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Saturated Control of an Uncertain Euler-Lagrange System with Input Delay

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***Abstract*—This paper examines saturated control of a general class of uncertain nonlinear Euler-Lagrange systems with time-delayed actuation and additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. A Lyapunov-based stability analysis utilizing Lyapunov-Krasovskii functionals is provided to prove uniformly ultimately bounded tracking despite uncertainties in the dynamics.**

I. Introduction

Time delays are present in many industrial and biological processes and frequently exist in the transmission of information between different parts of a system. Motivated by performance and stability problems with time delayed systems and inspired by the classic results of Smith [1] and Artstein [2], solutions for input delayed control problems typically exploit predictive-based control methods. While several results have used variations of these methods to solve the input delay problem for linear systems with certain and uncertain dynamics [3]–[9], and nonlinear systems with exact model knowledge [10]–[14], there presently exists only a small number of results that examine the input delay problem for uncertain nonlinear systems. Specifically, recent results in [15] proposed the development of a predictor-based controller for a time-delayed actuation system with parametric uncertainty and/or additive bounded disturbances using Lyapunov-Krasovskii functionals to achieve a semi-global uniformly ultimately bounded tracking result.

Control signals are a function of the system states, and large initial conditions or unmodeled disturbances may cause the controller to exceed physical limitations. For systems with input delays, errors can build over the delay interval leading to large actuator demands, exacerbating potential problems with actuator saturation. Because degraded control performance and the potential risk of thermal or mechanical failure can occur when unmodeled actuator constraints are violated, control schemes which can ensure performance while operating within actuator limitations are motivated.

Saturated controllers for state-delay systems have been rigorously studied for both linear and nonlinear systems [16]–[20]. However, the majority of saturated controllers presently available for systems with input delays are based on linear plant models [19], [21]–[23] and only a few results are present for nonlinear systems (especially those with uncertainties).

The authors of [22] proposed a parametric Lyapunov equation-based low-gain feedback law which guarantees stability of a linear system with delayed and saturated control input. In [24], global uniform asymptotic stabilization is obtained with bounded feedback of a strict-feedforward linear system with delay in the control input. The authors were able to extend the result to an uncertain but disturbance-free strict-feedforward nonlinear system with delays in the control input in [25] using a system of nested saturation functions. The controller requires a nonlinear strict-feedforward dynamic system with parametric uncertainty, $h(t)$, which satisfies the following condition: $|h(x_{i+1}, x_{i+2}, ..., x_n)| \leq M\left(x_{i+1}^2, x_{i+2}^2, ..., x_n^2\right)$ where M denotes a positive real number when $|x_j| \leq 1, j = i+1, ..., n$. Unlike compensation-based delay methods, the design in [25] cleverly exploits the inherent robustness to delay in the particular structure of the feedback law and the plant. Krstic proposed a saturated compensator based approach in [26] which results in a nonlinear version of the Smith Predictor [1] with nested saturation functions. The controller is able to achieve quantifiable closed-loop performance by using an infinite dimensional compensator for strict-feedforward nonlinear systems with no uncertainties.

Although the work in [24]–[27] provides fundamental contributions to the input delay problem in feedforward systems, the applicability of these methods to general uncertain mechanical systems (e.g., modeled by Euler-Lagrange dynamics) is not clear. An attempt at designing a transformation to convert an Euler-Lagrange system into a feedforward system in [28] required exact model knowledge; thus, the technique is not applicable when the system parameters are unknown or the dynamics are uncertain, which implies that methods developed for feedforward systems with input delays may not be applicable to uncertain Euler-Lagrange dynamics.

The work presented in this paper introduces a new saturated control design that can predict/compensate for input delays in uncertain nonlinear Euler-Lagrange systems. Based on our previous non-saturated feedback work in [29], a continuous saturated controller is developed which allows the bound on the control to be known a priori and be adjusted by changing the feedback gains. The saturated controller is shown to guarantee uniformly ultimately bounded tracking despite a known, constant input delay, parametric uncertain-

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ties and additive disturbances, without the use of acceleration measurements. Unlike previous results that apply nonlinear combinations of saturated functions for stabilizing the closed loop system, the proposed controller is based on smooth hyperbolic functions that can easily be implemented in real-time applications. Efforts focus on developing a delay compensating auxiliary signal to obtain a time delay free open-loop error system and the construction of LK functionals to cancel the time delayed terms.

