

Infinite Matrix Representations of Robust Stability Conditions for Discrete-Time Systems

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Abstract— This paper is motivated by the study on clarifying further relationship between the conventional lifting-free scaling and lifting-based noncausal linear periodically time-varying scaling approaches to robust stability analysis. To facilitate such a study, this paper gives the infinite matrix representation counterparts of the robust stability conditions in the separator-type robust stability theorems for these approaches. These counterparts lead to the idea of infinite-dimensional separators, and provide us with a unified framework for studying the mutual relationship between these two approaches. Through the derivation and comparison of explicit forms of infinite-dimensional separators in these two approaches, it is demonstrated that the infinite matrix representation framework leads to a very comprehensible and intuitive study on the mutual relationship between these approaches.

I. INTRODUCTION

Since modeling of real objects inevitably gives rise to uncertainties, robustness is quite important in the application of control theory. Robust stability analysis problems have been studied intensively, and some practical frameworks, e.g., multiplier [1],[2] and integral quadratic constraints (IQC) [3], have been developed. Both multiplier and IQC approaches ensure robust stability of the closed-loop system if an appropriate matrix exists satisfying some inequalities for a given class of uncertainties. The relationship between these two approaches has also been discussed in [4].

This paper is concerned with the separator-type robust stability theorem [5] for robust stability analysis of linear time-invariant (LTI) discrete-time systems. This theorem is closely related with the IQC approach through the topological separation notion [2], and gives a necessary and sufficient condition for robust stability with respect to a general class of LTI uncertainties. Various types of robust stability conditions [6]–[8] (e.g., the small-gain and passivity theorems, D -scaling, (D, G) -scaling and multiplier methods) are covered by this theorem (as well as the IQC theorem [3]) by appropriately confining the matrices in the theorem called separators. Robust stability can be analyzed by searching for separators satisfying the robust stability condition therein (such separators are said to be eligible in the following). To achieve nonconservative robust stability analysis, however, such a search must work on all frequency-dependent (i.e., dynamic) separators without any constraint, but this is not feasible from computational viewpoints. Thus, a tractable class of separators is introduced in practice, only on which the search of eligible separators is pursued. This generally

results in conservativeness in robust stability analysis, as is the case with the IQC and multiplier approaches.

For reducing the conservativeness, discrete-time noncausal linear periodically time-varying (LPTV) scaling was proposed in [9] recently. This approach can be naturally introduced when we employ the separator-type robust stability theorem under the lifting treatment [10] of discrete-time systems. This lifting-based scaling has been shown to generally induce dynamic scaling if it is interpreted in the framework of the lifting-free (i.e., conventional) scaling approach. More precisely, if a (possibly static) eligible separator $\hat{\theta}$ is found in the lifting-based framework of noncausal LPTV scaling, it immediately leads to an eligible dynamic separator $\theta = \varphi(\hat{\theta})$ in the conventional lifting-free framework, where the explicit form of the (non-injective) mapping $\varphi(\cdot)$ has also been clarified. With this mapping, we can formally introduce the class $\Theta := \{\varphi(\hat{\theta}) \mid \hat{\theta} \in \hat{\Theta}\}$, where $\hat{\Theta}$ denotes the class of tractable separators taken in noncausal LPTV scaling. However, it has not been clarified whether every eligible $\theta \in \Theta$ is ensured to have an eligible $\hat{\theta} \in \hat{\Theta}$ such that $\theta = \varphi(\hat{\theta})$; in other words, the subset of such eligible separators in Θ that can indeed be found through the alternative search of eligible $\hat{\theta} \in \hat{\Theta}$ has not been characterized. Even though this does not necessarily imply that noncausal LPTV scaling is less effective than the conventional scaling (since the former does generalize the latter from the viewpoint of causality and time-varying nature), advantages (or drawbacks) of the former over the latter have not been clarified enough especially when the classes of separators are confined to general ones. More fundamentally, in previous studies such as [9], the mutual relationship between the two scaling approaches has not been revealed completely.

This paper aims at developing a new unified framework for dealing with these two approaches that can facilitate their mutual comparison and relevant studies in a very comprehensible and intuitive manner. More explicitly, by means of infinite matrix representations of systems [11], [12], restatements of the robust stability conditions associated with the conventional scaling and the lifting-based noncausal LPTV scaling, reviewed in Section II, are established in a unified fashion in Section III. This leads to the notion of infinite-dimensional separators, whose specific forms are studied in Section IV both in the conventional scaling and noncausal LPTV scaling approaches. Section V combines these arguments to address the main problem. Specifically, it is demonstrated that the infinite matrix framework developed in this paper is very comprehensible and useful in the theoretical study on the mutual relationship between

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the conventional and noncausal LPTV scaling approaches. This is particularly true in comparison with the closely related study in [13] carried out in the frequency domain. Theoretical benefit of the infinite matrix framework is further demonstrated in its usefulness in isolating the effects of time dependence and frequency dependence in scaling through the structure of infinite-dimensional separators.

We use the following notation in this paper. \mathbf{N}_0 denotes the set of nonnegative integers, and $l_2(\mathbf{N}_0, \mathbf{R}^q)$ denotes the set of unilateral infinite series of vectors $v_k \in \mathbf{R}^q$ such that $\sum_{k=0}^{\infty} \|v_k\|^2 < \infty$. For a matrix (\cdot) , its complex conjugate transpose is denoted by $(\cdot)^*$; this is used also for a scalar.

