can conclude that as long as

$$\lambda \geq (1+\delta^2)tr(HU^-_{KF,LO}(k)) + tr(MU^-_{LO,RO}(k)) + tr(HR) - tr(MN), \forall k = \tau^\ell + 1, \cdots, \tau^\ell + T_r - 1,$$

the communication requirement (1) can be guaranteed. Because during the inter sample interval, no transmission occurs, $U^-_{KF,LO}(k)$ and $U^-_{LO,RO}(k)$ is iteratively calculated under the condition that there is no transmission from $\tau^\ell + 1$ to $\tau^\ell + T_r - 1$. Let $t = k - \tau^\ell$. $U^-_{KF,LO}(k) = \sum_{i=1}^{t-1} A^{t-i} R (A^T)^{t-i}$, and $U^-_{LR,RO}(k) = A^t N (A^T)^t$. Therefore, if the inequality (12) holds, the communication requirement (1) can be satisfied.

B. Proof of part 2)

To prove part 2), it is sufficient to show that the inequality (6) holds for any k with f_k defined in Theorem 3.6 and

$$\overline{J}_k = E(f_k(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = 0).$$

In the case of $\|e^-_{KF,LO}(k)\|^2_H \leq \lambda + tr(MN) - \zeta(k)$, no transmission occurs at step k, so the right hand side of (6)

$$\begin{aligned}
&\leq \|e^-_{KF,LO}(k)\|^2_H + \|e^-_{KF,LO}(k)\|^2_{A^THA-H+M} + \zeta(k+1) \\
&\quad + tr(HR) + tr(MU^-_{LO,RO}(k)) \\
&\leq \|e^-_{KF,LO}(k)\|^2_H + \delta^2(\lambda + tr(MN) - \zeta(k)) \\
&\quad + tr(HR) + tr(MU^-_{LO,RO}(k)) + \zeta(k+1) \\
&\leq f_k(e^-_{KF,LO}(k) + \overline{J}_{k+1}.
\end{aligned}$$

The first step is taken from (3) and the fact that $E(\min(f,g)) \leq \min(E(f), E(g))$, the second step is derived from the fact that $\|e^-_{KF,LO}(k)\|^2_H \leq \lambda + tr(MN) - \zeta(k)$, and the third step is derived from how we define the $\zeta(k)$.

In the case of $\|e^-_{KF,LO}(k)\|^2_H > \lambda + tr(MN) - \zeta(k)$, transmission occurs. It is obvious to see that the right side of inequality (6) is less or equal to $f_k(e^-_{KF,LO}(k) + \overline{J}_{k+1}$.

Since the inequality (6) holds in any condition, from Lemma 3.4, we know that $J_a(\{S(k)\}^\infty_{k=0})$ is bounded above by $\overline{J}_a(\{S(k)\}^\infty_{k=0})$ defined in (13).

C. Proof of part 3)

If a lower bound on the optimal cost is found, an upper bound of the difference between the cost of quadratic sets and the optimal sets can be derived. The lower bound on the optimal cost is given in Lemma 6.1.

Lemma 6.1: The optimal cost J^*_a is bounded below by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E(g_k(e^-_{KF,LO}(k)) | e_{KF,LO}(k-1) = 0),$$

where $g_k(\theta) = \min\{\|\theta\|^2_M + tr(MU^-_{LO,RO}(k)), \lambda + tr(MN)\}$.

Proof: Let's define

$$\underline{J}_{k+1} = E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = 0).$$

By Lemma 3.5, Lemma 6.1 is true if

$$\underline{J}_{k+1}(e^-_{KF,LO}(k)) + g_k(e^-_{KF,LO}(k)) \leq G\left(g_{k+1}(e^-_{KF,LO}(k))\right).$$

The left side of the inequality equals to

$$\begin{aligned}
&\min\{\|e^-_{KF,LO}(k)\|^2_M + tr(MU^-_{LO,RO}(k)) \\
&\; + E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = 0), \\
&\lambda + tr(MN) + E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = 0)\},
\end{aligned}$$

and the right side of the inequality equals

$$\begin{aligned}
&\min\{\|e^-_{KF,LO}(k)\|^2_M + tr(MU^-_{LO,RO}(k)) \\
&\; + E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = e^-_{KF,LO}(k)), \\
&\lambda + tr(MN) + E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = 0)\}.
\end{aligned}$$

With the fact that

$$p(Z + \frac{\beta_1+\beta_2}{2} b_2\varepsilon_2 + \rho\varepsilon_2) = -p(Z + \frac{\beta_1+\beta_2}{2} b_2\varepsilon_2 - \rho\varepsilon_2),$$

where $Z^T R^+ \varepsilon_2 = 0$, $b_2 = (Ad_2)^T \varepsilon_2 \geq 0$ and $\rho \geq 0$. Let $Ad_2 = \sum_{i=1}^{n} b_i \varepsilon_i$, we can show that

$$\begin{aligned}
&E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = 0) \\
\leq &E(g_{k+1}(e^-_{KF,LO}(k+1)) | e_{KF,LO}(k) = e^-_{KF,LO}(k)),
\end{aligned}$$

the inequality (8) holds, and hence Lemma 6.1 is true. ■

Now that both the upper and lower bounds are given, the difference between them can be shown to be bounded by $\min\{D_1, D_2\}$, where D_1 and D_2 are defined in Theorem 3.6.

First, we know from part 2) of Theorem 3.6 and Lemma 6.1 that

$$\begin{aligned}
&J_a(\{S(k)\}^\infty_{k=0}) - J^*_a \leq \overline{J}_a(\{S(k)\}^\infty_{k=0}) - J^*_a \\
\leq &tr(YR) + \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \max\{\zeta(k) - tr(MU^-_{LO,RO}(k)), 0\}.
\end{aligned}$$

Next, noticing that $J(\{S(k)\}^\infty_{k=0}) \leq \lambda + tr(MN)$, and $J^* \geq \min\{tr(MU^-_{LO,RO}(k)), \lambda + tr(MN)\}$, $J(\{S(k)\}^\infty_{k=0}) - J^*$ is bounded above by D_2.

Therefore, part 3) is proven.

References

[1] K. Årzén, "A simple event-based PID controller," in *Proc. 14th IFAC World Congress*, vol. 18, 1999, pp. 423–428.

[2] K. Åström and B. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 2, 2002, pp. 2011–2016.

[3] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *Automatic Control, IEEE Transactions on*, vol. 52, no. 9, pp. 1680–1685, 2007.

[4] X. Wang and M. Lemmon, "Self-triggered feedback control systems with finite-gain L2 stability," *IEEE transactions on Automatic Control*, vol. 45, no. 3, pp. 452–467, 2009.

[5] Y. Xu and J. Hespanha, "Optimal communication logics in networked control systems," in *Decision and Control, 2005. CDC. 43rd IEEE Conference on*, vol. 4. IEEE, 2005, pp. 3527–3532.

[6] R. Cogill, S. Lall, and J. Hespanha, "A constant factor approximation algorithm for event-based sampling," in *American Control Conference, 2007. ACC'07.* IEEE, 2007, pp. 305–311.

[7] A. Arapostathis, V. Borkar, E. Fernández-Gaucherand, M. Ghosh, and S. Marcus, "Discrete-time controlled Markov processes with average cost criterion: a survey," 1991.