II. Dynamic Model and Properties

Consider the following input delayed Euler-Lagrange dynamics

$$M(q)\ddot{q}+V_m(q,\dot{q})\dot{q}+G(q)+F(\dot{q})+d(t)=u(t-\tau) \quad (1)$$

where $M(q)\in\mathbb{R}^{n\times n}$ denotes a generalized inertia matrix, $V_m(q,\dot{q})\in\mathbb{R}^{n\times n}$ denotes a generalized centripetal-Coriolis matrix, $G(q)\in\mathbb{R}^n$ denotes a generalized gravity vector, $F(\dot{q})\in\mathbb{R}^n$ denotes generalized friction, $d(t)\in\mathbb{R}^n$ denotes an exogenous disturbance, $u(t-\tau)\in\mathbb{R}^n$ represents the generalized delayed input control vector, where $\tau\in\mathbb{R}$ is a constant time delay, and $q(t),\dot{q}(t),\ddot{q}(t)\in\mathbb{R}^n$ denote the generalized states. The subsequent development is based on the assumptions that $q(t)$ and $\dot{q}(t)$ are measurable, $M(q)$, $V_m(q,\dot{q})$, $G(q)$, $F(\dot{q})$, $d(t)$ are unknown, the time delay constant, τ, is known and the control input vector, $u(t)$, and its past values (i.e., $u(t-\theta)\,\forall\theta\in[0\ \tau]$) are measurable. Throughout the paper, a time dependent delayed function is denoted as $\zeta(t-\tau)$ or ζ_τ. The following properties and assumptions are used in the subsequent development.

Property 1: The inertia matrix $M(q)$ is symmetric positive-definite, and satisfies the following inequality:

$$\underline{m}\,\|\xi\|^2\leq\xi^TM\xi\leq\bar{m}\,\|\xi\|^2,\quad\forall\xi\in\mathbb{R}^n$$

where $\underline{m},\bar{m}\in\mathbb{R}^+$ are known constants and $\|\cdot\|$ denotes the standard Euclidean norm.

Property 2: The inertia and centripetal-Coriolis matrices satisfy the following skew symmetric relationship

$$\xi^T\left(\frac{1}{2}\dot{M}(q)-V_m(q,\dot{q})\right)\xi=0,\quad\forall\xi\in\mathbb{R}^n$$

where $\dot{M}(q)$ is the time derivative of the inertia matrix.

Property 3: The centripetal-Coriolis, gravity, and friction terms in (1) can be bounded in the following manner

$$\|V_m(q,\dot{q})\|\leq\zeta_v\|\dot{q}\|,\quad\|G(q)\|\leq\zeta_g,\quad\|F\|\leq\zeta_f$$

where $\zeta_v,\zeta_g,\zeta_f\in\mathbb{R}$ are positive bounding constants.

Property 4: The centripetal-Coriolis matrix satisfies the following relationship

$$V_m(q,\xi)\upsilon=V_m(q,\upsilon)\xi\quad\forall\xi,\nu\in\mathbb{R}^n.$$

Assumption 1: The desired trajectory $q_d(t)\in\mathbb{R}^n$ is designed such that $q_d(t),q_d^{(i)}(t)\in\mathcal{L}_\infty$, where $q_d^{(i)}(t)$ denotes the ith time derivative for $i=1,2,3$.

Assumption 2: The nonlinear disturbance term and its first time derivative are bounded, i.e., $d(t),\dot{d}(t)\in\mathcal{L}_\infty$.