II. SEPARATOR-TYPE ROBUST STABILITY THEOREMS

This section first states the robust stability analysis problem studied in this paper, and reviews the separator-type robust stability theorem in the lifting-free framework [5], which can be viewed as a specialized form of the IQC theorem [3] for LTI systems. These theorems have been recognized to give a unified basis for conventional approaches to robust stability problems, e.g., the multiplier approach and causal LTI scaling. The separator-type theorem can further lead to the idea called noncausal LPTV scaling [9] when it is combined with the discrete-time lifting technique [10], and the present paper is strongly motivated by this new type of scaling approach. Therefore, as a preliminary section, these two scaling approaches are also reviewed here so that the significance of the central discussions in the present paper can be better demonstrated in the following sections.

A. Robust stability analysis problem

This paper studies the robust stability analysis problem of the discrete-time closed-loop system Σ shown in Fig. 1 consisting of the nominal system G and the uncertainty Δ . The nominal system G is assumed to be internally stable, finite-dimensional, LTI, and represented by

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k \quad (1)$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^p$, and $y_k \in \mathbf{R}^p$. The uncertainty Δ is assumed to belong to some given set $\mathbf{\Delta}$ satisfying the following assumption.

Assumption 1: Every $\Delta \in \mathbf{\Delta}$ is stable, finite-dimensional and LTI, and $\mathbf{\Delta}$ is star-convex with a center at $\Delta = 0$ (i.e., $k\Delta \in \mathbf{\Delta}$ whenever $\Delta \in \mathbf{\Delta}$ and $0 \leq k \leq 1$).

We represent the transfer matrices of the LTI systems G and Δ by $G(\zeta)$ and $\Delta(\zeta)$, respectively, where ζ denotes the variable for z -transform (in the lifting-free framework).

B. Separator-type robust stability theorem and causal LTI scaling

The following separator-type robust stability theorem [3], [5] plays a fundamental role in the robust stability analysis problem for Σ .

Theorem 1: Suppose that G is internally stable and $\mathbf{\Delta}$ satisfies Assumption 1. If Σ is well-posed for every $\Delta \in \mathbf{\Delta}$, then Σ is robustly stable with respect to $\mathbf{\Delta}$ if and only if there exists $\theta(\zeta) = \theta(\zeta)^*$ ($\zeta \in \partial\mathbf{D}$) such that

$$\begin{bmatrix} I \\ G(\zeta) \end{bmatrix}^* \theta(\zeta) \begin{bmatrix} I \\ G(\zeta) \end{bmatrix} \leq 0 \quad (\forall \zeta \in \partial\mathbf{D}) \quad (2)$$

$$\begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix}^* \theta(\zeta) \begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix} > 0 \quad \left(\begin{array}{l} \forall \Delta \in \mathbf{\Delta}, \\ \forall \zeta \in \partial\mathbf{D} \end{array} \right) \quad (3)$$

where $\partial\mathbf{D} := \{\zeta \in \mathbf{C} : |\zeta|=1\}$ denotes the unit circle.

The Hermitian matrix $\theta(\zeta)$ in (2) and (3) is called a separator. The above theorem implies that robust stability of Σ can be analyzed by searching for separators $\theta(\zeta)$ satisfying (2) and (3) against a given class $\mathbf{\Delta}$. For practical reasons related to such a search, separators are usually confined to have tractable forms conforming to the following definition of causal LTI separators.

Definition 1: A separator given by $\theta(\zeta) = V(\zeta)^* \Lambda V(\zeta)$ is called a causal LTI separator, where $V(\zeta)$ is the transfer matrix of a causal LTI system V with $2p$ inputs and $\Lambda = \Lambda^*$ is a constant matrix of compatible size with $V(\zeta)$. In particular, if V is static, then the corresponding separator is called a static causal LTI separator.

The approach to robust stability analysis based on causal LTI separators is called causal LTI scaling in the following.



Fig. 1. Closed-loop system Σ . Fig. 2. Lifted closed-loop system $\widehat{\Sigma}$.

C. Lifting-based treatment and noncausal LPTV scaling

The separator-type robust stability theorem reviewed in the preceding subsection can naturally lead to the idea called noncausal LPTV scaling [9] by introducing the discrete-time lifting technique [10]. The present paper is strongly motivated by this new type of scaling approach, and the discussions in the following sections are indeed intended to facilitate further studies on the relationship between causal LTI scaling and noncausal LPTV scaling. Hence, this subsection is devoted to reviewing this particular idea of noncausal LPTV scaling for robust stability analysis.

Let us begin with the discrete-time lifting technique. The operation of constructing new signal representations

$$\widehat{u}_\kappa := [u_{\kappa N}^T, u_{\kappa N+1}^T, \dots, u_{\kappa N+N-1}^T]^T, \quad (4)$$

$$\widehat{y}_\kappa := [y_{\kappa N}^T, y_{\kappa N+1}^T, \dots, y_{\kappa N+N-1}^T]^T \quad (5)$$

from the discrete-time signals u and y and a positive integer N is called the lifting of signals, where N is called the lifting period. This operation induces the conversion of the treatment of the system with input u and output y into that of the system with lifted input \widehat{u} and lifted output \widehat{y} , and such treatment is called the lifting of systems. The resulting lifted representations of systems are called N -lifted systems. By defining $\widehat{x}_\kappa := x_{\kappa N}$, we can describe the N -lifted nominal system \widehat{G} by

$$\widehat{x}_{\kappa+1} = \widehat{A}\widehat{x}_\kappa + \widehat{B}\widehat{u}_\kappa, \quad \widehat{y}_\kappa = \widehat{C}\widehat{x}_\kappa + \widehat{D}\widehat{u}_\kappa. \quad (6)$$

All the coefficient matrices of \widehat{G} can be constructed with the coefficient matrices in (1), but their explicit representations are irrelevant in the following. We denote the transfer matrix of \widehat{G} by $\widehat{G}(z)$; it is called the N -lifted transfer matrix of G , where z denotes the z -variable in the lifted framework and corresponds to ζ^N . We can also obtain the N -lifted representation $\widehat{\Delta}$ and the N -lifted transfer matrix $\widehat{\Delta}(z)$ from Δ . Through these ideas, we can obtain the lifted representation $\widehat{\Sigma}$ (Fig. 2) from the closed-loop system Σ .

