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Generalized Kalman-Yakubovich-Popov Lemma for 2-D FM LSS Model and Its Application to Finite Frequency Positive Real Control

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Abstract— **This paper is concerned with the development of the generalized Kalman-Yakubovich-Popov (KYP) lemma for two-dimensional (2-D) Fornasini-Marchesini local state-space (FM LSS) systems and its application to state-feedback positive realness control with finite frequency specifications. An linear matrix inequality (LMI) characterization for a rectangular finite frequency region is firstly technically constructed and then a generalized KYP lemma is proposed for 2-D FM LSS models. This lemma provides sufficient conditions in terms of LMI for general quadratic properties of the transfer function over a rectangular finite frequency region, including the extensively investigated bounded realness and positive realness as special cases. Based on this result, a new condition is further derived for designing controllers guaranteeing the finite frequency positive realness of the closed-loop systems. The presented numerical example shows the advantage of the proposed design method.**

I. Introduction

In the field of control theory and signal processing, transfer function method and state space method are two fundamental approaches to describe, analyze and design dynamic systems, respectively, from the frequency-domain and time/space-domain points of view. A well-known result bridging the two methods is the Kalman-Yakubovich-Popov (KYP) lemma [1], [14]. Many properties of a transfer function, such as bounded realness (H_∞) [8] and positive realness (passivity) [16], can be tackled via utilizing the KYP lemma to convert an infinite-dimensional problem to a convex optimization problem with LMI constraints.

In engineering practice, each property specification is often required for a *finite frequency* range. However, the standard KYP lemma can *only* treat the properties over the entire frequency domain, which is the main drawback of the KYP lemma when it is applied to engineering practice. A milestone in exploring solution to overcoming this obstacle is the generalized KYP lemma in [10], which elegantly generalizes the KYP lemma from the infinite frequency range to a finite frequency range. Hence, the generalized version is more suitable for practical requirements [11], [9], [10].

On the other hand, two-dimensional (2-D) systems also have been widely investigated over the past decades due to their significance in both theory and practical applications [13], [4]. Two commonly used state space models for 2-D systems are the Roesser model [15] and Fornasini-Marchesini local state-space (FM LSS) model [7]. For these two models, properties such as bounded realness [5], [6], [17] and positive realness [19] have been extensively studied and a great number of related results have been reported. However, these results can only tackle certain specific property for 2-D systems. Until recently, some general results with regard to the KYP lemma for 2-D systems have just appeared for both entire frequency and finite frequency cases. In [2], the authors proposed a KYP lemma for hybrid 2-D Roesser systems, which includes the existing bounded real lemma (BRL) in [6] and positive real lemma (PRL) in [18] as special cases; independent of [2], [21] developed a generalized KYP lemma that can directly treat properties of 2-D systems over a rectangular region for Roesser models. However, it should be noted that the 2-D KYP lemmas mentioned above are only for Roesser models. To the authors' knowledge, there is *no (generalized) KYP lemma for FM LSS models* existing, which motivates us to make efforts to fill the void of gap.

Hence, the first main results of the paper will focus on developing a generalized KYP lemma for 2-D FM LSS systems over a rectangular finite frequency region. To this end, an LMI characterization of rectangular finite frequency regions specified for 2-D FM LSS models is firstly derived, and then by combining this new characterization with S-procedure [20], LMI conditions are obtained guaranteeing a general quadratic property of the transfer function, which is the generalized KYP lemma. The obtained results not only can directly imply the existing BRL [6] and PRL [19] for 2-D FM LSS models, but also include the standard KYP lemma as a special case.

To further show the effectiveness of the proposed generalized KYP lemma, we also apply it to the finite frequency positive realness control problem for FM LSS models. The authors in [9] pointed out that positive realness property is crucial for achieving good control performance, and they still provided compelling evidences to support that in practice, it is not necessary to require this property for the entire frequency range due to the limitation on the control bandwidth. By the developed generalized KYP lemma, we will present a new state-feedback controller design method. Numerical results will show that the proposed synthesis method is advantageous over the standard one when finite frequency specifications are considered.

Notation: $\mathcal{N}_X$ is arbitrary matrices whose columns form a basis of the nullspace of X. $\mathbb{R}$ and $\mathbb{C}$ denote the set of reals and complex numbers, respectively. The notation $P > 0$ (≥ 0) means that matrix P is positive (semi)definite. $\mathbf{I}$ denotes an identity matrix with appropriate dimension. In addition, sym$\{A\}$ indicates $A^* + A$, diag$\{\cdots\}$ stands for a block-

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diagonal matrix, and $\sigma_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$, respectively, denote the maximum singular value and minimum eigenvalue of a transfer function.

II. Generalized KYP Lemma for FM LSS Model

A. Problem Statement

Consider the following 2-D FM LSS model [7]:

$$\begin{aligned} x(i+1,j+1) &= A_1x(i,j+1) + A_2x(i+1,j) \\ &\quad + B_1u(i,j+1) + B_2u(i+1,j), \\ y(i,j) &= Cx(i,j) + Du(i,j), \end{aligned} \tag{1}$$

where $x(i,j) \in \mathbb{C}^n$ is the state vector; $u(i,j) \in \mathbb{C}^{n_u}$ is the external (noise/disturbance) input signal; $y(i,j) \in \mathbb{C}^{n_y}$ is the concerned output signal; $A_1, A_2 \in \mathbb{C}^{n\times n}$, $B_1, B_2 \in \mathbb{C}^{n\times n_u}$, $C \in \mathbb{C}^{n_y\times n}$ and $D \in \mathbb{C}^{n_y\times n_u}$ are system matrices.

