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Small-gain results for discrete-time networks of systems with delay

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***Abstract*— In this paper input-to-state stability for large-scale discrete-time networks of systems with delay is considered. Firstly, interconnection delays between systems, which can arise due to the propagation of signals over large distances, are treated. It is established that such delays cannot cause the instability of the network if a delay-independent small-gain condition holds. Secondly, local delays in each system, which can arise due to inherent delays in each local dynamical process, are considered. For this set-up, using a small-gain condition, the stability of the network of systems is established via the Razumikhin method and the Krasovskii approach, respectively. By combining the results for networks of systems with communication and local delays, respectively, a framework for ISS analysis for general networks with delay is obtained.**

I. INTRODUCTION

Large-scale networks of systems, such as electrical power networks, chemical processes and urban water supply networks, form an important topic within the field of control systems, see, e.g., [1], [2] and the references therein. The stability analysis, e.g., based on Lyapunov theory, of such systems is generally complicated by the large size and complexity of the network. Therefore, a stability analysis for the separate systems in the network is typically performed first. Then, the stability of the overall network is studied. To this end, small-gain theorems, such as the ones presented in [3] and [4], can be used. Alternatively, vector Lyapunov functions [5] or dissipativity theory [6] can also be used to perform a stability analysis for the network.

In practice, networks of systems, such as, for example, electrical power networks, often show a geographical separation of the systems. Hence, the propagation of signals takes place over large distances which can induce interconnection delays. Furthermore, due to inherent delays in the dynamical processes, local delays can also arise in the systems in the network. As delays can cause the instability of a dynamical system [7], they need to be taken into account in the stability analysis. To this end, the Razumikhin method and the Krasovskii approach, see, e.g., [7], [8], were proposed as extensions of Lyapunov theory to delay systems. In fact, as indicated in [9], the Razumikhin method is a type of small-gain approach that handles delays. Recently, based on the aforementioned extensions of Lyapunov theory, several small-gain theorems for networks of systems with delay were proposed. For example, based on the Krasovskii approach, the ISS analysis for a network of two systems with both interconnection and local delays was performed in [10]. This result was then extended to a network with an arbitrary number of systems in [11]. Furthermore, also in [11], it was established that a similar result can be obtained based on the Razumikhin method. Alternatively, the relation of the Razumikhin method to the small-gain theorem established in [9] was used in [12] to formulate a small-gain theorem for networks with delays. A different approach was taken in [13]. Therein, a small-gain theorem for networks with delays was established using standard small-gain arguments without the use of Lyapunov theory. Unfortunately, none of the above results applies to *discrete-time* systems with delay.

As most modern controllers are implemented via a computer and hence discrete-time systems form an important modeling class, in this paper, the stability of networks of discrete-time systems with delay is studied. Firstly, the stability of networks of systems with interconnection delays is considered. Based on the transformation of a delay difference equation into a network of difference equations proposed in [14], it is established that interconnection delays in a network of systems cannot cause the instability of that network if a delay-independent small-gain condition holds. Then, networks of systems with local delays are considered. It is established that such a network admits a so-called Lyapunov-Razumikhin function if each system in the network admits a Lyapunov-Razumikhin function and a small-gain condition is satisfied. A similar result is established using Lyapunov-Krasovskii functions and the same small-gain condition as for the Razumikhin method. By combining all of the above results, a framework for ISS analysis for networks with both local and interconnection delays is obtained.

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$, define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$. Furthermore, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a vector $x \in \mathbb{R}^n$, let $[x]_i$, $i \in \mathbb{Z}_{[1,n]}$ denote the i-th component of x and let $\|x\|$ denote an arbitrary norm. Let $\mathbf{x} := \{x(l)\}_{l\in\mathbb{Z}_+}$ with $x(l) \in \mathbb{R}^n$ for all $l \in \mathbb{Z}_+$ denote an arbitrary sequence and define $\|\mathbf{x}\| := \sup\{\|x(l)\| \mid l \in \mathbb{Z}_+\}$. Furthermore, $\mathbf{x}_{[c_1,c_2]} := \{x(l)\}_{l\in\mathbb{Z}_{[c_1,c_2]}}$, with $c_1, c_2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c_1,c_2]}$. Let $\mathrm{col}(\{x(l)\}_{l\in\mathbb{Z}_{[1,N]}}) :=$

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$\left[\, x(1)^\top \, \ldots \, x(N)^\top \,\right]^\top$ for some $N \in \mathbb{Z}_{\geq 1}$. Let $\mathbb{S}^h := \mathbb{S} \times \ldots \times \mathbb{S}$ for any $h \in \mathbb{Z}_{\geq 1}$ denote the h-times cross-product of an arbitrary set $\mathbb{S} \subseteq \mathbb{R}^n$. Let $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ denote the identity function. For two functions $\varphi_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $\varphi_2 : \mathbb{R}^l \to \mathbb{R}^n$, let $\varphi_1 \circ \varphi_2(x) := \varphi_1(\varphi_2(x))$ for all $x \in \mathbb{R}^l$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Define $\varphi^k(s) := \varphi \circ \varphi^{k-1}(s)$ for all $k \in \mathbb{Z}_{\geq 1}$ and all $s \in \mathbb{R}_+$, where $\varphi^0(s) := s$. Furthermore, $\varphi \in \mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. $\varphi \in \mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. The notation $\varphi \in \mathcal{K} \cup \{0\}$ ($\varphi \in \mathcal{K}_\infty \cup \{0\}$) is used to denote that either $\varphi \in \mathcal{K}$ ($\varphi \in \mathcal{K}_\infty$) or $\varphi(s) = 0$ for all $s \in \mathbb{R}_+$. Let $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. $\beta \in \mathcal{KL}$ if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$.