It follows from the property of lifting [10] that Σ is robustly stable with respect to $\mathbf{\Delta}$ if and only if $\widehat{\Sigma}$ is with

respect to $\widehat{\Delta} := \{\widehat{\Delta} \mid \Delta \in \mathbf{\Delta}\}$. This leads to the following robust stability theorem [9], which is essentially the same as Theorem 1 but is stated in the lifting-based framework.

Theorem 2: Suppose that G is internally stable and $\mathbf{\Delta}$ satisfies Assumption 1. If Σ is well-posed for every $\Delta \in \mathbf{\Delta}$, then Σ is robustly stable with respect to $\mathbf{\Delta}$ if and only if there exists $\widehat{\Theta}(z) = \widehat{\Theta}(z)^*$ ($z \in \partial\mathbf{D}$) such that

$$\begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix} \leq 0 \quad (\forall z \in \partial\mathbf{D}) \quad (7)$$

$$\begin{bmatrix} \widehat{\Delta}(z) \\ I \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} \widehat{\Delta}(z) \\ I \end{bmatrix} > 0 \quad \left(\begin{array}{l} \forall \Delta \in \mathbf{\Delta}, \\ \forall z \in \partial\mathbf{D} \end{array} \right). \quad (8)$$

By Theorem 2, the robust stability problem of Σ reduces to searching for separators $\widehat{\Theta}(z)$ satisfying (7) and (8) against the given $\mathbf{\Delta}$. Such an approach is called noncausal LPTV scaling, and its effectiveness has been discussed in [9]. The present paper aims at giving a useful framework for further studies on the relationship between causal LTI scaling and noncausal LPTV scaling, where the latter includes as a special case a somewhat restrictive approach called causal LPTV scaling. As in causal LTI scaling, the separators $\widehat{\Theta}(z)$ are usually confined to have tractable forms, and causal LPTV scaling corresponds to confining to a more restrictive class of separators than in noncausal LPTV scaling. The distinction of causal and noncausal separators in the following definitions [9] is crucial throughout the paper.

Definition 2: A separator given by

$$\widehat{\Theta}(z) = [\widehat{V}_1(z) \quad \widehat{V}_2(z)]^* \widehat{\Lambda} [\widehat{V}_1(z) \quad \widehat{V}_2(z)] \quad (9)$$

is called a causal LPTV separator, where $\widehat{V}_1(z)$ and $\widehat{V}_2(z)$ are the N -lifted transfer matrices of causal N -periodic systems V_1 and V_2 with p inputs, respectively, and $\widehat{\Lambda} = \widehat{\Lambda}^*$ is a constant matrix of the form $\widehat{\Lambda} = \text{diag}[\Lambda_1, \dots, \Lambda_N]$ with the size of Λ_i being the same for all $i = 1, \dots, N$ and compatible with V_1 and V_2 . In particular, if V_1 and V_2 are static, then the corresponding separator is called a static causal LPTV separator.

Definition 3: A separator given by $\widehat{\Theta}(z) = \widehat{V}(z)^* \Gamma \widehat{V}(z)$ is called a noncausal LPTV separator, where $\widehat{V}(z)$ is the transfer matrix of a causal LTI system \widehat{V} with $2Np$ inputs defined on the lifted time axis and $\Gamma = \Gamma^*$ is a constant matrix of compatible size. In particular, if \widehat{V} is static, then the corresponding separator is called a static noncausal LPTV separator.

The approach to robust stability analysis based on noncausal (resp. causal) LPTV separators is called noncausal (resp. causal) LPTV scaling. The structure of noncausal LPTV separators is more general than that of causal LPTV separators, and thus noncausal LPTV scaling is more general than causal LPTV scaling.

III. ROBUST STABILITY CONDITIONS BASED ON INFINITE MATRIX REPRESENTATIONS

The preceding section stated the robust stability analysis problem studied in this paper, and reviewed separator-type robust stability theorems for such a problem. This section employs an infinite matrix representations of the systems G and $\mathbf{\Delta}$ [11],[12], and introduces a different form of robust stability conditions under such treatment of Σ . The infinite

matrix framework of such new conditions will turn out to provide a unified medium for directly comparing the two types of robust stability conditions reviewed in the preceding section; one was stated in the lifting-free framework while the other in the lifting-based framework. Hence, such a unified framework facilitates further studies on clarifying the relationship between the conventional causal LTI scaling (defined in the lifting-free framework) and noncausal LPTV scaling (defined through the lifting-based arguments). At the same time, such a framework is also effective for comparing the effect of frequency dependence in scaling (i.e., dealing with dynamic separators, as is often the case in the conventional lifting-free framework) and that of time dependence in scaling (which includes noncausal operations in time, naturally introduced by the application of noncausal LPTV scaling). The discussions in this section about introducing robust stability conditions through infinite matrix representations will constitute the basis of the subsequent arguments in this paper.