Let z_1 and z_2 be the z-transform operator of the i- and j-orientations, respectively, and $\hat{x}$ denotes the z-transform of the state vector x. Then, the state equation of the FM LSS model (1) can be expressed in the frequency-domain as

$$z_1z_2\hat{x} = z_2A_1\hat{x} + z_1A_2\hat{x} + z_2B_1\hat{u} + z_1B_2\hat{u}, \tag{2}$$

and accordingly the transfer function from $\hat{u}$ to $\hat{y}$ is

$$H(z_1,z_2) = C(z_1z_2\mathbf{I} - z_2A_1 - z_1A_2)^{-1}(z_2B_1 + z_1B_2) + D. \tag{3}$$

Motivated by the generalized KYP lemma for 1-D systems [10] and 2-D Roesser systems [21], one of the main objectives of this paper is to find an LMI condition ensuring a general quadratic property of the 2-D FM LSS model over a rectangular finite frequency region, which is the generalized 2-D KYP lemma.

B. Finite Frequency Region and Its Characterization

To develop the generalized KYP lemma, a key procedure is to establish an equivalent LMI condition for a complex vector set $\mathcal{G}$ defined as

$$\begin{aligned} \mathcal{G} \triangleq \{ &\operatorname{col}\{f, g_1, g_2\} \in \mathbb{C}^{3n} \colon f = \mathrm{e}^{\mathrm{j}\omega_1}g_1, f = \mathrm{e}^{\mathrm{j}\omega_2}g_2, \\ &\text{for some } \omega_1,\omega_2 \in [-\pi,\pi], |\omega_1| \le \bar{\omega}_1, |\omega_2| \le \bar{\omega}_2\}, \end{aligned} \tag{4}$$

where $\bar{\omega}_1, \bar{\omega}_2 \in [0,\pi]$ are given real scalars. Noting $z_1 = \mathrm{e}^{\mathrm{j}\omega_1}, z_2 = \mathrm{e}^{\mathrm{j}\omega_2}$ and regarding $f = z_1z_2\hat{x}, g_1 = z_2\hat{x}$ and $g_2 = z_1\hat{x}$, one can easily see that $\mathcal{G}$ describes the relationship of signals $(z_1z_2\hat{x}, z_2\hat{x}, z_1\hat{x})$ within a rectangular finite frequency region $[-\bar{\omega}_1, \bar{\omega}_1] \times [-\bar{\omega}_2, \bar{\omega}_2]$. First, let us introduce the following fundamental lemma.

Lemma 1 ([14]): Let $F, G \in \mathbb{C}^{n\times n}$ and $f, g \in \mathbb{C}^n$. Then, $ff^* = gg^*$ if and only if there exists a scalar $\omega \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{j}\omega}g$.

Then, we have the following result, which reveals that the set $\mathcal{G}$ can be equivalently characterized by an LMI.

Lemma 2: Let $\bar{\omega}_1, \bar{\omega}_2 \in [0,\pi]$ and $f, g_1, g_2 \in \mathbb{C}^n$ be given. The following statements are equivalent.

(i) There exist two real scalars $\omega_1, \omega_2 \in [-\pi,\pi]$ such that

$$f = \mathrm{e}^{\mathrm{j}\omega_1}g_1, f = \mathrm{e}^{\mathrm{j}\omega_2}g_2, |\omega_1| \le \bar{\omega}_1, |\omega_2| \le \bar{\omega}_2.$$

(ii) For all Hermitian matrices $P_k \in \mathbb{C}^{n\times n}$ and $0 < Q_k \in \mathbb{C}^{n\times n}, k = 1, 2$, the following matrix inequality holds:

$$\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} P & Q \\ Q^* & \Delta \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \ge 0, \tag{5}$$

where

$$\begin{aligned} &P \triangleq P_1 + P_2,\ Q \triangleq [\ Q_1 \quad Q_2\ ],\ g = \operatorname{col}\{g_1, g_2\}, \\ &\Delta \triangleq \operatorname{diag}\{-P_1 - 2\cos\bar{\omega}_1Q_1, -P_2 - 2\cos\bar{\omega}_2Q_2\}. \end{aligned}$$

Proof: The proof is omitted due to space limitation. ■

Remark 1: In [10], the characteristic set $\mathcal{G}$ in (4) only includes two vectors, for instance, $f = \mathrm{j}\omega g$ for the 1-D continuous-time case; in [21], this set $\mathcal{G}$ actually consists of two pairs of such vectors as $f_h = \mathrm{e}^{\mathrm{j}\omega_h}g_h$ and $f_v = \mathrm{e}^{\mathrm{j}\omega_v}g_v$, with each pair corresponding to the horizontal or vertical state vector in the discrete Roesser model. For the discrete FM LSS model, since $x(i+1, j+1)$ can be viewed as the one-step forward shift from $x(i, j+1)$ or $x(i+1, j)$, we need three vectors to identify the relationship of $x(i+1, j+1), x(i, j+1)$ and $x(i+1, j)$, which exactly corresponds to that of f, g_1 and g_2 of $\mathcal{G}$ in (4). This can well explain how to technically explore such $\mathcal{G}$ along the road of [10] and [21] and also is the most important underlying idea and motivation of this paper. Thus, Lemma 2 further extends the results of 1-D systems and 2-D Roesser models to 2-D FM LSS models.

C. Generalized KYP Lemma for FM LSS Model

Based on Lemma 2, we present the generalized 2-D KYP lemma for the FM LSS model as follows.