Remark 1. To aid the subsequent control design and analysis, the vector $Tanh(\cdot)\in\mathbb{R}^n$ and the matrix $Cosh(\cdot)\in\mathbb{R}^{n\times n}$ are defined as follows

$$Tanh(\xi)\triangleq[tanh(\xi_1),...,tanh(\xi_n)]^T, \quad (2)$$

$$Cosh(\xi)\triangleq diag\{cosh(\xi_1),...,cosh(\xi_n)\} \quad (3)$$

where $\xi=[\xi_1,...,\xi_n]^T\in\mathbb{R}^n$. Based on the definition of (2) and (3), the following inequalities hold $\forall\xi\in\mathbb{R}^n$ [30]:

$$\|\xi\|^2\geq\sum_{i=1}^n ln(cosh(\xi_i))\geq\frac{1}{2}tanh^2(\|\xi\|),$$

$$\|\xi\|>\|Tanh(\xi)\|,\quad\|Tanh(\xi)\|^2\geq tanh^2(\|\xi\|),$$

$$\xi^TTanh(\xi)\geq Tanh^T(\xi)Tanh(\xi). \quad (4)$$

III. Control Objective

The objective is to design an amplitude-limited, continuous controller that will enable the input delayed system in (1) to track a desired trajectory, q_d, despite uncertainties and bounded disturbances in the dynamic model. To quantify the control objective, a tracking error, denoted $e(t)\in\mathbb{R}^n$, is defined as

$$e\triangleq q_d-q. \quad (5)$$

To facilitate the subsequent analysis, a measurable filtered tracking error, denoted by $r(e,e_f,e_z,t)\in\mathbb{R}^n$, is defined as

$$r\triangleq\dot{e}+\alpha Tanh(e)+Tanh(e_f)-Be_z \quad (6)$$

where $\alpha\in\mathbb{R}^+$ is a known gain constant, $e_f(e,r,t)\in\mathbb{R}^n$ is an auxiliary signal whose dynamics are given by

$$\dot{e}_f\triangleq Cosh^2(e_f)(-kr+Tanh(e)-\gamma Tanh(e_f)) \quad (7)$$

where $e_f(0)=0$, $k,\gamma\in\mathbb{R}^+$ are constant control gains, and $e_z(t)\in\mathbb{R}^n$ is an auxiliary signal containing the time delays in the system, defined as

$$e_z\triangleq\int_{t-\tau}^t u(\theta)\,\mathrm{d}\theta. \quad (8)$$

From the definition in (8), the finite integral can be upper bounded as $\|e_z\|\leq\zeta_z$, where $\zeta_z\in\mathbb{R}^+$ is a known bounding constant provided the subsequently designed control input, $u(\cdot)$, is bounded. In (6), $B\in\mathbb{R}^{n\times n}$ is a known, symmetric, positive-definite constant gain matrix that satisfies the following inequality

$$\|B\|_\infty\leq\bar{b} \quad (9)$$

where $\bar{b}\in\mathbb{R}^+$ is a known constant. The error between B and $M^{-1}(q)$ is denoted by $\eta(q)\in\mathbb{R}^{n\times n}$ and is defined as

$$\eta\triangleq B-M^{-1} \quad (10)$$

that satisfies the following inequality

$$\|\eta\|_\infty\leq\bar{\eta} \quad (11)$$

where $\bar{\eta}\in\mathbb{R}^+$ is a known constant.

IV. CONTROL DEVELOPMENT

The open-loop error system can be obtained by premultiplying the time derivative of (6) by $M(q)$ and utilizing the expressions in (1), (5) and (10) to yield

$$\begin{aligned} M\dot{r} &= M\ddot{q}_d + V_m\dot{q} + G + F + d - u \\ &\quad -M\eta(u - u_\tau) + M\alpha Cosh^{-2}(e)\dot{e} \\ &\quad +MCosh^{-2}(e_f)\dot{e}_f. \end{aligned} \tag{12}$$

From (12) and the subsequent stability analysis, the control input, $u(e, e_f, t)$, is designed as

$$u \triangleq -kTanh(e_f) + Tanh(e) \tag{13}$$

where k was introduced in (7).

Remark 2. An important feature of the controller given by (13) is its applicability to the case where constraints exist on the available actuator commands. Note that the control law is bounded since an upper bound can be explicitly obtained as

$$\|u\| \leq (k+1) \cdot n$$

where n is the degree of u.