A. Infinite matrix representations of systems

We begin with infinite vector representations of input/output signals and the associated infinite matrix representations of systems [11],[12]. They will be a basis for introducing a different form of robust stability condition in this section. Let us consider the infinite vector representation of u , the input of the nominal system G , and denote it as follows.

$$\tilde{u} = [u_0^T, u_1^T, u_2^T, \dots]^T \quad (10)$$

We also define \tilde{y} similarly. Assuming that the initial state of G is zero, these representations \tilde{u} and \tilde{y} lead us to the formal description

$$\tilde{y} = \tilde{G}\tilde{u} \quad (11)$$

of the input-output relation of the nominal system G , where the infinite matrix \tilde{G} is given by

$$\tilde{G} = \begin{bmatrix} D & 0 & \dots & 0 & \dots \\ CB & D & \ddots & \vdots & \\ CAB & CB & D & 0 & \\ CA^2B & CAB & CB & D & \\ \vdots & & & & \ddots \end{bmatrix}. \quad (12)$$

with block Toeplitz and lower triangular structure. The infinite matrix representation $\tilde{\Delta}$ of the uncertainty $\mathbf{\Delta}$ is defined similarly.

B. Robust stability condition based on infinite matrix representations

This subsection is devoted to introducing a robust stability condition based on the infinite matrix representations of G and $\mathbf{\Delta}$, as well as V and Λ in the definition of causal LTI scaling. The condition introduced here may be, in a sense, simply a restatement of that in Theorem 1 leading to causal LTI scaling. Moreover, the restated infinite matrix condition might be less useful if its value were assessed only from a practical point of view. Nonetheless, significance of the extension toward such a direction with infinite matrix

representations lies in the fact that a parallel and unified extension can also be achieved about Theorem 2 leading to noncausal LPTV scaling. We will indeed see this parallelism in the following subsection, and further see in Section V that such extensions effectively facilitate us to have a fresh and clear insight into the relationship among causal LTI scaling and causal/noncausal LPTV scaling.

We begin our discussions with the following lemma. This is a key lemma in this paper and follows by applying the Fourier expansion, but requires rigorous arguments to circumvent mathematical subtleties; the proof, however, is omitted due to limited space.

Lemma 1: Suppose that $M(\zeta)$ is a stable transfer matrix with q columns, and $L = L^*$ is a constant matrix. For a given $\alpha \in \mathbf{R}$,

$$M(\zeta)^* L M(\zeta) \geq \alpha I \quad (\forall \zeta \in \partial \mathbf{D}) \quad (13)$$

holds if and only if

$$\widetilde{M}^* \widetilde{L} \widetilde{M} \geq \alpha \widetilde{I} \quad (14)$$

holds on $l_2(\mathbf{N}_0, \mathbf{R}^q)$, where \widetilde{M} , \widetilde{L} and \widetilde{I} are the infinite matrix representations of $M(\zeta)$, L and I , respectively.

In (14), the inequality is in terms of non-negativeness of quadratic forms on $l_2(\mathbf{N}_0, \mathbf{R}^q)$. An implication of the above theorem is that the ζ -dependence in the inequality (13) may be removed if one accepts to work on the infinite matrix inequality (14). A direct application of the above lemma leads immediately to the following proposition.

Proposition 1: Suppose that G is internally stable and the separator $\theta(\zeta)$ is given by

$$\theta(\zeta) = [V_1(\zeta) \quad V_2(\zeta)]^* \Lambda [V_1(\zeta) \quad V_2(\zeta)] \quad (15)$$

with a constant matrix $\Lambda = \Lambda^*$ and stable transfer matrices $V_1(\zeta)$ and $V_2(\zeta)$. Then, the robust stability conditions (2) and (3) hold if and only if the infinite matrix inequalities

$$\begin{bmatrix} \widetilde{I} \\ \widetilde{G} \end{bmatrix}^* \widetilde{\theta} \begin{bmatrix} \widetilde{I} \\ \widetilde{G} \end{bmatrix} \leq 0 \quad (16)$$

$$\begin{bmatrix} \widetilde{\Delta} \\ \widetilde{I} \end{bmatrix}^* \widetilde{\theta} \begin{bmatrix} \widetilde{\Delta} \\ \widetilde{I} \end{bmatrix} \geq \epsilon(\Delta) \widetilde{I} \quad \left(\begin{array}{l} \forall \Delta \in \mathbf{\Delta}, \\ \exists \epsilon(\Delta) > 0 \end{array} \right) \quad (17)$$

hold on $l_2(\mathbf{N}_0, \mathbf{R}^p)$ for the infinite matrix

$$\widetilde{\theta} = [\widetilde{V}_1 \quad \widetilde{V}_2]^* \widetilde{\Lambda} [\widetilde{V}_1 \quad \widetilde{V}_2] \quad (18)$$

where \widetilde{V}_1 , \widetilde{V}_2 and $\widetilde{\Lambda}$ are the infinite matrix representations of $V_1(\zeta)$, $V_2(\zeta)$ and Λ , respectively.

In view of the role of the infinite matrix $\widetilde{\theta}$ in the inequalities (16) and (17), we call it a separator in the framework of infinite matrix representations, or simply an infinite-dimensional separator. Furthermore, we call the particular separator $\widetilde{\theta}$ given in (18) the infinite matrix representation of the separator (15).