A. Delay difference equations

Consider the delay difference equation

$$x(k+1) = F(\mathbf{x}_{[k-h,k]}, u(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $\mathbf{x}_{[k-h,k]} \in (\mathbb{R}^n)^{h+1}$ is a sequence of (delayed) states, $h \in \mathbb{Z}_+$ is the maximal delay and $u(k) \in \mathbb{R}^m$ is a disturbance input. Furthermore, $F : (\mathbb{R}^n)^{h+1} \times \mathbb{R}^m \to \mathbb{R}^n$ is a function with the origin as equilibrium point, i.e., $F(\mathbf{0}_{[-h,0]}, 0) = 0$. The notation $\{x(k, \mathbf{x}_{[-h,0]}, \mathbf{u}_{[0,k-1]})\}_{k \in \mathbb{Z}_{\geq 1}}$ is used to denote a trajectory of system (1) from $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ with disturbance $\mathbf{u}_{[0,k-1]} := \{u(i)\}_{i \in \mathbb{Z}_{[0,k-1]}}$, $u(i) \in \mathbb{R}^m$.

Definition 1 System (1) is called *input-to-state stable* (ISS) if there exist a $\beta \in \mathcal{KL}$ and a $\gamma_u \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_{\geq 1}$ it holds that

$$\begin{aligned}&\|x(k, \mathbf{x}_{[-h,0]}, \mathbf{u}_{[0,k-1]})\| \\ &\qquad \leq \max\{\beta(\|\mathbf{x}_{[-h,0]}\|, k), \gamma_u(\|\mathbf{u}_{[0,k-1]}\|)\},\end{aligned}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\mathbf{u}_{[0,k-1]} := \{u(i)\}_{i \in \mathbb{Z}_{[0,k-1]}}$, with $u(i) \in \mathbb{R}^m$. □

Note that, ISS, cf. Definition 1, is a global property which is often referred to as global ISS. Furthermore, as $\max\{r, s\} \leq r + s$ and $r + s \leq \max\{2r, 2s\}$ for all $r, s \in \mathbb{R}_+$, Definition 1 is equivalent to Definition 2.2 in [15], which was also indicated therein.

B. A small-gain theorem for networks of systems

Consider a set of $N \in \mathbb{Z}_{\geq 2}$ interconnected systems. The dynamics of the i-th system, $i \in \mathbb{Z}_{[1,N]}$, is given by

$$x_i(k+1) = g_i(x_1(k), \ldots, x_N(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{2}$$

where $x_i(k) \in \mathbb{R}^{n_i}$, $u(k) \in \mathbb{R}^m$ is a disturbance input and $g_i : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i \in \mathbb{Z}_{[1,N]}$, is a function with the origin as equilibrium point. The network of systems is described via the state vector $x := \mathrm{col}(\{x_l\}_{l \in \mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$, which yields

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{3}$$

where $n := \sum_{i=1}^N n_i$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is obtained from the functions g_i, $i \in \mathbb{Z}_{[1,N]}$.

Next, let $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$, $\mu_i \in \mathcal{K} \cup \{0\}$, for all $i \in \mathbb{Z}_{[1,N]}$, and let $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$ for all $i, j \in \mathbb{Z}_{[1,N]}$. Then, consider a set of functions $W_j : \mathbb{R}^{n_j} \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,N]}$, that satisfy

$$\alpha_{1,j}(\|x_j\|) \leq W_j(x_j) \leq \alpha_{2,j}(\|x_j\|), \quad \forall x_j \in \mathbb{R}^{n_j}.$$

Definition 2 Let $\gamma_{i,i}(s) < s$ for all $s \in \mathbb{R}_{>0}$. A function W_i, $i \in \mathbb{Z}_{[1,N]}$, that satisfies

$$\begin{aligned}&W_i(g_i(x_1, \ldots, x_N, u)) \\ &\qquad \leq \max\{\max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i,j} \circ W_j(x_j), \mu_i(\|u\|)\},\end{aligned}$$

for all $x_j \in \mathbb{R}^{n_j}$, $j \in \mathbb{Z}_{[1,N]}$, and all $u \in \mathbb{R}^m$ is called an *ISS-Lyapunov function* (ISS-LF) for system (2). □

Note that Definition 2 requires that W_i is an ISS-LF for system (2), i.e., for $x_j = 0$ for all $j \neq i$. Moreover, it also requires that the input of the other systems to (2) can be bounded via $\gamma_{i,j}$. Next, consider the following nonlinear small-gain theorem for interconnected difference equations.

Theorem 3 *Suppose that all systems* (2)*,* $i \in \mathbb{Z}_{[1,N]}$*, admit an ISS-LF. Furthermore, suppose that for all* $y \in \mathbb{R}_+^N \setminus \{0\}$*, there exists a* $i(y) \in \mathbb{Z}_{[1,N]}$ *such that*

$$\max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i,j}([y]_j) < [y]_i. \tag{4}$$

Then, the following claims hold:

(i) *There exist* $\sigma_i \in \mathcal{K}_\infty$*,* $i \in \mathbb{Z}_{[1,N]}$*, such that*

$$W(x) = \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(W_i(x_i)) \tag{5}$$

is an ISS-LF for the network of systems (3)*;*

(ii) *The network of systems* (3) *is ISS.* □

The proof of Theorem 3 can be obtained *mutatis mutandis* from the proof of Corollary 5.7 in [3] and is omitted here for brevity. Alternative small-gain theorems for interconnected difference equations, which also parallel the continuous-time results in [3], can be found in, e.g., [16], [17].

In what follows, Theorem 3 and the techniques presented in [14] will be used to obtain small-gain theorems for networks of discrete-time systems with delay.