To facilitate the subsequent stability analysis, an auxiliary signal, $N_d(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$, is defined as

$$N_d \triangleq M_d\ddot{q}_d + V_{md}\dot{q}_d + G_d + F_d$$

where M_d, V_{md}, G_d, F_d denote $M(q_d) \in \mathbb{R}^{n\times n}$, $V_m(q_d, \dot{q}_d) \in \mathbb{R}^{n\times n}$, $G(q_d) \in \mathbb{R}^n$, $F(\dot{q}_d) \in \mathbb{R}^n$, respectively. The closed-loop error system is obtained by adding and subtracting $N_d(q_d, \dot{q}_d, \ddot{q}_d, t)$ to (12) and utilizing (6), (7), and (13) to yield

$$\begin{aligned} M\dot{r} &= -V_m r + \chi + S + kTanh(e_f) - Tanh(e) \\ &\quad -Mkr - M\eta(-kTanh(e_f) + Tanh(e)) \\ &\quad -M\eta(kTanh(e_{f\tau}) - Tanh(e_\tau)) \end{aligned} \tag{14}$$

where the auxiliary terms $\chi(e_1, e_f, r, e_z, t)$, $S(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$ are defined as

$$\begin{aligned} \chi \triangleq\ & \alpha MCosh^{-2}(e)(r - \alpha Tanh(e) - Tanh(e_f) + Be_z) \\ & +MTanh(e) - \gamma MTanh(e_f) \\ & +V_m(q, \dot{q}_d + \alpha Tanh(e) + Tanh(e_f) - Be_z) \quad (15) \\ & \times(\alpha Tanh(e) + Tanh(e_f) - Be_z) \\ & -V_m(q, r)(\dot{q}_d + \alpha Tanh(e) + Tanh(e_f) - Be_z) \\ & +V_m(q, \dot{q}_d)(\alpha Tanh(e) + Tanh(e_f)) \\ & +M\ddot{q}_d + V_m(q, \dot{q}_d)\dot{q}_d + G + F - N_d, \end{aligned}$$

$$S \triangleq N_d + d. \tag{16}$$

Using Assumptions 1 and 2, the following inequality can be developed based on the expression in (16)

$$\|S\| \leq \bar{s} \tag{17}$$

where $\bar{s} \in \mathbb{R}^+$ is a known constant. The structure of (14) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. Using the Mean Value Theorem, Properties 1, 3, 4, and (4), the expression in (15) can be upper bounded as [31]

$$\|\chi\| \leq \bar{\chi}\|z\| \tag{18}$$

where $\bar{\chi} \in \mathbb{R}^+$ is a known bounding constant, and $z(e, e_f, r, e_z, P, Q, R) \in \mathbb{R}^{4n+3}$ is defined as

$$z \triangleq \begin{bmatrix} e^T & Tanh^T(e_f) & r^T & e_z^T & \sqrt{P} & \sqrt{Q} & \sqrt{R} \end{bmatrix}^T. \tag{19}$$

In (19), $P(t), Q(e_f), R(e) \in \mathbb{R}$ denote LK functionals defined as [3]

$$P \triangleq \omega\int_{t-\tau}^{t}\left(\int_{s}^{t}\|u(\theta)\|^2 \mathrm{d}\theta\right)\mathrm{d}s \tag{20}$$

$$Q \triangleq \frac{k\bar{m}\bar{\eta}}{2}\int_{t-\tau}^{t}\|Tanh(e_f)\|^2 \mathrm{d}\theta \tag{21}$$

$$R \triangleq \frac{\bar{m}\bar{\eta}}{2}\int_{t-\tau}^{t}\|Tanh(e)\|^2 \mathrm{d}\theta \tag{22}$$

where $\omega \in \mathbb{R}^+$ is a known constant.

To facilitate the subsequent stability analysis, let k, introduced in (7) and (13), be selected as

$$k \triangleq k_1 + k_2 + k_3. \tag{23}$$

Additionally, let the auxiliary constant $\beta \in \mathbb{R}^+$ be defined by

$$\begin{aligned} \beta \triangleq\ & min\{\underline{m}k_1 - \bar{m}\bar{\eta}k - \bar{m}\bar{\eta}, \\ & \alpha - \bar{m}\bar{\eta} - \frac{\bar{b}^2\psi^2}{4} - \omega\tau(1+k), \\ & \gamma - \bar{m}\bar{\eta}k - k\omega\tau(1+k)\}. \end{aligned} \tag{24}$$