We have assumed in the above proposition that $V_1(\zeta)$ and $V_2(\zeta)$ are stable. This obviously leads to restricting the class of separators with respect to the search of $\theta(\zeta)$ satisfying (2) and (3). Fortunately, however, such restriction is known to lead to no conservativeness in the robust stability analysis [5]. Hence, Theorem 1 and Proposition 1 lead immediately to the following robust stability theorem.

Theorem 3: Suppose that G is internally stable and Σ is well-posed for every $\Delta \in \mathbf{\Delta}$. Then, Σ is robustly stable with respect to $\mathbf{\Delta}$ if and only if there exists $\widetilde{\theta}$ that satisfies (16) and (17) on $l_2(\mathbf{N}_0, \mathbf{R}^p)$ and is in the form of (18), where \widetilde{V}_1 and \widetilde{V}_2 are the infinite matrix representations of stable transfer matrices $V_1(\zeta)$ and $V_2(\zeta)$, respectively, and $\widetilde{\Lambda}$ is the infinite matrix representation of a constant matrix Λ .

C. Generalization of the class of infinite-dimensional separators

The preceding subsection studied introducing the infinite matrix representation counterpart to causal LTI scaling supported by Theorem 1, and gave infinite matrix representations of causal LTI separators. This subsection extends the study to noncausal LPTV scaling supported by Theorem 2, gives infinite matrix representations of noncausal LPTV separators (including causal ones as a special case), and observes how these representations are different from those of causal LTI separators.

The extension to noncausal LPTV scaling is essentially just to repeat in the lifted framework the same arguments as those in the preceding subsection. In other words, the key in the extension is to apply Lemma 1 with $M(\zeta)$ replaced by the transfer matrix $\widetilde{M}(z)$ defined on the lifted time axis. This naturally leads to an infinite matrix inequality on $l_2(\mathbf{N}_0, \mathbf{R}^{Np})$, but this space is isometrically isomorphic to $l_2(\mathbf{N}_0, \mathbf{R}^p)$ used in that lemma. Hence, these two spaces may be identified with each other, and thus it is not always necessary to distinguish an infinite matrix inequality on one space from the one on another. Similarly, the infinite vector representation \widetilde{u} of u and that of the lifted counterpart \widehat{u} are essentially the same, and they need not be distinguished; we denote both of them by \widetilde{u} without introducing the bothering notation \widehat{u} . Similar comments apply also to the lifted representations of systems; the infinite matrix representation of G and that of the lifted counterpart of G may also be identified, and both of them are denoted by \widetilde{G} without introducing the notation \widehat{G} .

With the preceding arguments and notation, we are led to the following infinite matrix representation counterpart of Theorem 2 about noncausal LPTV scaling.

Theorem 4: Suppose that G is internally stable and Σ is well-posed for every $\Delta \in \mathbf{\Delta}$. Then, Σ is robustly stable with respect to $\mathbf{\Delta}$ if and only if there exists $\widetilde{\theta}$ that satisfies (16) and (17) on $l_2(\mathbf{N}_0, \mathbf{R}^{Np})$ and is in the form of

$$\widetilde{\theta} = [\widetilde{V}_1 \quad \widetilde{V}_2]^* \widetilde{I} [\widetilde{V}_1 \quad \widetilde{V}_2] \quad (19)$$

where \widetilde{V}_1 and \widetilde{V}_2 are the infinite matrix representations of stable transfer matrices $V_1(z)$ and $V_2(z)$ defined on the lifted time axis, respectively, and \widetilde{I} is the infinite matrix representation of a constant matrix I .

Since $l_2(\mathbf{N}_0, \mathbf{R}^p)$ and $l_2(\mathbf{N}_0, \mathbf{R}^{Np})$ may essentially be identified as stated earlier, the difference between Theorem 3 about causal LTI scaling and Theorem 4 lies only in the forms of the infinite-dimensional separators $\widetilde{\theta}$ in these theorems. In Theorem 3 (or (18)), \widetilde{V}_1 and \widetilde{V}_2 are related to stable LTI systems V_1 and V_2 (defined on the lifting-free time axis), and $\widetilde{\Lambda}$ is related to a constant matrix Λ

whose size is compatible with the number of rows of V_1 and V_2 . In Theorem 4, on the other hand, \tilde{V}_1 and \tilde{V}_2 are related to stable LTI systems V_1 and V_2 defined on the lifted time axis, and \tilde{T} is related to a constant matrix T whose size is compatible with the number of rows of V_1 and V_2 . Therefore, in comparison with Theorem 3, Theorem 4 can be interpreted as relaxing the assumptions on V_1 and V_2 by allowing them to have N -periodicity. Moreover, in the case of noncausal (rather than causal) LPTV scaling, some sort of noncausality is also allowed by introducing V_1 and V_2 directly on the lifted time axis. These differences lead to the enhanced ability of causal/noncausal LPTV scaling, but they are simply natural consequences since the ideas of causal/noncausal LPTV scaling underlying Theorem 4 intend nothing but such enhancement. However, it should be further noted that the size of the underlying matrix T in Theorem 4 is N times as large as that of the matrix A in Theorem 3. This difference is also closely related to the enhanced ability of noncausal LPTV scaling. In fact, we can see that causal LTI scaling corresponds to taking $T = I_N \otimes A$, where \otimes denotes the Kronecker product of matrices, while causal LPTV scaling corresponds to taking $T = \text{diag}[A_1, \dots, A_N]$. In each of these two scaling approaches T is a block diagonal matrix, but it is allowed to be an arbitrary matrix in noncausal LPTV scaling.