Theorem 1: Consider the FM LSS model in (1) and suppose that $\det(z_1z_2\mathbf{I} - z_2A_1 - z_1A_2) \ne 0$ for all $(z_1, z_2) \in \{(z_1,z_2) \in \mathbb{C}\times\mathbb{C}\colon |z_1| \ge 1, |z_2| \ge 1\}$. Let Hermitian matrices $\Theta_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{13} \end{bmatrix} \in \mathbb{C}^{n+n_u}, \Theta_2 = \begin{bmatrix} \Theta_{21} & \Theta_{22} \\ \Theta_{22}^* & \Theta_{23} \end{bmatrix} \in \mathbb{C}^{n+n_u}$ and scalars $\bar{\omega}_1, \bar{\omega}_2 \in [0,\pi]$ be given. If there exist Hermitian matrices $P_k \in \mathbb{C}^{n\times n}$ and $0 < Q_k \in \mathbb{C}^{n\times n}, k = 1, 2$, satisfying

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} P & Q \\ Q^* & \Delta \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} + \mathbf{\Theta} < 0, \tag{6}$$

where P, Q, Δ are defined in (5) and

$$\mathbf{A} \triangleq [\ A_1 \quad A_2\ ],\ \mathbf{B} \triangleq [\ B_1 \quad B_2\ ],$$

$$\mathbf{\Theta} \triangleq \begin{bmatrix} \Theta_{11} & \mathbf{0} & \Theta_{12} & \mathbf{0} \\ \mathbf{0} & \Theta_{21} & \mathbf{0} & \Theta_{22} \\ \Theta_{12}^* & \mathbf{0} & \Theta_{13} & \mathbf{0} \\ \mathbf{0} & \Theta_{22}^* & \mathbf{0} & \Theta_{23} \end{bmatrix},$$

then the following finite frequency condition holds:

$$\begin{bmatrix} \mathbf{G}(\omega_1,\omega_2) \\ \mathbf{I}(\omega_1,\omega_2) \end{bmatrix}^* \mathbf{\Theta} \begin{bmatrix} \mathbf{G}(\omega_1,\omega_2) \\ \mathbf{I}(\omega_1,\omega_2) \end{bmatrix} < 0,\ \forall(\omega_1,\omega_2) \in \bar{\mathbf{\Omega}} \tag{7}$$

where

$$\mathbf{G}(\omega_1,\omega_2) \triangleq \begin{bmatrix} \mathrm{e}^{\mathrm{j}\omega_2}G(\omega_1,\omega_2) \\ \mathrm{e}^{\mathrm{j}\omega_1}G(\omega_1,\omega_2) \end{bmatrix},\ \mathbf{I}(\omega_1,\omega_2) \triangleq \begin{bmatrix} \mathrm{e}^{\mathrm{j}\omega_2}\mathbf{I}_n \\ \mathrm{e}^{\mathrm{j}\omega_1}\mathbf{I}_n \end{bmatrix},$$

$$\begin{aligned} G(\omega_1,\omega_2) \triangleq &[\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2)}\mathbf{I}_n - \mathrm{e}^{\mathrm{j}\omega_2}A_1 - \mathrm{e}^{\mathrm{j}\omega_1}A_2]^{-1} \\ &\times [\mathrm{e}^{\mathrm{j}\omega_2}B_1 + \mathrm{e}^{\mathrm{j}\omega_1}B_2], \end{aligned}$$

$$\bar{\Omega} \triangleq [-\bar{\omega}_1, \bar{\omega}_1] \times [-\bar{\omega}_2, \bar{\omega}_2].$$

Proof: The proof is omitted for brevity. ■

Theorem 1 provides a sufficient condition in terms of LMI in (6) to guarantee the finite frequency specification in (7) for the FM LSS model. Note that the specified frequency region, $\bar{\mathbf{\Omega}}$, is a rectangular low frequency domain with the center being the origin of the $\omega_1\omega_2$-plane. In the following, the result in Theorem 1 is further developed to tackle the case of any given rectangular finite frequency domain.

Corollary 1: Consider the FM LSS model in (1) and suppose that $\det(z_1z_2\mathbf{I}-z_2A_1-z_1A_2) \neq 0$ for all $(z_1,z_2) \in \{(z_1,z_2) \in \mathbb{C}\times\mathbb{C}\colon |z_1| \geq 1, |z_2| \geq 1\}$. Let Hermitian matrices $\Theta_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{13} \end{bmatrix} \in \mathbb{C}^{n+n_u}, \Theta_2 = \begin{bmatrix} \Theta_{21} & \Theta_{22} \\ \Theta_{22}^* & \Theta_{23} \end{bmatrix} \in \mathbb{C}^{n+n_u}$ and scalars $\omega_{Mk}, \omega_{mk} \in [-\pi,\pi], k=1,2$, satisfying $\omega_{mk} < \omega_{Mk}$ be given. If there exist Hermitian matrices $P_k \in \mathbb{C}^{n\times n}$ and $0 < Q_k \in \mathbb{C}^{n\times n}, k=1,2$, satisfying