Based on the subsequent stability analysis, the control gains $\alpha, \gamma, k_1, k_2, k_3$ are selected according to the following sufficient conditions

$$\begin{aligned} & k_1 > \frac{(k_2+k_3)2\bar{m}\bar{\eta}}{1 - \frac{2\bar{m}\bar{\eta}}{\underline{m}}}, \\ & \alpha > \bar{m}\bar{\eta} + \frac{\bar{b}^2\psi^2}{4} + \omega\tau(1+k), \\ & \gamma > \bar{m}\bar{\eta}k + k\omega\tau(1+k), \quad \omega\psi^2 > 2\tau, \; k_2 > \frac{\beta\underline{m}}{\bar{\chi}^2} \end{aligned} \tag{25}$$

where $\psi \in \mathbb{R}^+$ is a subsequently defined constant. If the sufficient conditions in (25) are satisfied, then $\beta > 0$. The sufficient gain conditions indicate that ω can be selected sufficiently small and k_1 sufficiently large provided $1 - \frac{2\bar{m}\bar{\eta}}{\underline{m}} > 0$. This condition indicates that the constant approximation matrix B must be chosen sufficiently close to $M^{-1}(q)$ so that $\left\|B - M^{-1}(q)\right\|_\infty < \frac{\bar{m}}{2\underline{m}}$.

V. Stability Analysis

Theorem: Given the dynamics in (1), the controller in (13) ensures uniformly ultimately bounded tracking provided the conditions in (25) are satisfied.

Proof: Let $V_L(z,t) : \mathcal{D} \times [0,\, \infty) \to \mathbb{R}$ be a continuously differentiable, positive-definite functional defined as

$$\begin{aligned} V_L \;\triangleq\; & \frac{1}{2}r^T M r + \sum_{i=1}^{n} ln\left(cosh\left(e_i\right)\right) \qquad (26) \\ & + \frac{1}{2}Tanh^T\left(e_f\right)Tanh\left(e_f\right) + P + Q + R \end{aligned}$$

which, using (4), can be bounded as

$$\phi_1\left(\|z\|\right) \le V_L \le \phi_2\left(\|z\|\right) \qquad (27)$$

where $\phi_1(\cdot), \phi_2(\cdot) : \mathbb{R} \to \mathbb{R}$ are strictly increasing non-negative functions defined as

$$\begin{aligned} \phi_1\left(\|z\|\right) \;&\triangleq\; \lambda_1 ln\left(cosh\left(\|z\|\right)\right), \\ \phi_2\left(\|z\|\right) \;&\triangleq\; \lambda_2 \|z\|^2, \qquad (28) \end{aligned}$$

and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ are known constants defined as

$$\lambda_1 \triangleq \frac{1}{2}min\left[\underline{m}, 1\right], \quad \lambda_2 \triangleq max\left[\frac{1}{2}\bar{m}, 1\right].$$

After utilizing (6), (7), (14), Property 2, and canceling similar terms, the time derivative of (26) can be expressed as

$$\begin{aligned} \dot{V}_L \;=\; & -Mkr^T r - \alpha Tanh^T(e)\,Tanh(e) \\ & -\gamma Tanh^T\left(e_f\right)Tanh\left(e_f\right) \\ & -r^T\left(M\eta\left(-kTanh\left(e_f\right) + Tanh(e)\right)\right) \\ & +r^T\left(M\eta\left(kTanh\left(e_{f\tau}\right) - Tanh\left(e_\tau\right)\right)\right) \\ & +r^T(X+S) + Tanh(e)^T Be_z \qquad (29) \\ & +\omega\tau\|u\|^2 - \omega\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta \\ & +\frac{M\eta k}{2}\left(\|Tanh\left(e_f\right)\|^2 - \|Tanh\left(e_{f\tau}\right)\|^2\right) \\ & +\frac{M\eta}{2}\left(\|Tanh(e)\|^2 - \|Tanh\left(e_\tau\right)\|^2\right) \end{aligned}$$

where the Leibniz integral rule was applied to determine the time derivative of (20), (21) and (22). Using Property 1, (4), (9), (11), (13), (17), and (18), (29) can be upper bounded by