IV. INFINITE MATRIX REPRESENTATIONS OF DYNAMIC LTI AND NONCAUSAL LPTV SEPARATORS

This section first considers a dynamic LTI separator in the lifting-free framework characterized by finite impulse response (FIR), for simplicity, and gives its infinite matrix representation. The arguments reveal an implication of frequency dependence of separators interpreted in the framework of infinite matrix representations. The arguments are then extended to a separator in the lifted framework characterized by FIR, and the infinite matrix representation of a noncausal LPTV separator is also given. Such a separator has not only frequency dependence but also time dependence resulting from its N -periodicity, in general. An implication of such N -periodicity in contrast with frequency dependence will also be interpreted in the infinite matrix framework. These interpretations will be crucial as preliminary observations for the arguments in the following section. In fact, the mutual relationship between causal LTI separators and causal/noncausal LPTV separators is discussed in an intuitive fashion by the use of the infinite matrix framework.

A. Dynamic causal LTI separator characterized by finite impulse response

In this subsection, we consider the dynamic causal LTI separator $\Theta_{\text{FIR}}(\zeta) = \Theta_{\text{FIR}}(\zeta)^*$ ($\zeta \in \partial\mathcal{D}$) (defined in the lifting-free framework) given by

$$\begin{aligned} \Theta_{\text{FIR}}(\zeta) &= (\Theta_{\text{FIR},ij}(\zeta))_{i,j=1,2}, \\ \Theta_{\text{FIR},ij}(\zeta) &= \theta_{ij}^{[0]} + \sum_{k=1}^K \left(\theta_{ij}^{[-k]} \zeta^{-k} + \theta_{ij}^{[k]} (\zeta^*)^{-k} \right), \\ \theta_{ij}^{[k]} &= \left(\theta_{ji}^{[-k]} \right)^T \in \mathbf{R}^{p \times p} \quad (k = 0, \pm 1, \dots, \pm K). \end{aligned} \quad (20)$$

We call it a causal LTI separator characterized by FIR, or simply a causal LTI FIR-separator; a causal LTI separator $V(\zeta)^*TV(\zeta)$ is in the form of (20) if and only if $V(\zeta)$ has FIR. Since we have assumed in Theorem 3 that $V_1(\zeta)$ and $V_2(\zeta)$ are stable and thus their impulse responses converge to zero, it would not be extremely restrictive to confine ourselves to such a class of separators by taking a large enough K . This subsection is devoted to giving an explicit form of the infinite matrix representation of the dynamic causal LTI FIR-separator (20). The arguments will be a crucial basis for comparing different scaling approaches as we shall see in the following section.

The separator (20) defined on $\partial\mathcal{D}$ can be rearranged as

$$\Theta_{\text{FIR}}(\zeta) = \left(T_p(\zeta)^* \theta_{ij}^{[\text{FIR}]} T_p(\zeta) \right)_{i,j=1,2} \quad (21)$$

where

$$T_p(\zeta) = \begin{bmatrix} \zeta^{-K} I_p \\ \vdots \\ \zeta^{-1} I_p \\ I_p \end{bmatrix}, \quad \theta_{ij}^{[\text{FIR}]} = \begin{bmatrix} \theta_{ij}^{[0]} & \theta_{ij}^{[1]} & \dots & \theta_{ij}^{[K]} \\ \theta_{ij}^{[-1]} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{ij}^{[-K]} & 0 & \dots & 0 \end{bmatrix}. \quad (22)$$

Hence by (18), the infinite matrix representation of $\Theta_{\text{FIR}}(\zeta)$ is given by

$$\tilde{\Theta}_{\text{FIR}} = \left(\tilde{T}_p^* \tilde{\theta}_{ij}^{[\text{FIR}]} \tilde{T}_p \right)_{i,j=1,2}, \quad (23)$$

where \tilde{T}_p and $\tilde{\theta}_{ij}^{[\text{FIR}]} = \text{diag}[\theta_{ij}^{[\text{FIR}]}], \theta_{ij}^{[\text{FIR}]}], \dots]$ are the infinite matrix representations of $T_p(\zeta)$ and $\theta_{ij}^{[\text{FIR}]}$, respectively. By defining the matrices

$$T_{p,i} = \begin{bmatrix} \delta_{Ki} I_p \\ \vdots \\ \delta_{1i} I_p \\ \delta_{0i} I_p \end{bmatrix}, \quad i = 0, 1, \dots, K \quad (24)$$

with the Kronecker delta δ_{ij} , we have

$$\tilde{T}_p = \begin{bmatrix} T_{p,0} & 0 & \dots \\ \vdots & T_{p,0} & \ddots \\ T_{p,K} & \vdots & \ddots \\ 0 & T_{p,K} & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}. \quad (25)$$

Substituting the above \tilde{T}_p into (23) leads to $\tilde{\Theta}_{\text{FIR}} = \left(\tilde{\Theta}_{\text{FIR},ij} \right)_{i,j=1,2}$ with

$$\tilde{\Theta}_{\text{FIR},ij} = \begin{bmatrix} \theta_{ij}^{[0]} & \theta_{ij}^{[1]} & \dots & \theta_{ij}^{[K]} & 0 & \dots \\ \theta_{ij}^{[-1]} & \theta_{ij}^{[0]} & \theta_{ij}^{[1]} & \dots & \theta_{ij}^{[K]} & \ddots \\ \vdots & \theta_{ij}^{[-1]} & \ddots & \ddots & \ddots & \ddots \\ \theta_{ij}^{[-K]} & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \theta_{ij}^{[-K]} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (26)$$