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} P & Q\mathbf{V}^* \\ \mathbf{V}Q^* & W \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} + \mathbf{\Theta} < 0, \quad (8)$$

where P, Q are defined in (5), $\mathbf{A}, \mathbf{B}, \mathbf{\Theta}$ in (6) and

$$W \triangleq \operatorname{diag}\{-P_1 - 2\cos\omega_1^a Q_1, -P_2 - 2\cos\omega_2^a Q_2\}$$
$$\mathbf{V} \triangleq \operatorname{diag}\{\mathrm{e}^{-\mathrm{j}\omega_1^c}\mathbf{I}, \mathrm{e}^{-\mathrm{j}\omega_2^c}\mathbf{I}\},$$
$$\omega_k^c \triangleq (\omega_{Mk}+\omega_{mk})/2,\ \omega_k^a \triangleq (\omega_{Mk}-\omega_{mk})/2, k=1,2$$

then the following finite frequency condition holds:

$$\begin{bmatrix} \mathbf{G}(\omega_1,\omega_2) \\ \mathbf{I}(\omega_1,\omega_2) \end{bmatrix}^* \mathbf{\Theta} \begin{bmatrix} \mathbf{G}(\omega_1,\omega_2) \\ \mathbf{I}(\omega_1,\omega_2) \end{bmatrix} < 0,\ \forall(\omega_1,\omega_2) \in \mathbf{\Omega} \quad (9)$$

where $\mathbf{G}(\omega_1,\omega_2), \mathbf{I}(\omega_1,\omega_2)$ are defined in (7) and

$$\mathbf{\Omega} \triangleq [\omega_{m1}, \omega_{M1}] \times [\omega_{m2}, \omega_{M2}].$$

D. Finite Frequency BRL and PRL of FM LSS Model

In this subsection, we consider two important properties for the FM LSS model: bounded realness and positive realness. A great number of results in the sense of these two indices have been proposed for the control and/or filtering of 2-D systems. By applying the Roesser model version of generalized KYP lemma, [21] extended the standard BRL [6] and PRL [18] to the finite frequency domain. Here, based on the generalized KYP lemma obtained in the paper, we can also obtain the corresponding finite frequency version of BRL [6] and PRL [19] for the FM LSS model.

1) Finite Frequency BRL: In contrast with the standard H_∞ norm, the finite frequency bounded realness property of the FM LSS model indicates that $H(\mathrm{e}^{\mathrm{j}\omega_1},\mathrm{e}^{\mathrm{j}\omega_2})$ in (3) satisfies

$$\|H\|_\infty^{\mathbf{\Omega}} \triangleq \sup_{(\omega_1,\omega_2)\in\mathbf{\Omega}} \sigma_{\max}[H(\mathrm{e}^{\mathrm{j}\omega_1},\mathrm{e}^{\mathrm{j}\omega_2})] < \gamma \quad (10)$$

where $\mathbf{\Omega}$ is a given finite frequency region and $\gamma > 0$ a scalar. Then, we have the following finite frequency BRL.

Corollary 2: Consider the FM LSS model in (1) and suppose that $\det(z_1z_2\mathbf{I}-z_2A_1-z_1A_2) \neq 0$ for all $(z_1,z_2) \in \{(z_1,z_2) \in \mathbb{C}\times\mathbb{C}\colon |z_1| \geq 1, |z_2| \geq 1\}$. Given scalars $\gamma > 0$ and $\omega_{Mk}, \omega_{mk} \in [-\pi,\pi], k=1,2$, satisfying $\omega_{mk} < \omega_{Mk}$, if there exist Hermitian matrices $P_k \in \mathbb{C}^{n\times n}$ and $0 < Q_k \in \mathbb{C}^{n\times n}, k=1,2$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} P & Q\mathbf{V}^* \\ \mathbf{V}Q^* & W \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{C}^*\mathbf{C} & \mathbf{C}^*\mathbf{D} \\ \mathbf{D}^*\mathbf{C} & \mathbf{D}^*\mathbf{D}-\gamma^2\mathbf{I} \end{bmatrix} < 0 \quad (11)$$

holds, where P, Q are in (5), $\mathbf{A}$,$\mathbf{B}$ in (6), $W, \mathbf{V}$ in (8) and

$$\mathbf{C} = \operatorname{diag}\{C, C\},\ \mathbf{D} = \operatorname{diag}\{D, D\},$$

then the finite frequency bounded realness condition in (10) is satisfied with $\mathbf{\Omega}$ defined in (9).

2) Finite Frequency PRL: Motivated by the usual positive realness definition in [19], the FM LSS system with $H(z_1,z_2)$ being analytic in $|z_1| \geq 1, |z_2| \geq 1$ is said to be finite frequency positive real if the following condition holds

$$\operatorname{sym}[H(\mathrm{e}^{\mathrm{j}\omega_1},\mathrm{e}^{\mathrm{j}\omega_2})] > 0,\ \forall(\omega_1,\omega_2) \in \mathbf{\Omega} \quad (12)$$

with $\mathbf{\Omega}$ being a given finite frequency domain. Based on Corollary 1, the following finite frequency PRL can be obtained for the FM LSS model.

Corollary 3: Consider the FM LSS model in (1) and suppose that $\det(z_1z_2\mathbf{I}-z_2A_1-z_1A_2) \neq 0$ for all $(z_1,z_2) \in \{(z_1,z_2) \in \mathbb{C}\times\mathbb{C}\colon |z_1| \geq 1, |z_2| \geq 1\}$. Given scalars $\omega_{Mk}, \omega_{mk} \in [-\pi,\pi], k=1,2$, satisfying $\omega_{mk} < \omega_{Mk}$, if there exist Hermitian matrices $P_k \in \mathbb{C}^{n\times n}$, $0 < Q_k \in \mathbb{C}^{n\times n}, k=1,2$, and $M \in \mathbb{C}^{n_u\times n_u}$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^* \begin{bmatrix} P & Q\mathbf{V}^* \\ \mathbf{V}Q^* & W \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{C}^* \\ -\mathbf{C} & M_d - \operatorname{sym}(\mathbf{D}) \end{bmatrix} < 0 \quad (13)$$

holds, where P, Q are in (5), $\mathbf{A}$,$\mathbf{B}$ in (6), $W, \mathbf{V}$ in (8), $\mathbf{C}, \mathbf{D}$ in (11) and $M_d = \operatorname{diag}\{M, -M\}$, then the finite frequency positive realness condition in (12) is satisfied with $\mathbf{\Omega}$ defined in (9).