III. NETWORKS OF SYSTEMS WITH INTERCONNECTION DELAYS

In practice, the systems (2) can be located in different geographical places. This geographical separation introduces delays on the interconnection channels, called interconnection delays, as it is the case for hydro-thermal power networks. Therefore, consider the set of $N \in \mathbb{Z}_{\geq 2}$ interconnected systems (2) with interconnection delays, i.e.,

$$\begin{aligned}x_i(k+1) = g_i(x_1(k - h_{i,1}), \ldots,& \\ x_N(k - h_{i,N}), u(k)),& \quad k \in \mathbb{Z}_+,\end{aligned} \tag{6}$$

where $x_i(k) \in \mathbb{R}^{n_i}$, $u(k) \in \mathbb{R}^m$ and g_i, $i \in \mathbb{Z}_{[1,N]}$, as defined in (2). Moreover, $h_{i,j} \in \mathbb{Z}_+$, $i, j \in \mathbb{Z}_{[1,N]}$, is the interconnection delay from system j to system i. In this section, *it is assumed that* $h_{i,i} = 0$ *for all* $i \in \mathbb{Z}_{[1,N]}$, i.e., the systems are not affected by local delay. To describe the network of systems, let $x := \mathrm{col}(\{x_l\}_{l\in\mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$, which yields a network of the form (1), where $n := \sum_{i=1}^N n_i$, $h := \max_{(i,j)\in\mathbb{Z}_{[1,N]}\times\mathbb{Z}_{[1,N]}} h_{i,j}$ and F is obtained from the functions g_i and the delays $h_{i,j}$, $i,j \in \mathbb{Z}_{[1,N]}$.

The ISS analysis for the network of systems with interconnection delays, i.e., the network (1) obtained from (6), using traditional techniques for delay systems is hampered by the complexity of the network. In what follows, it is established that the ISS analysis can be greatly simplified when small-gain arguments are used.

Theorem 4 *Suppose that the systems* (2) *satisfy the hypothesis of Theorem 3. Then, the network of systems* (1) *obtained from* (6)*, is ISS.*

Proof: The proof consists of three parts. Firstly, the network with interconnection delays is transformed into an augmented network of systems without delay. Therefore, let $\bar{h} := \sum_{i=1}^N \sum_{j=1}^N h_{i,j}$ and consider any $I, J \in \mathbb{Z}_{[1,N]}$. Throughout this proof the numbers I and J refer to an interconnection in the network from system J to system I. If $h_{I,J} \geq 1$, let $x_{N+1}(k) := x_J(k-1)$ for all $k \in \mathbb{Z}_+$, which yields $\hat{g}_{N+1}(x_1,\dots,x_{N+\bar{h}},u) = x_J$. Moreover, if $h_{I,J} \geq 2$ let $x_{N+l+1}(k) := x_{N+l}(k-1)$ for all $k \in \mathbb{Z}_+$ and all $l \in \mathbb{Z}_{[1,h_{I,J}-1]}$, which yields $\hat{g}_{N+l+1}(x_1,\dots,x_{N+\bar{h}},u) = x_{N+l}$. A similar procedure is applied for all $h_{i,j} \in \mathbb{Z}_+$ with $i,j \in \mathbb{Z}_{[1,N]}$. Therein, the number N is replaced by $\hat{N}$ which denotes the size of the augmented vector that was obtained so far. Thus, the delayed states in (6) can be replaced by newly introduced states such that a network of $N+\bar{h}$ interconnected systems without delay is obtained, i.e.,

$$x_i(k+1) = \hat{g}_i(x_1(k),\dots,x_{N+\bar{h}}(k),u(k)), \quad k \in \mathbb{Z}_+, \quad (7)$$

with $i \in \mathbb{Z}_{[1,N+\bar{h}]}$. To describe the augmented network, let $\xi := \mathrm{col}(\{x_l\}_{l\in\mathbb{Z}_{[1,N+\bar{h}]}})$, which yields

$$\xi(k+1) = \hat{G}(\xi(k),u(k)), \quad k \in \mathbb{Z}_+. \quad (8)$$

A graphical depiction of the transformation of the network with interconnection delays into the augmented network without delay (8) is shown in Figure 1.

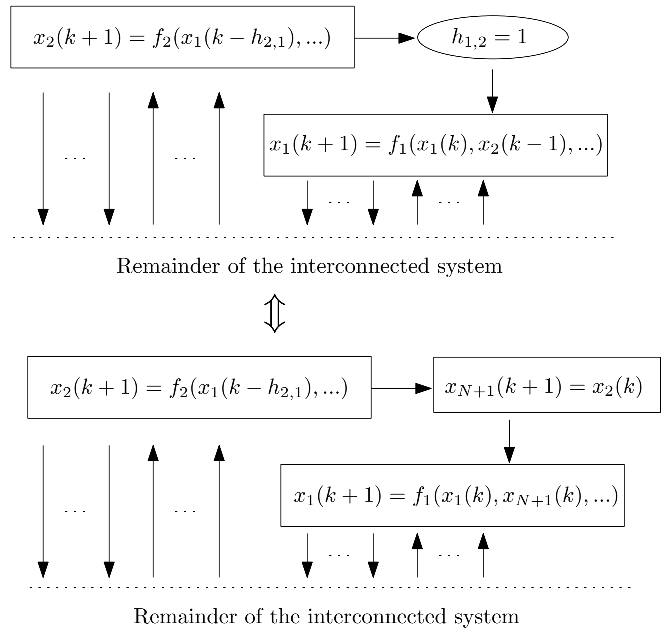

Fig. 1. A graphical depiction of the transformation of a network with interconnection delays into an augmented network without delay.