It should be noted that the above infinite matrix is block Toeplitz, which is obtained by repeatedly shifting the matrix Θ_{ij}^{FIR} toward the right-lower direction by one block and then superposing the resulting matrices. Such an operation will be denoted by $\text{repeat}(\cdot, \cdot, \cdot)$ in the following; by its definition, we have

$$\tilde{\Theta}_{\text{FIR},ij} = \text{repeat} \left(\Theta_{ij}^{[-1]}, \Theta_{ij}^{[0]}, \Theta_{ij}^{[+]} \right), \quad (27)$$

where

$$\begin{aligned} \Theta_{ij}^{[-]} &:= \left[(\Theta_{ij}^{[-1]})^T \quad \dots \quad (\Theta_{ij}^{[-K]})^T \right]^T, \\ \Theta_{ij}^{[+]} &:= \left[\Theta_{ij}^{[1]} \quad \dots \quad \Theta_{ij}^{[K]} \right] \end{aligned} \quad (28)$$

Since only $\Theta_{ij}^{[k]} \in \mathbf{R}^{p \times p}$, $k = 0, \pm 1, \dots, \pm K$ are nonzero submatrices in $\tilde{\Theta}_{\text{FIR},ij}$, we simply say that $\tilde{\Theta}_{\text{FIR}}$ has band structure with one-side width K with respect to the size of $\mathbf{R}^{p \times p}$.

In the special case of a static causal LTI separator, i.e., when $K = 0$ so that $\Theta_{\text{FIR},ij} = \Theta_{ij}^{[0]}$, each block of $\tilde{\Theta}_{\text{FIR}}$ reduces to an infinite block diagonal matrix. Increasing K leads to the increase in the one-side width of (or the freedom in) the band structure associated with the infinite-dimensional separator $\tilde{\Theta}_{\text{FIR}}$. We could formally regard the limit of such band structure for $K \rightarrow \infty$ to be a general block Toeplitz structure.

B. Dynamic noncausal LPTV separator characterized by finite impulse response

Next, let us consider the dynamic noncausal LPTV FIR-separator $\hat{\Theta}_{\text{FIR}}^{\text{noncausal}}(z) = \tilde{\Theta}_{\text{FIR}}^{\text{noncausal}}(z)^*$ ($z \in \partial\mathbf{D}$) (defined in the lifted framework) given by

$$\begin{aligned} \hat{\Theta}_{\text{FIR}}^{\text{noncausal}}(z) &= \left(\hat{\Theta}_{\text{FIR},ij}^{\text{noncausal}}(z) \right)_{i,j=1,2}, \\ \hat{\Theta}_{\text{FIR},ij}^{\text{noncausal}}(z) &= \hat{\Theta}_{ij}^{[0]} + \sum_{k=1}^K \left(\hat{\Theta}_{ij}^{[-k]} z^{-k} + \hat{\Theta}_{ij}^{[k]} (z^*)^{-k} \right), \\ \hat{\Theta}_{ij}^{[k]} &= \left(\hat{\Theta}_{ji}^{[-k]} \right)^T \in \mathbf{R}^{Np \times Np} \quad (k = 0, \pm 1, \dots, \pm K). \end{aligned} \quad (29)$$

By parallel arguments to those in the preceding subsection, the infinite matrix representation of this separator is given by

$$\begin{aligned} \tilde{\Theta}_{\text{FIR}}^{\text{noncausal}} &= \left(\text{repeat} \left(\hat{\Theta}_{ij}^{[-1]}, \hat{\Theta}_{ij}^{[0]}, \hat{\Theta}_{ij}^{[+]} \right) \right)_{i,j=1,2}, \\ \hat{\Theta}_{ij}^{[-]} &= \left[(\hat{\Theta}_{ij}^{[-1]})^T \quad \dots \quad (\hat{\Theta}_{ij}^{[-K]})^T \right]^T, \\ \hat{\Theta}_{ij}^{[+]} &= \left[\hat{\Theta}_{ij}^{[1]} \quad \dots \quad \hat{\Theta}_{ij}^{[K]} \right]. \end{aligned} \quad (30)$$

This infinite-dimensional separator also has band structure with one-side width K with respect to the size of $\mathbf{R}^{Np \times Np}$. That is, the structure depends on both K and N , which are related to frequency dependence and time dependence, respectively.

The preceding subsection observed that the infinite matrix representation of the causal LTI FIR-separator (20) has band structure with one-side width K with respect to the size of $\mathbf{R}^{p \times p}$. The degree of freedom in its structure can be increased by taking larger K (the factor for frequency dependence). The size of matrices constituting the band,

however, remains the same regardless of K . For the noncausal LPTV FIR-separator (29), on the other hand, the role of K remains the same, while the size of matrices constituting the band structure (30) depends on N (the factor for time dependence). Hence, even under fixed K (e.g., $K = 0$ leading to static separators), the degree of freedom in the associated band structure can be increased as the lifting period N increases. An important question about noncausal LPTV scaling would be whether there can be established some explicit relationship between the freedom with respect to K (relevant to frequency dependence) and that with respect to N (relevant to time dependence). We demonstrate in the following section that the framework of infinite matrix representations developed in this paper provides a very clear and intuitive insight that is helpful to studying such issues.

V. COMPARISON OF DIFFERENT SCALING APPROACHES THROUGH INFINITE MATRIX FRAMEWORK

The purpose of this section is to demonstrate the usefulness of the framework of infinite matrix representations developed in this paper, as a unified medium for dealing with causal/noncausal LPTV scaling approaches to robust stability analysis.