Remark 2: From the system dimension point of view, the obtained generalized KYP lemma (Theorem 1 or Corollary 1), finite frequency BRL (Corollary 2) and finite frequency PRL (Corollary 3) can be seen as the extension of the 1-D generalized KYP lemma [10] to discrete 2-D systems; from the modeling point of view, these results can be regarded as the extension of the Roesser model version of 2-D generalized KYP lemma [21] to the FM LSS model; from the frequency domain point of view, they also can be deemed as the extension of the full frequency (BRL and PRL) results [6], [19] to the finite frequency domain.

III. Application to Finite Frequency Positive Real Control

The positive realness is one of the basic properties that can be addressed by KYP lemmas. For 2-D systems, the problem of positive real control has been considered in [19], [18]; however, the results proposed there were specialized for the entire frequency domain. In this section, in order to show the applicative significance of the proposed generalized KYP lemma in the paper, we concentrate on solving the problem of finite frequency positive real control for the FM LSS model.

A. Problem Formulation

In this section, consider the following open-loop 2-D FM LSS system (Σ_o) with control input:

$$\begin{aligned}(\Sigma_o): x(i+1,j+1) &= A_1x(i,j+1) + A_2x(i+1,j)\\ &\quad + B_{11}u_1(i,j+1) + B_{12}u_1(i+1,j)\\ &\quad + B_{21}u_2(i,j+1) + B_{22}u_2(i+1,j),\\ y_1(i,j) &= C_{11}x(i,j) + D_{11}u_1(i,j)\\ &\quad + D_{12}u_2(i,j), \end{aligned} \quad (14)$$

where $x(i,j) \in \mathbb{C}^{n_p}$ is the state vector; $u_1(i,j) \in \mathbb{C}^{n_{u1}}$ is the external disturbance signal and $u_2(i,j) \in \mathbb{C}^{n_{u2}}$ is the control input signal; $y_1(i,j) \in \mathbb{C}^{n_{u1}}$ is the interested output signal and $y_2(i,j) \in \mathbb{C}^{n_{y2}}$ is the measurement output signal; $A_k, B_{1k}, B_{2k}, C_{11}, C_{21}, D_{11}, D_{12}$ and $D_{21}, k = 1,2$, are known system matrices with appropriate dimensions.

Suppose all the states $x_2(i,j)$ are measurable and we are interested in designing a state feedback control law

$$(\Sigma_{c1}): u_2(i,j) = K_cx_2(i,j). \quad (15)$$

Then, by connecting controller (Σ_{c1}) to the open-loop system (Σ_o), the resulting closed-loop system (Σ_{cl}) from u_1 to y_1 can be obtained as follows:

$$\begin{aligned}(\Sigma_{cl}): \bar{x}(i+1,j+1) &= \bar{A}_1\bar{x}(i,j+1) + \bar{A}_2\bar{x}(i+1,j)\\ &\quad + \bar{B}_1u_1(i,j+1) + \bar{B}_2u_1(i+1,j),\\ y_1(i,j) &= \bar{C}\bar{x}(i,j) + \bar{D}u_1(i,j), \end{aligned} \quad (16)$$

where

$$\left[\begin{array}{cc} \bar{A}_k & \bar{B}_k \\ \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{c|c} A_k + B_{2k}K_c & B_{1k} \\ \hline C_{11} + D_{12}K_c & D_{11} \end{array}\right], k = 1,2.$$

Let H_{cl} denote the transfer function of (Σ_{cl}), i.e.,

$$H_{cl}(z_1,z_2) = \bar{C}(z_1z_2\mathbf{I} - z_2\bar{A}_1 - z_1\bar{A}_2)^{-1}(z_2\bar{B}_1 + z_1\bar{B}_2) + \bar{D}.$$

The objective of finite frequency positive real control is to design a state feedback controller (Σ_{c1}) for system (Σ_o) such that the closed-loop system (Σ_{cl}) is asymptotically stable and satisfies

$$\mathrm{sym}[H_{cl}(e^{j\omega_1}, e^{j\omega_2})] > 0, \ \forall(\omega_1,\omega_2) \in \mathbf{\Omega} \quad (17)$$

where $\mathbf{\Omega}$ is a given finite frequency region defined in (9).

B. Multiplier Expansion for Finite Frequency PRL

By virtue of the finite frequency PRL, Corollary 3, system (Σ_{cl}) satisfies (17) if there exist Hermitian matrices P_k, $0 < Q_k, k = 1,2$, and M such that

$$\left[\begin{array}{cc} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \mathbf{I} & \mathbf{0} \end{array}\right]^* \left[\begin{array}{cc} P & Q\mathbf{V}^* \\ \mathbf{V}Q^* & W \end{array}\right] \left[\begin{array}{cc} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \mathbf{I} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathbf{0} & -\bar{\mathbf{C}}^* \\ -\bar{\mathbf{C}} & M_d - \mathrm{sym}\,(\bar{\mathbf{D}}) \end{array}\right] < 0 \quad (18)$$

with $\bar{\mathbf{A}} \triangleq [\bar{A}_1, \bar{A}_2]$ and other notations similarly defined as in (13). So the analysis of the specification (17) can be processed by testing an LMI in (18).