Secondly, appropriate ISS-LF candidates are selected for the newly introduced systems (7) and it is shown that if the network (1) obtained from (6) satisfies the small-gain condition (4), then the augmented network (8) satisfies a similar small-gain condition. Therefore, note that it follows from the hypotheses of Theorem 3 that, *for all* $i \in \mathbb{Z}_{[1,N]}$, the systems (7) admit an ISS-LF, i.e., W_i. Next, if $h_{1,2} \geq 1$, let $W_{N+1}(x_{N+1}) := W_2(x_{N+1})$, which yields $W_{N+1}(\hat{g}_{N+1}(x_1,\dots,x_{N+\bar{h}},u)) \leq W_2(x_2)$. Moreover, if $h_{1,2} \geq 2$ let $W_{N+l+1}(x_{N+l+1}) := W_2(x_{N+l+1})$ for all $l \in \mathbb{Z}_{[1,h_{1,2}-1]}$, which yields $W_{N+l+1}(\hat{g}_{N+l+1}(x_1,\dots,x_{N+\bar{h}},u)) \leq W_{N+l}(x_{N+l})$. Again, the same procedure is applied for all $h_{i,j} \in \mathbb{Z}_+$, $i,j \in \mathbb{Z}_{[1,N]}$. Thus, an ISS-LF is obtained for all the systems (7), $i \in \mathbb{Z}_{[1,N+\bar{h}]}$. Next, the corresponding gain functions $\gamma_{i,j}$ are defined recursively and it is shown that they satisfy a small-gain condition. Therefore, let $\gamma_{i,j}^0 := \gamma_{i,j}$ for all $i,j \in \mathbb{Z}_{[1,N]}$. Furthermore, let $(I,J) \in \mathbb{Z}_{[1,N+l]} \times \mathbb{Z}_{[1,N+l]}$ correspond to the interconnection with delay between system J and I for which the new state x_{N+l+1} was introduced and define

$$\gamma_{i,j}^{l+1} := \begin{cases} 0, & i = I,\ j = J \text{ or } i \neq I,\ j = N+l+1 \\ & \text{or } i = N+l+1,\ j \neq J \\ \gamma_{I,J}^l, & i = I,\ j = N+l+1 \\ \mathrm{Id}, & i = N+l+1,\ j = J \\ \gamma_{i,j}^l, & \text{otherwise,} \end{cases}$$

for all $i,j \in \mathbb{Z}_{[1,N+l+1]}$ and all $l \in \mathbb{Z}_{[0,\bar{h}-1]}$. In what follows, it is proven, by induction, that for all $l \in \mathbb{Z}_{[0,\bar{h}]}$ and all $y \in \mathbb{R}_+^{N+l} \setminus \{0\}$, there exists a $i(y) \in \mathbb{Z}_{[1,N+l]}$ such that

$$\max_{j\in\mathbb{Z}_{[1,N+l]}} \gamma_{i,j}^l([y]_j) < [y]_i. \quad (9)$$

Therefore, let $l = 0$ and let $y := \begin{bmatrix}\bar{y}^\top & \tilde{y}\end{bmatrix}^\top$ for any $\bar{y} \in \mathbb{R}_+^N$ and $\tilde{y} \in \mathbb{R}_+$ such that $y \neq 0$. If $\tilde{y} \leq [\bar{y}]_J$, then it follows from (4) that (9) with $l = 1$ holds for $i(y) = i(\bar{y})$. Conversely, if $\tilde{y} > [\bar{y}]_J$, then

$$\max_{j\in\mathbb{Z}_{[1,N+1]}} \gamma_{N+1,j}^1([y]_j) = [y]_J < \tilde{y} = [y]_{N+1}.$$

Thus, it has been established that (9) with $l = 1$ holds for $i(y) = N+1$. Next, consider any $\ell \in \mathbb{Z}_{[0,\bar{h}-1]}$ and suppose that (9) with $l = \ell$ holds, i.e., for all $\bar{y} \in \mathbb{R}_+^{N+\ell}$ there exists some $i(\bar{y})$ such that (9) holds. Let $y := \begin{bmatrix}\bar{y}^\top & \tilde{y}\end{bmatrix}^\top$ for any $\bar{y} \in \mathbb{R}_+^{N+\ell}$ and $\tilde{y} \in \mathbb{R}_+$ such that $y \neq 0$. If $\tilde{y} \leq [\bar{y}]_J$, then it follows from (9) with $l = \ell$ that (9) with $l = \ell+1$ also holds for $i(y) = i(\bar{y})$. Conversely, if $\tilde{y} > [\bar{y}]_J$, then

$$\max_{j\in\mathbb{Z}_{[1,N+\ell+1]}} \gamma_{N+\ell+1,j}^{\ell+1}([y]_j) = [y]_J < \tilde{y} = [y]_{N+\ell+1}.$$

Hence, (9) with $l = \ell + 1$ holds for $i(y) = N + \ell$. Thus, it has been established, by induction, that (9) holds for any $l \in \mathbb{Z}_{[0,\bar{h}]}$. Therefore, the small-gain condition (4) holds for the augmented network (8) and it follows from Theorem 3 that (8) is ISS.

Thirdly, it has to be shown that the network of systems with interconnection delays is ISS when the augmented network of systems (8) is ISS. The proof of this claim, which is omitted here for brevity, can be obtained using the arguments used in the proof of Lemma III.3 in [18]. □

Theorem 4 establishes that *finite interconnection delays cannot cause the instability of a network of systems* if the delay-independent small-gain condition (4) holds. Hence, the ISS analysis for a network with interconnection delays can be reduced to the ISS analysis for a standard network without delay. A similar relation was also mentioned for the interconnection of *two systems only* in [10]. The above discussion indicates an important advantage of considering interconnection and local delays separately as opposed to considering them both at once, as done in [11]–[13].

Note that, Theorem 4 does not assume any knowledge about the interconnection delay. If the interconnection delay is assumed to be known, potentially less conservative delay-dependent small-gain conditions can be derived, see, e.g., [10] for the continuous-time case.

IV. NETWORKS WITH LOCAL DELAYS

Another cause for delay in networks of systems are inherent delays in the dynamical process in one or more of the systems. Therefore, consider a set of $N \in \mathbb{Z}_{\geq 2}$ interconnected systems affected by local delays. The dynamics of the i-th system, $i \in \mathbb{Z}_{[1,N]}$, is given by

$$x_i(k+1) = f_i(\mathbf{x}_{[k-\hat{h},k];i}, x_1(k), \ldots, x_N(k), u(k)), \quad (10)$$

where $\mathbf{x}_{[k-\hat{h},k];i} := \{x_i(k-j)\}_{j\in\mathbb{Z}_{[0,\hat{h}]}}$. Furthermore, $k \in \mathbb{Z}_+$ and $f_i : (\mathbb{R}^{n_i})^{\hat{h}+1} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i \in \mathbb{Z}_{[1,N]}$. Note that, with a slight abuse of notation, to simplify the exposition, $x_i(k)$ appears twice as an argument of f_i. Above, $\hat{h} \in \mathbb{Z}_+$ is the maximal delay affecting the systems (10), $i \in \mathbb{Z}_{[1,N]}$. The complete network of systems can again be described by (1) with $h = \hat{h}$ and where F is obtained from f_i, $i \in \mathbb{Z}_{[1,N]}$.