$$\begin{aligned} \dot{V}_L \;\le\; & -\underline{m}k\|r\|^2 - \alpha\|Tanh(e)\|^2 - \gamma\|Tanh\left(e_f\right)\|^2 \\ & +\bar{m}\bar{\eta}k\|r\|\|Tanh\left(e_f\right)\| + \bar{m}\bar{\eta}\|r\|\|Tanh(e)\| \\ & +\bar{m}\bar{\eta}k\|r\|\|Tanh\left(e_{f\tau}\right)\| + \bar{m}\bar{\eta}\|r\|\|Tanh\left(e_\tau\right)\| \\ & +\|r\|\bar{\chi}\|z\| + \|r\|\bar{s} + \bar{b}\|Tanh(e)\|\|e_z\| \\ & +\omega\tau\|Tanh\left(e_f\right)\|^2 + \omega\tau\|Tanh(e)\|^2 \\ & +2k\omega\tau\|Tanh\left(e_f\right)\|\|Tanh(e)\| \qquad (30) \\ & -\omega\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta \\ & +\frac{\bar{m}\bar{\eta}k}{2}\left(\|Tanh\left(e_f\right)\|^2 - \|Tanh\left(e_{f\tau}\right)\|^2\right) \\ & +\frac{\bar{m}\bar{\eta}}{2}\left(\|Tanh(e)\|^2 - \|Tanh\left(e_\tau\right)\|^2\right). \end{aligned}$$

Young's Inequality can be used to upper bound each of the terms in the second, third and sixth lines of (30), as well as

$$\bar{b}\|Tanh(e)\|\|e_z\| \le \frac{\bar{b}^2\psi^2}{4}\|e\|^2 + \frac{1}{\psi^2}\|e_z\|^2 \qquad (31)$$

where $\psi \in \mathbb{R}^+$ is a known constant. Utilizing the Cauchy Schwartz inequality, the following term in (31) can be upper bounded as

$$\|e_z\|^2 \le \tau\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta. \qquad (32)$$

By adding and subtracting $\frac{\tau}{\psi^2}\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta$ and by using (31) and (32), (30) becomes

$$\begin{aligned} \dot{V}_L \;\le\; & -\left(\underline{m}k_1 - \bar{m}\bar{\eta}k - \bar{m}\bar{\eta}\right)\|r\|^2 \qquad (33) \\ & -\left(\alpha - \bar{m}\bar{\eta} - \frac{\bar{b}^2\psi^2}{4} - \omega\tau - k\omega\tau\right)\|Tanh(e)\|^2 \\ & -\left(\gamma - \bar{m}\bar{\eta}k - k^2\omega\tau - k\omega\tau\right)\|Tanh\left(e_f\right)\|^2 \\ & -\frac{1}{\tau}\left(\omega - \frac{2\tau}{\gamma^2}\right)\|e_z\|^2 - \underline{m}k_2\|r\|^2 + \bar{\chi}\|z\|\|r\| \\ & -\underline{m}k_3\|r\|^2 + \bar{s}\|r\| - \frac{\tau}{\psi^2}\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta \end{aligned}$$

where $k_1, k_2, k_3 \in \mathbb{R}^+$ are constant control gains defined in (23). After completing the squares, the expression in (33) can be upper bounded by

$$\begin{aligned} \dot{V}_L \;\le\; & -\beta\|x\|^2 - \frac{1}{\tau}\left(\omega - \frac{2\tau}{\gamma^2}\right)\|e_z\|^2 + \frac{\bar{\chi}^2}{\underline{m}k_2}\|z\|^2 \\ & -\frac{\tau}{\psi^2}\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta + \frac{\bar{s}^2}{4\underline{m}k_3} \qquad (34) \end{aligned}$$

where β was defined in (24), and $x(t) \in \mathbb{R}^{3n}$ is defined as

$$x \triangleq \left[Tanh^T(e) \;\; Tanh^T\left(e_f\right) \;\; r^T\right]^T. \qquad (35)$$