When controller synthesis problem is considered, (18) is not an LMI anymore due to the unknown realization of controller (Σ_{c1}). Moreover, since both P and Q are multiplied by system matrices, the usual change-of-variables method applied to P (see [3]) is unable to transform (18) to an LMI and further explicitly present the controller realization, which is one of the main differences between Corollary 3 and the entire frequency PRL [19], [18], and also is one of the main drawbacks when it is applied to synthesis problems. To overcome this, inspired by the strategy for 1-D systems in [11] and with the aid of the projection lemma [8], we introduce an additional matrix multiplier for (18) and obtain the following condition.

Theorem 2: Denote $n = n_p$. Let system (Σ_{cl}) in (16), Hermitian matrices $P_k \in \mathbb{C}^{n\times n}$, $0 < Q_k \in \mathbb{C}^{n\times n}, k = 1,2$, $M \in \mathbb{C}^{n_{u1}\times n_{u1}}$, general matrix $R \in \mathbb{C}^{n\times(3n+2n_{u1})}$ and scalars $\omega_{Mk}, \omega_{mk} \in [-\pi,\pi], k = 1,2$, satisfying $\omega_{mk} < \omega_{Mk}$ be given. The following statements are equivalent.

(i) The conditions in (18) and in the following hold:

$$\mathcal{N}_R^*\Psi\mathcal{N}_R < 0, \ \Psi = \left[\begin{array}{ccc} P & Q\mathbf{V}^* & \mathbf{0} \\ \mathbf{V}Q^* & W & -\bar{\mathbf{C}}^* \\ \mathbf{0} & -\bar{\mathbf{C}} & M_d - \bar{\mathbf{D}} - \bar{\mathbf{D}}^* \end{array}\right]. \quad (19)$$

(ii) There exists general matrix $F \in \mathbb{C}^{n\times n}$ such that

$$\Psi + \mathrm{sym}(R^*F[\ -\mathbf{I} \quad \bar{\mathbf{A}} \quad \bar{\mathbf{B}}\ ]) < 0. \quad (20)$$

Compared with (18), the system matrices in (20) are multiplied only by F, which relatively facilitates converting (20) to an LMI when the controller realization is unknown. It should be noted that in general, (20) is only sufficient for (18), unless R is chosen to satisfy (19). However, as is pointed out by [11], for finite frequency conditions, it turns out to be impossible to find an R such that (19) is satisfied from (18) in general. According to the discussion in [11], to make the synthesis problem tractable, a reasonable choice of R is of the form

$$R = [\ \mathbf{I}_n \quad \alpha\mathbf{I}_n \quad \beta\mathbf{I}_n \quad \mathbf{0}_{n\times 2n_{u1}}\ ], \quad (21)$$

where α and β are two prescribled scalar parameters. In what follows, we also adopt this specification for R.

Before proceeding further, it should be noted that, however, the satisfaction of (18) or (20) does not imply the stability of (Σ_{cl}), which is another difference of finite frequency PRL compared with the entire frequency one in [19]. Hence, it is necessary to separately provide a condition for the asymptotic stability of system (Σ_{cl}). To conclude this subsection, the following lemma is presented for this purpose, where a multiplier is also introduced through the projection lemma.

Lemma 3: Denote $n = n_p$ and $\bar{R} = [\ \mathbf{I}_n \quad \mathbf{0}_{n\times 2n}\ ]$. System (Σ_{cl}) is asymptotically stable if there exist Hermitian matrices $0 < \bar{P}_1, \bar{P}_2 \in \mathbb{C}^{n\times n}$ and general matrix $F \in \mathbb{C}^{n\times n}$ such that the following LMI holds:

$$\bar{\Psi} + \mathrm{sym}(\bar{R}^*F[\ -\mathbf{I} \quad \bar{\mathbf{A}}\ ]) < 0 \quad (22)$$

where $\bar{\Psi} = \left[\begin{array}{cc} \bar{P} & \mathbf{0} \\ \mathbf{0} & -\bar{P}_d \end{array}\right]$, $\bar{P} = \bar{P}_1 + \bar{P}_2$, $\bar{P}_d = \mathrm{diag}\{\bar{P}_1, \bar{P}_2\}$.

C. Controller Design

Based on Theorem 2 and Lemma 3, we have the following result guaranteeing the existence of a state feedback controller for system (Σ_o).

Theorem 3: Consider system (Σ_o) in (14) and denote $n = n_p$. Given R as in (21), $\bar{R} = [\ \mathbf{I}_n \quad \mathbf{0}_{n\times 2n}\ ]$ and scalars $\omega_{Mk}, \omega_{mk} \in [-\pi, \pi], k = 1, 2$, satisfying $\omega_{mk} < \omega_{Mk}$, a state feedback controller (Σ_{c1}) exists such that the closed-loop system (Σ_{cl}) in (16) is asymptotically stable and satisfies the finite frequency positive realness specification in (17) if there exist Hermitian matrices $\mathbf{P}_k \in \mathbb{C}^{n\times n}$, $0 < \bar{\mathbf{P}}_k \in \mathbb{C}^{n\times n}$, $0 < \mathbf{Q}_k \in \mathbb{C}^{n\times n}, k = 1, 2$, $\mathbf{M} \in \mathbb{C}^{n_{u1}\times n_{u1}}$ and general matrices $\mathbf{F} \in \mathbb{C}^{n\times n}, \mathbf{K}_c \in \mathbb{C}^{n_{u2}\times n}$ satisfying the following LMIs:

$$\Phi + \mathrm{sym}(R^*\Gamma) < 0,\ \bar{\Phi} + \mathrm{sym}(\bar{R}^*\bar{\Gamma}) < 0, \tag{23}$$

where

$$\Phi \triangleq \begin{bmatrix} \mathbf{P}_1 + \mathbf{P}_2 & \mathrm{e}^{\mathrm{j}\omega_1^c}\mathbf{Q}_1 & \mathrm{e}^{\mathrm{j}\omega_2^c}\mathbf{Q}_2 & \mathbf{0} & \mathbf{0} \\ \mathrm{e}^{-\mathrm{j}\omega_1^c}\mathbf{Q}_1 & \Phi_{11} & \mathbf{0} & \Phi_{21}^* & \mathbf{0} \\ \mathrm{e}^{-\mathrm{j}\omega_2^c}\mathbf{Q}_2 & \mathbf{0} & \Phi_{12} & \mathbf{0} & \Phi_{22}^* \\ \mathbf{0} & \Phi_{21} & \mathbf{0} & \Phi_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi_{22} & \mathbf{0} & \Phi_{32} \end{bmatrix},$$

$$\bar{\Phi} \triangleq \mathrm{diag}\{\bar{\mathbf{P}}_1 + \bar{\mathbf{P}}_2, -\bar{\mathbf{P}}_1, -\bar{\mathbf{P}}_2\},$$

$$\Gamma \triangleq [\ -\mathbf{F} \quad A_1\mathbf{F} + B_{21}\mathbf{K}_c \quad A_2\mathbf{F} + B_{22}\mathbf{K}_c \quad B_{11} \quad B_{12}\ ],$$

$$\bar{\Gamma} \triangleq [\ -\mathbf{F} \quad A_1\mathbf{F} + B_{21}\mathbf{K}_c \quad A_2\mathbf{F} + B_{22}\mathbf{K}_c\ ],$$

$$\Phi_{1k} \triangleq -\mathbf{P}_k - 2\cos\omega_k^a\mathbf{Q}_k,\ \Phi_{2k} \triangleq -C_{11}\mathbf{F} - D_{12}\mathbf{K}_c,$$

$$\Phi_{31} \triangleq \mathbf{M} - \mathrm{sym}(D_{11}),\ \Phi_{32} \triangleq -\mathbf{M} - \mathrm{sym}(D_{11}),$$

and $\omega_k^c, \omega_k^a, k = 1, 2$, are defined in (8). Moreover, if the previous conditions are feasible, the state feedback gain matrix in (15) is $K_c = \mathbf{K}_c\mathbf{F}^{-1}$.

If letting $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{0}$, Theorem 3 can be applied to designing state feedback controllers in the sense of entire frequency positive realness, and is reduced to the following.

Corollary 4: Denote $n = n_p$. A state feedback controller (Σ_{c1}) with $K_c = \mathbf{K}_c\mathbf{F}^{-1}$ exists such that the closed-loop system (Σ_{cl}) in (16) is asymptotically stable and satisfies $\mathrm{sym}[H_{cl}(\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2})] > 0$ for all frequencies if there exist Hermitian matrices $\mathbf{P}_k \in \mathbb{C}^{n\times n}$, $0 < \bar{\mathbf{P}}_k \in \mathbb{C}^{n\times n}, k = 1, 2$, $\mathbf{M} \in \mathbb{C}^{n_{u1}\times n_{u1}}$ and general matrices $\mathbf{F} \in \mathbb{C}^{n\times n}, \mathbf{K}_c \in \mathbb{C}^{n_{u2}\times n}$ satisfying the following LMIs:

$$\hat{\Phi} + \mathrm{sym}(R^*\Gamma) < 0,\ \bar{\Phi} + \mathrm{sym}(\bar{R}^*\bar{\Gamma}) < 0, \tag{24}$$

where

$$\hat{\Phi} \triangleq \begin{bmatrix} \mathbf{P}_1 + \mathbf{P}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Phi_{11} & \mathbf{0} & \Phi_{21}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi_{12} & \mathbf{0} & \Phi_{22}^* \\ \mathbf{0} & \Phi_{21} & \mathbf{0} & \Phi_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi_{22} & \mathbf{0} & \Phi_{32} \end{bmatrix}$$

and other notations are defined in (23).

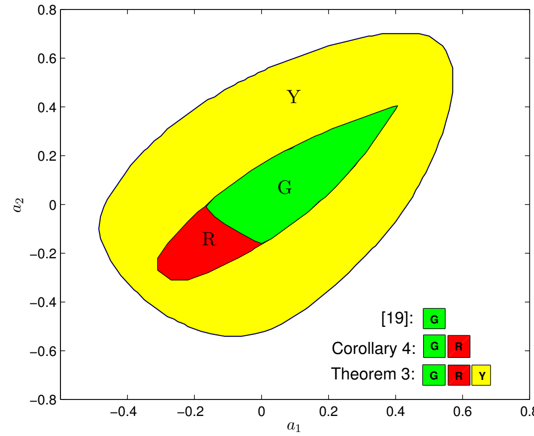

Fig. 1. Feasible areas of a_1a_2 for different methods

D. An Illustrative Example

In this subsection, we present an example to illustrate the effectiveness and merits of the proposed results. Consider a stationary random field described by the following 2-D system [12]:

$$\begin{aligned} \eta(i+1, j+1) = a_1\eta(i, j+1) + a_2\eta(i+1, j) \\ - a_1a_2\eta(i, j) + \omega(i, j), \end{aligned} \tag{25}$$

where $i, j \in \mathbb{Z}^+ \cup \{0\}$ are, respectively, the horizontal and vertical position variables, $\eta(i, j)$ is the state of the random field at spatial coordinate (i, j), $\omega(i, j)$ is a noise input; a_1 and a_2 are, respectively, the vertical and horizontal correlations of the random field.