Next, let $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$, $\mu_i \in \mathcal{K} \cup \{0\}$, for all $i \in \mathbb{Z}_{[1,N]}$, and let $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$ for all $i, j \in \mathbb{Z}_{[1,N]}$. Then, consider the functions $W_j : \mathbb{R}^{n_j} \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,N]}$, satisfying

$$\alpha_{1,j}(\|x_j\|) \leq W_j(x_j) \leq \alpha_{2,j}(\|x_j\|), \quad \forall x_j \in \mathbb{R}^{n_j}.$$

Definition 5 Let $\gamma_{i,i}(s) < s$ for all $s \in \mathbb{R}_{>0}$. A function W_i, $i \in \mathbb{Z}_{[1,N]}$, that satisfies

$$W_i(f_i(\mathbf{x}_{[-h,0];i}, x_1, \ldots, x_N, u)) \leq \max\{\max_{\theta\in\mathbb{Z}_{[-h,0]}} \gamma_{i,i} \circ W_i(x_i(\theta)), \max_{j\in\mathbb{Z}_{[1,N]}, j\neq i} \gamma_{i,j} \circ W_j(x_j), \mu_i(\|u\|)\},$$

for all $\mathbf{x}_{[-h,0];i} \in (\mathbb{R}^{n_i})^{h+1}$, $x_j \in \mathbb{R}^{n_j}$, $j \in \mathbb{Z}_{[1,N]}$ and $j \neq i$, $x_i := x_i(0)$ and all $u \in \mathbb{R}^m$ is called an *ISS-Lyapunov-Razumikhin function* (ISS-LRF) for system (10). □

The interested reader is referred to [14], [15] for a detailed discussion on the ISS analysis for single delay difference equations based on the existence of an ISS-LRF.

In what follows, a counterpart to Theorem 3 for networks of systems with local delays, is established.

Theorem 6 *Suppose that all systems* (10)*, $i \in \mathbb{Z}_{[1,N]}$, admit an ISS-LRF. Furthermore, suppose that for all $y \in \mathbb{R}^N_+ \setminus \{0\}$, there exists a $i(y) \in \mathbb{Z}_{[1,N]}$ such that*

$$\max_{j\in\mathbb{Z}_{[1,N]}} \gamma_{i,j}([y]_j) < [y]_i. \quad (11)$$

Then, the following claims hold:

(i) *There exist $\sigma_i \in \mathcal{K}_\infty$, $i \in \mathbb{Z}_{[1,N]}$, such that*

$$W(x) = \max_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ W_i(x_i) \quad (12)$$

is an ISS-LRF for the network (1) *obtained from* (10)*;*

(ii) *The network* (1) *obtained from* (10) *is ISS.*

Proof: The proof of claim (i) consists of two parts. Firstly, it is established that the small-gain condition (11) implies the existence of a set functions σ_i, $i \in \mathbb{Z}_{[1,N]}$, satisfying a particular condition. Secondly, these functions and the corresponding condition are used to construct an ISS-LRF for the network (1) obtained from (10).

It follows from the small-gain condition (11) that the hypothesis of Theorem 5.2, claim (iii) in [3] is satisfied. Therefore, there exist $\sigma_i \in \mathcal{K}_\infty$, $i \in \mathbb{Z}_{[1,N]}$, such that

$$\max_{j\in\mathbb{Z}_{[1,N]}} \gamma_{i,j} \circ \sigma_j([y]_j) < \sigma_i([y]_i), \quad (13)$$

for all $y \in \mathbb{R}^N_+ \setminus \{0\}$ and all $i \in \mathbb{Z}_{[1,N]}$.

Next, consider the candidate ISS-LRF (12) and consider any $(\mathbf{x}_{[-h,0]}, u) \in (\mathbb{R}^n)^{h+1} \times \mathbb{R}^m$. Let $\mu(s) := \max_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \mu_i(s)$ for all $s \in \mathbb{R}_+$ and let

$$\gamma(s) := \max_{i\in\mathbb{Z}_{[1,N]}} \max_{j\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \gamma_{i,j} \circ \sigma_j(s).$$

Then, $\gamma \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and it follows from (13) that $\gamma(s) < s$ for all $s \in \mathbb{R}_{>0}$. Furthermore,

$$\begin{aligned}
&W(F(\mathbf{x}_{[-h,0]}, u)) \\
&= \max_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ W_i(f_i(\mathbf{x}_{[-h,0];i}, x_1(0), \ldots, x_N(0), u)) \\
&\leq \max_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \max\{\max_{\theta\in\mathbb{Z}_{[-h,0]}} \gamma_{i,i} \circ W_i(x_i(\theta)), \\
&\qquad \max_{j\in\mathbb{Z}_{[1,N]}, j\neq i} \gamma_{i,j} \circ W_j(x_j(0)), \mu_i(\|u\|)\} \\
&\leq \max\{\max_{\theta\in\mathbb{Z}_{[-h,0]}} \max_{i\in\mathbb{Z}_{[1,N]}} \max_{j\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \gamma_{i,j} \circ \sigma_j \circ \sigma_j^{-1} \circ \\
&\qquad \circ W_j(x_j(\theta)), \mu(\|u\|)\} \\
&\leq \max\{\max_{\theta\in\mathbb{Z}_{[-h,0]}} \gamma \circ \max_{i'\in\mathbb{Z}_{[1,N]}} \sigma_{i'}^{-1} \circ W_{i'}(x_{i'}(\theta)), \mu(\|u\|)\} \\
&\leq \max\{\max_{\theta\in\mathbb{Z}_{[-h,0]}} \gamma \circ W(x(\theta)), \mu(\|u\|)\}.
\end{aligned}$$