The inequality

$$\int_{t-\tau}^{t}\left(\int_{s}^{t}\|u(\theta)\|^2 \mathrm{d}\theta\right)\mathrm{d}s \le$$

$$\tau\sup_{s\in[t,\,t-\tau]}\left[\int_{s}^{t}\|u(\theta)\|^2 \mathrm{d}\theta\right] = \tau\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta$$

can be used to show that by separating the integral term, the expression in (34) can be rewritten as

$$\begin{aligned} \dot{V}_L \;\le\; & -\beta\|x\|^2 - \frac{1}{\tau}\left(\omega - \frac{2\tau}{\gamma^2}\right)\|e_z\|^2 + \frac{\bar{\chi}^2}{\underline{m}k_2}\|z\|^2 \\ & -\frac{\tau}{2\psi^2}\int_{t-\tau}^{t}\|u(\theta)\|^2 \mathrm{d}\theta \qquad (36) \\ & -\frac{1}{2\psi^2}\int_{t-\tau}^{t}\left(\int_{s}^{t}\|u(\theta)\|^2 \mathrm{d}\theta\right)\mathrm{d}s + \frac{\bar{s}^2}{4\underline{m}k_3}. \end{aligned}$$

The motivation for separating terms in (36) is provided by the need to have $\dot{V}_L$ in terms of z, which contains P, Q, R

terms. The expression in (36) can be reduced by utilizing (13), (19), and (35) to yield

$$\dot{V}_L \leq -\phi_3(\|z\|) + \frac{\bar{s}^2}{4\underline{m}k_3} \quad (37)$$

where $\phi_3(\|z\|) \in \mathbb{R}$ is defined as

$$\phi_3(\|z\|) \triangleq \left(\beta_2 - \frac{\bar{\chi}^2}{\underline{m}k_2}\right) tanh^2(\|z\|),$$

and $\beta_2(\|z\|) \in \mathbb{R}^+$ is denoted as

$$\begin{aligned}\beta_2 \quad &\triangleq \quad min\{\left(\beta - \frac{\bar{\chi}^2}{\underline{m}k_2}\right),\ \frac{1}{\tau}\left(\omega - \frac{2\tau}{\gamma^2}\right),\\ &\frac{\tau k}{\psi^2 \bar{m}\bar{\eta}},\ \frac{\tau}{\psi^2 \bar{m}\bar{\eta}},\ \frac{1}{2\omega\psi^2}\}.\end{aligned}$$

Given (28) and (37), where $\phi_i(\cdot)$ are scalar, strictly increasing functions, the following conditions hold [32]

$$\phi_i(0) = 0, \quad \forall i = 1, 2, 3$$

$$\lim_{\|z\|\to\infty} \phi_i(\|z\|) = \infty, \quad \forall i = 1, 2$$

$$\lim_{\|z\|\to\infty} \phi_3(\|z\|) \triangleq l < \infty, \qquad \frac{\bar{s}^2}{4\underline{m}k_3} < l$$

where l is a positive scalar constant provided the sufficient conditions in (25) are satisfied. Based on these conditions, $z(\cdot)$ is uniformly ultimately bounded [32] in the sense that

$$\|e(t)\| \leq \|z(t)\| < \bar{d}, \quad \forall t \geq T\left(\bar{d},\ \|z(0)\|\right)$$

where $\bar{d}$ is a positive constant that defines the radius of the ball and is selected according to

$$\bar{d} > \left(\phi_1^{-1} \circ \phi_2\right)\left(\phi_3^{-1}\left(\frac{\bar{s}^2}{4\underline{m}k_3}\right)\right),$$

and $T\left(\bar{d},\ \|z(0)\|\right)$ is a positive constant that denotes the ultimate time and is given by

$$T = \begin{cases} 0 & \|z_0\| \leq \left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right) \\ \frac{\phi_2(\|z_0\|) - \phi_1\left(\left(\phi_2^{-1} \circ \phi_1\right)(\bar{d})\right)}{\phi_3\left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right) - \frac{\bar{s}^2}{4\underline{m}k_3}} & \|z_0\| > \left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right). \end{cases}$$

Remark 3. Based on (37), the size of the ultimate bound can be made arbitrarily small by selecting k_3 arbitrarily large.

VI. Conclusion

This paper provides a continuous saturated controller for uncertain nonlinear systems which include input delays and additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. The saturated controller utilizes smooth hyperbolic functions and is shown to guarantee uniformly ultimately bounded tracking in the presence of model uncertainty and/or unmodeled effects. Extending the result to include uncertain, time-varying time delays will enhance the applicability of the controller, as is the focus of on-going efforts.

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