To represent system in (25) into the FM LSS model in (14), let $x(i, j) = [\ \eta(i, j+1)^T - a_2\eta(i, j)^T \quad \eta(i, j)\ ]^T$ be the state vector, $u_1(i, j) = \omega(i, j)$ and assume that the output signal is $y_1(i, j) = 0.5\eta(i, j) + 0.6\omega(i, j)$ and the control input $u_2(i, j)$ is with matrices $B_{21} = B_{22} = [\ 0.1 \quad 0.1\ ]^T$, which result in the representation in (14) with parameters

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 1 & a_2 \end{bmatrix},\ B_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$B_{21} = B_{22} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ C_{11} = [\ 0 \quad 0.5\ ],$$

$$B_{12} = 0,\ D_{11} = 0.6,\ D_{12} = 0.$$

Suppose that the interested frequency region is $\bar{\mathbf{\Omega}} = [-\frac{3}{4}\pi, \frac{3}{4}\pi] \times [-\frac{3}{4}\pi, \frac{3}{4}\pi]$. For given a_1 and a_2, we design state-feedback controllers for (25) by the developed methods, Theorem 3 and Corollary 4 ($\alpha = \beta = 0$), and the one in [19] such that the closed-loop system is stable with positive real property in $\bar{\mathbf{\Omega}}$. To illustrate the improvement of our methods, we compare the area on a_1a_2-plane, where the corresponding method has a feasible solution, shown in Fig. 1. It is found that the feasible area of a_1a_2 for Theorem 3 (Green, Red and Yellow) is much larger than that for Corollary 4 (Green and Red) and the one in [19] (Green), which verifies the advantage of Theorem 3 when being applied to state-feedback positive realness control for the FM LSS model with finite frequency specifications. Even for the standard positive realness control problem, our method,

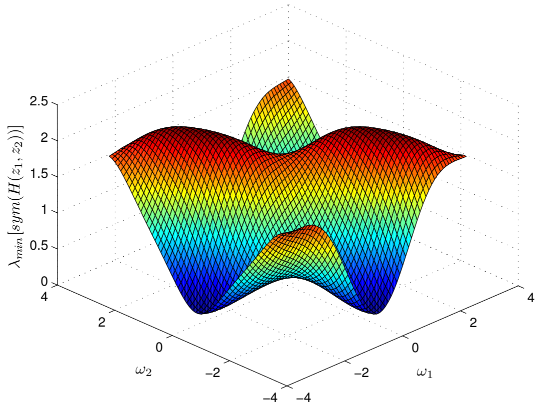

Fig. 2. $\lambda_{\min}[\text{sym}(H(z_1, z_2))]$ with state-feedback controller $K_{c\text{EF}}$

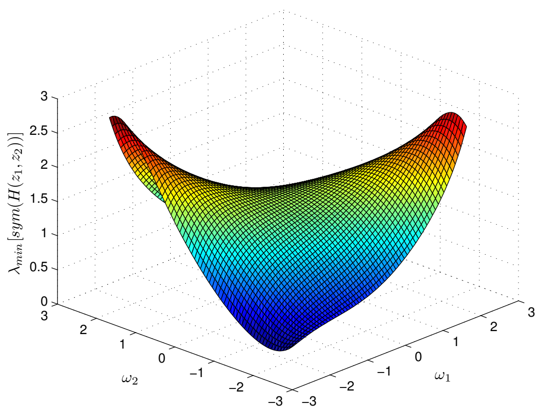

Fig. 3. $\lambda_{\min}[\text{sym}(H(z_1, z_2))]$ with state-feedback controller $K_{c\text{FF}}$

Corollary 4, also has larger feasible area than the one in [19].

By solving the conditions in Corollary 4 and Theorem 3, the resulting entire frequency (EF) and finite frequency (FF) positive realness controllers are, respectively, given by

$$K_{c\text{EF}} = \begin{bmatrix} -2.2055 & 0.2300 \end{bmatrix},\ K_{c\text{FF}} = \begin{bmatrix} -1.2287 & 0.1236 \end{bmatrix}.$$

To illustrate the effectiveness of the designed controllers, we depict the frequency responses $\lambda_{\min}[\text{sym}(H(z_1, z_2))]$ of the closed-loop systems, presented in Fig. 2 and 3. It is displayed that the resulting closed-loop systems have positive realness property within the corresponding frequency regions, which demonstrates the effectiveness of the developed methods.

IV. Conclusion

In the paper, a generalized KYP lemma has been proposed for the 2-D FM LSS models. The proposed KYP lemma provides sufficient conditions in terms of LMI for a general quadratic property of the transfer function over a rectangular finite frequency region and includes the existing BRLs and PRLs for FM LSS models as special cases. To obtain this results, an equivalent LMI characterization for a rectangular finite frequency region has been technically constructed. By the developed KYP lemma, an existence condition has been developed for state-feedback controllers guaranteeing the asymptotic stability and finite frequency positive realness of the closed-loop system. As a fundamental result, the 2-D KYP lemma can be potentially applied to many other promising areas such as 2-D signal processing and stability analysis of 2-D systems, which deserve further investigation.

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