Furthermore, the equivalence of norms [19] yields that there exist some $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$c_1 \max_{i\in\mathbb{Z}_{[1,N]}} \|x_i\| \leq \|x\| \leq c_2 \max_{i\in\mathbb{Z}_{[1,N]}} \|x_i\|.$$

Hence, the $\mathcal{K}_\infty$ bounds for the functions W_i, $i \in \mathbb{Z}_{[1,N]}$, yield $\min_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_{1,i}(c_2^{-1}\|x\|) \leq W(x) \leq \max_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_{2,i}(c_1^{-1}\|x\|)$. Therefore, let $\alpha_1(s) = \min_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_{1,i}(c_2^{-1}s)$ and let $\alpha_2(s) = \max_{i\in\mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_{2,i}(c_1^{-1}s)$. As $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ it follows that the function W is an ISS-LRF for the network (1) obtained from (10), which establishes claim (i).

Observing that claim (ii) follows directly from Theorem 6 in [14] completes the proof. □

Thus, it was established that if all systems (10) admit an ISS-LRF and the small-gain condition (11) holds, then the network of systems with local delays admits an ISS-LRF. Moreover, as a direct consequence, it also follows that the network of systems with local delays is ISS.

Next, a result similar to Theorem 6 is established using the Krasovskii approach. Therefore, let $\bar{\alpha}_{1,i}, \bar{\alpha}_{2,i} \in \mathcal{K}_\infty$, $\bar{\mu}_i \in \mathcal{K} \cup \{0\}$, for all $i \in \mathbb{Z}_{[1,N]}$, and let $\bar{\gamma}_{i,j} \in \mathcal{K}_\infty \cup \{0\}$ for all $i, j \in \mathbb{Z}_{[1,N]}$. Then, consider the functions $\bar{W}_j : (\mathbb{R}^{n_j})^{(h+1)} \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,N]}$, satisfying

$$\bar{\alpha}_{1,j}(\|\mathbf{x}_{[-h,0];j}\|) \leq \bar{W}_j(\mathbf{x}_{[-h,0];j}) \leq \bar{\alpha}_{2,j}(\|\mathbf{x}_{[-h,0];j}\|),$$

for all $\mathbf{x}_{[-h,0];j} \in (\mathbb{R}^{n_j})^{h+1}$.

Definition 7 Let $\bar{\gamma}_{i,i}(s) < s$ for all $s \in \mathbb{R}_{>0}$. A function $\bar{W}_i : (\mathbb{R}^{n_i})^{h+1} \to \mathbb{R}_+$ that satisfies

$$\begin{aligned}\bar{W}_i(\{&\mathbf{x}_{[-h+1,0];i}, f_i(\mathbf{x}_{[-h,0];i}, x_1(0), \ldots, x_N(0), u)\}) \\ &\leq \max\{\max_{j\in\mathbb{Z}_{[1,N]}} \bar{\gamma}_{i,j} \circ \bar{W}_j(\mathbf{x}_{[-h,0];j}), \bar{\mu}_i(\|u\|)\},\end{aligned}$$

for all $\mathbf{x}_{[-h,0];j} \in (\mathbb{R}^{n_j})^{h+1}$, $j \in \mathbb{Z}_{[1,N]}$, and all $u \in \mathbb{R}^m$ is called an *ISS-Lyapunov-Krasovskii function* (ISS-LKF) for system (10). □

Theorem 8 *Suppose that all systems* (10)*, $i \in \mathbb{Z}_{[1,N]}$, admit an ISS-LKF. Furthermore, suppose that for all $y \in \mathbb{R}_+^N \setminus \{0\}$, there exists a $i(y) \in \mathbb{Z}_{[1,N]}$ such that*

$$\max_{j\in\mathbb{Z}_{[1,N]}} \bar{\gamma}_{i,j}([y]_j) < [y]_i. \tag{14}$$

Then, the following claims hold:

(i) *There exist $\bar{\sigma}_i \in \mathcal{K}_\infty$, $i \in \mathbb{Z}_{[1,N]}$, such that*

$$\bar{W}(\mathbf{x}_{[-h,0]}) = \max_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{W}_i(\mathbf{x}_{[-h,0];i}) \tag{15}$$

is an ISS-LKF for the network (1) *obtained from* (10)*;*

(ii) *The network* (1) *obtained from* (10) *is ISS.*

Proof: The proof of Theorem 8 proceeds along similar lines as the proof of Theorem 6.

It follows from the small-gain condition (14) that the hypothesis of Theorem 5.2, claim (iii) in [3] is satisfied. Therefore, there exist $\bar{\sigma}_i \in \mathcal{K}_\infty$, $i \in \mathbb{Z}_{[1,N]}$, such that

$$\max_{j\in\mathbb{Z}_{[1,N]}} \bar{\gamma}_{i,j} \circ \bar{\sigma}_j([y]_j) < \bar{\sigma}_i([y]_i), \tag{16}$$

for all $y \in \mathbb{R}_+^N \setminus \{0\}$ and all $i \in \mathbb{Z}_{[1,N]}$.

Next, consider the candidate ISS-LKF (15) and consider any $(\mathbf{x}_{[-h,0]}, u) \in (\mathbb{R}^n)^{h+1} \times \mathbb{R}^m$. Let $\bar{\mu}(s) := \max_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{\mu}_i(s)$ for all $s \in \mathbb{R}_+$ and let

$$\bar{\gamma}(s) := \max_{i\in\mathbb{Z}_{[1,N]}} \max_{j\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{\gamma}_{i,j} \circ \bar{\sigma}_j(s).$$

Then, $\bar{\gamma} \in \mathcal{K}_\infty$, $\bar{\mu} \in \mathcal{K}$ and it follows from (16) that $\bar{\gamma}(s) < s$ for all $s \in \mathbb{R}_{>0}$. Furthermore,

$$\begin{aligned}&\bar{W}(\{\mathbf{x}_{[-h+1,0]}, F(\mathbf{x}_{[-h,0]}, u)\}) \\ &= \max_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{W}_i(\{\mathbf{x}_{[-h+1,0];i}, f_i(\mathbf{x}_{[-h,0];i}, \ldots, u)\}) \\ &\leq \max_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \max\{\max_{j\in\mathbb{Z}_{[1,N]}} \bar{\gamma}_{i,j} \circ \bar{W}_j(\mathbf{x}_{[-h,0];j}), \bar{\mu}_i(\|u\|)\} \\ &\leq \max\{\max_{i\in\mathbb{Z}_{[1,N]}} \max_{j\in\mathbb{Z}_{[1,N]}} \\ &\qquad \bar{\sigma}_i^{-1} \circ \bar{\gamma}_{i,j} \circ \bar{\sigma}_j \circ \bar{\sigma}_j^{-1} \circ \bar{W}_j(\mathbf{x}_{[-h,0];j}), \bar{\mu}(\|u\|)\} \\ &\leq \max\{\bar{\gamma} \circ \max_{i'\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_{i'}^{-1} \circ \bar{W}_{i'}(\mathbf{x}_{[-h,0];i'}), \bar{\mu}(\|u\|)\} \\ &\leq \max\{\bar{\gamma} \circ \bar{W}(\mathbf{x}_{[-h,0]}), \bar{\mu}(\|u\|)\}.\end{aligned}$$

Furthermore, the equivalence of norms [19] yields that there exist some $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$c_1 \max_{i\in\mathbb{Z}_{[1,N]}} \|\mathbf{x}_{[-h,0];i}\| \leq \|\mathbf{x}_{[-h,0]}\| \leq c_2 \max_{i\in\mathbb{Z}_{[1,N]}} \|\mathbf{x}_{[-h,0];i}\|.$$

Hence, the $\mathcal{K}_\infty$ bounds for the functions $\bar{W}_i$ yield

$$\begin{aligned}\min_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{\alpha}_{1,i}(c_2^{-1}\|\mathbf{x}_{[-h,0]}\|) &\leq \bar{W}(\mathbf{x}_{[-h,0]}) \\ &\leq \max_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{\alpha}_{2,i}(\|c_1^{-1}\mathbf{x}_{[-h,0]}\|).\end{aligned}$$

Therefore, let $\bar{\alpha}_1(s) = \min_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{\alpha}_{1,i}(c_2^{-1}s)$ and let $\bar{\alpha}_2(s) = \max_{i\in\mathbb{Z}_{[1,N]}} \bar{\sigma}_i^{-1} \circ \bar{\alpha}_{2,i}(c_1^{-1}s)$. Then, as $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ it follows that the function $\bar{W}$ is an ISS-LKF for the network (1) obtained from (10), which establishes claim (i).

Applying the inequality $\bar{W}(\{\mathbf{x}_{[-h+1,0]}, F(\mathbf{x}_{[-h,0]}, u)\}) \leq \max\{\bar{\gamma} \circ \bar{W}(\mathbf{x}_{[-h,0]}), \bar{\mu}(\|u\|)\}$ recursively and using the bounds $\bar{\alpha}_1$ and $\bar{\alpha}_2$ yields that

$$\begin{aligned}&\|x(k, \mathbf{x}_{[-h,0]}, \mathbf{u}_{[0,k-1]})\| \\ &\quad \leq \max\{\bar{\alpha}_1^{-1} \circ \bar{\gamma}^k \circ \bar{\alpha}_2(\|\mathbf{x}_{[-h,0]}\|), \bar{\alpha}_1^{-1} \circ \bar{\mu}(\|\mathbf{u}_{[0,k-1]}\|)\},\end{aligned}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, $\mathbf{u}_{[0,k-1]} \in (\mathbb{R}^m)^k$, and all $k \in \mathbb{Z}_{\geq 1}$. Letting $\beta(r, s) := \bar{\alpha}_1^{-1} \circ \bar{\gamma}^s \circ \bar{\alpha}_2(r)$ it follows, from the fact that $\gamma(s) < s$ for all $s \in \mathbb{R}_{>0}$, that $\beta \in \mathcal{KL}$. Therefore, the network (1) obtained from (10) is ISS with $\beta \in \mathcal{KL}$ and $\gamma_u(s) := \bar{\alpha}_1^{-1} \circ \bar{\mu}(s)$, which establishes claim (ii) and completes the proof. □

Observe that the advantages of standard Krasovskii and Razumikhin theorems when compared to each other, see, e.g., [7], also apply to Theorem 6 and Theorem 8. Therefore, Theorem 6 provides a method to verify stability of networks with delay that is computationally more attractive but conceptually more conservative than the method in Theorem 8.

Remark 9 Theorem 6 and Theorem 8 are discrete-time counterparts of Theorem 3.4 and Theorem 3.7 in [11], respectively. However, the reasoning required to prove the results for the discrete-time case differs significantly with respect to the continuous-time case, mainly due to the different conditions involved in the Razumikhin method. As such, Theorem 6 and Theorem 8 provide a valuable addition to the results presented in [11]. □

V. NETWORKS OF SYSTEMS WITH DELAY

If a network of systems contains one or more systems with delays in the dynamical process and the systems are located in different geographical places, a network of systems with both local and interconnection delays is obtained. The ISS analysis for such systems requires a combination of Theorem 4 with Theorem 6 or Theorem 8, respectively. In this section, such results are derived. Therefore, consider a set of $N \in \mathbb{Z}_{\geq 2}$ interconnected systems with both local and interconnection delays. Then, the dynamics of the i-th system, $i \in \mathbb{Z}_{[1,N]}$, is given by

$$x_i(k+1) = f_i(\mathbf{x}_{[k-\hat{h},k];i}, x_1(k-h_{i,1}), \ldots, x_N(k-h_{i,N}), u(k)), \quad (17)$$

with $k \in \mathbb{Z}_+$, $x_i(k) \in \mathbb{R}^{n_i}$, $u(k) \in \mathbb{R}^m$ and f_i, $i \in \mathbb{Z}_{[1,N]}$, as defined in (10). Above, $h_{i,j} \in \mathbb{Z}_+$, $i,j \in \mathbb{Z}_{[1,N]}$, is the interconnection delay from the j-th to the i-th system. It is assumed that $h_{i,i} = 0$ for all $i \in \mathbb{Z}_{[1,N]}$. Hence, as it was also the case for (10), for ease of notation, $x_i(k)$ appears twice as an argument of f_i. The network can again be described by (1) where $h := \max\{\hat{h}, \max_{(i,j)\in\mathbb{Z}_{[1,N]}\times\mathbb{Z}_{[1,N]}} h_{i,j}\}$, $n := \sum_{i=1}^N n_i$ and F is obtained from the functions f_i and the delays $h_{i,j}$, $i,j \in \mathbb{Z}_{[1,N]}$. The following corollary, which employs the Razumikhin method, can be obtained directly from Theorem 4 and Theorem 6.

Corollary 10 *Suppose that the systems* (10) *satisfy the hypothesis of Theorem 6. Then, the network of systems* (1) *obtained from* (17) *is ISS.* □

Moreover, a similar result can be obtained, using the Krasovskii approach, from Theorem 4 and Theorem 8.

Corollary 11 *Suppose that the systems* (10) *satisfy the hypothesis of Theorem 8. Then, the network of systems* (1) *obtained from* (17) *is ISS.* □

The above general results provide a framework for the ISS analysis for networks of systems with delay. It is also worth noting that Corollary 10 and Corollary 11 reduce, via a delay-independent small-gain condition, the ISS analysis for a network of systems with both local and interconnection delays to the ISS analysis for a network of systems with local delays only. As the ISS analysis for a network of systems with local delays only is in general less complex, Corollary 10 and Corollary 11 provide a simpler tool for the analysis of ISS for networks with delay, when compared to the continuous-time results in, e.g., [10]–[12].

VI. CONCLUSIONS

ISS analysis for networks of systems with delay and subject to external disturbances was considered. A set of small-gain theorems was derived which provides a means to verify stability of networks of systems with both local and interconnection delays. An important conclusion was that finite interconnection delays cannot cause the instability of the network of systems if a delay-independent small-gain condition holds. Future work deals with the application of these results to large-scale power systems.

REFERENCES

[1] A. N. Michel and R. K. Miller, *Qualatitive analysis of large scale dynamical systems*, ser. Mathematics in Science and Engineering. New York (NY): Academic Press, Inc., 1977, vol. 134.

[2] M. Vidyasagar, *Input-Output Analysis of Large-scale Interconnected Systems*, ser. Lecture Notes in Control and Information Sciences. Berlin, Germany: Springer-Verlag, 1981, vol. 29.

[3] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM Journal on Control and Optimization*, vol. 48, no. 6, pp. 4089–4118, 2010.

[4] I. Karafyllis and Z.-P. Jiang, "A vector small-gain theorem for general nonlinear control systems," in *48th IEEE Conference on Decision and Control*, Shanghai, China, 2009, pp. 7996–8001.

[5] V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram, *Vector Lyapunov functions and stability analysis of nonlinear systems*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1991.

[6] J. C. Willems, "Dissipative dynamical systems," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–393, 1972.

[7] V. Kolmanovskii and A. Myshkis, *Introduction to the theory and applications of functional differential equations*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1999.

[8] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of time-delay systems*. Boston, MA: Birkhäuser, 2003.

[9] A. R. Teel, "Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem," *IEEE Transactions on Automatic Control*, vol. 43, no. 7, pp. 960–964, 1998.

[10] H. Ito, P. Pepe, and Z.-P. Jiang, "A small-gain condition for iISS of interconnected retarded systems based on Lyapunov-Krasovskii functionals," *Automatica*, vol. 46, no. 10, pp. 1646–1656, 2010.

[11] S. N. Dashkovskiy and L. Naujok, "Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems for interconnected ISS time-delay systems," in *19th International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary, 2010, pp. 1179–1184.

[12] S. Tiwari and Y. Wang, "Razumikhin-type small-gain theorems for large-scale systems with delays," in *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, GA, 2010, pp. 7407–7412.

[13] S. Tiwari, Y. Wang, and Z.-P. Jiang, "A nonlinear small-gain theorem for large-scale time delay systems," in *Proceedings of the 48th IEEE Conference on Decision and Control*, Shanghai, China, 2009, pp. 7204–7209.

[14] R. H. Gielen, M. Lazar, and A. R. Teel, "On input-to-state stability of delay difference equations," in *Proceedings of the 18th IFAC World Congress*, Milano, Italy, 2011, pp. 3372–3377.

[15] B. Liu and D. J. Hill, "Input-to-state stability for discrete time-delay systems via the Razumikhin technique," *Systems & Control Letters*, vol. 58, pp. 567–575, 2009.

[16] D. M. Raimondo, L. Magni, and R. Scattolini, "Decentralized MPC of nonlinear systems: An input-to-state stability approach," *International Journal of Robust and Nonlinear Control*, vol. 17, pp. 1651–1667, 2007.

[17] T. Liu, D. J. Hill, and Z.-P. Jiang, "Lyapunov formulation of ISS cyclic-small-gain in discrete-time dynamical networks," in *Proceedings of the 8th WCICA*, Jinan, China, 2010, pp. 568–573.

[18] R. H. Gielen, M. Lazar, and I. V. Kolmanovsky, "On Lyapunov theory for delay difference inclusions," in *Proceedings of the American Control Conference*, Baltimore, MD, 2010, pp. 3697–3703.

[19] E. Kreyszig, *Introductory functional analysis with applications*. John Wiley & Sons, 1989.