In the following arguments, we occasionally refer to the schematic picture of (around the top-left corner of) the infinite-dimensional separator $\tilde{\Theta}_{ij}$ to ease descriptions and help intuitive understanding; this picture assumes the case of causal/noncausal LPTV scaling with $N = N_0 = 3$, but it applies at the same time to causal LTI scaling by considering $p \times p$ submatrices in this figure. An explicit description about our standing assumption (on the stability of the transfer matrices contained in separators) is suppressed for conciseness in this section.

A. Separators induced by a causal LTI separator

In this subsection, we assume that the infinite matrix representation (18) of a causal LTI separator $\Theta(\zeta)$ satisfies the robust stability conditions (16) and (17). Our purpose here is to discuss what implications will follow (what separators may be induced equivalently) under the interpretation from the causal/noncausal LPTV scaling viewpoint.

Here, recall that the infinite matrix representation of a system and that of its N -lifted description coincide with each other, regardless of N . This, together with the inspection of the infinite matrix representations (18) and (19), leads immediately to the following result.

Theorem 5: Suppose the infinite-dimensional separator (18) associated with the causal LTI separator $\Theta(\zeta) = V(\zeta)^* A V(\zeta)$ satisfies the robust stability conditions (16) and (17), where $V = [V_1, V_2]$. Then, the infinite matrix representation of the lifted counterpart (denoted by $\hat{\mathbb{E}}[\Theta(\zeta)]$) of $\Theta(\zeta)$ also satisfies the same conditions. Here, $\hat{\mathbb{E}}[\Theta(\zeta)]$ is defined as $\hat{\Theta}(z)$ in (9) with $\hat{V}_1(z)$ and $\hat{V}_2(z)$ given by the N -lifted transfer matrices of V_1 and V_2 , respectively, and \hat{A} given by $I_N \otimes A$.

We need no manipulations of equations to prove this theorem, and it just suffices to change the way to view infinite matrices by taking different partitioning. Indeed, if the band structure in Fig. 3 shown with dash lines is block Toeplitz in terms of submatrices in $\mathbf{R}^{p \times p}$, then it can also be viewed as

block Toeplitz in terms of submatrices in $\mathbf{R}^{Np \times Np}$ (consider the band structure in Fig. 3 shown with dash-dot lines), and this together with close inspection of the forms of the latter submatrices completes the proof. We can easily see that the converse assertion of the above theorem also holds.

We see that $\hat{\mathbf{E}}[\Theta(\zeta)]$ is a causal LPTV separator. Hence by Theorem 5, if (i) there exists a causal LTI separator satisfying the robust stability conditions, then (ii) there exists a causal LPTV separator satisfying the same conditions, while it is obvious that (ii) implies (iii) there exists a noncausal LPTV separator satisfying the same conditions. If we apply the technique with \tilde{S}_p introduced in the following subsection, we can readily establish that (iii) implies (i). Hence, these three conditions are in fact equivalent. Since we have started our arguments from two robust stability theorems, each of which gives an apparently different but *necessary and sufficient* condition for robust stability, this mere observation, clarifying no specific mutual correspondence among the separators in these conditions, is actually a trivial conclusion.

It is still very important, however, to note it and that it does not simply imply causal/noncausal LPTV scaling to offer no advantage over causal LTI scaling. Instead, what should be more substantial is to compare different scaling approaches under the practical situation in which the search of separators satisfying the robust stability conditions can only be carried out within some restricted but tractable class. Indeed, the most fundamental motivation of the present paper lies in the comparison of causal LTI, causal LPTV and noncausal LPTV scaling under such a viewpoint. The significance of the above theorem would be clearer in such a context in which the class of separators has to be somehow restricted (because otherwise the freedom in separators are too large to accept the associated computational load) whichever of these approaches we may take.

More precisely, suppose we restrict the causal LTI separators to some tractable class $\Theta(\zeta)$, and consider the associated equivalent class $\hat{\Theta}_E(z) := \{\hat{\mathbf{E}}[\Theta(\zeta)] \mid \Theta(\zeta) \in \Theta(\zeta)\}$ of causal LPTV separators. Then, by simply taking a class of noncausal LPTV separators that contains $\hat{\Theta}_E(z)$ as a subset, we can ensure the corresponding noncausal LPTV scaling to be more effective (rigorously speaking, not less effective) than causal LTI scaling under the class $\Theta(\zeta)$. In this regard, it would be worth mentioning that the arguments for establishing such explicit relationships between a separator in one approach and an “equivalent (or more

effective) separator viewed in another approach” is very involved without employing the infinite matrix framework; see, e.g., [13] for an alternative approach.

B. Separators induced by a causal/noncausal LPTV separator

We next discuss the opposite direction; we assume that the infinite matrix representation $\tilde{\Theta}$ of a causal/noncausal LPTV separator satisfies the robust stability conditions (16) and (17), and study what implications will follow under the interpretation from the causal LTI scaling viewpoint.

Let us introduce the infinite matrix

$$\tilde{S}_p = \begin{bmatrix} 0_{p \times \infty} \\ \tilde{I} \end{bmatrix} \quad (31)$$

where \tilde{I} denotes the infinite matrix representation of the identity matrix in $\mathbf{R}^{p \times p}$. Since G is LTI and thus \tilde{G} is block Toeplitz, it is easy to see that

$$\tilde{G}\tilde{S}_p = \begin{bmatrix} 0_{p \times \infty} \\ \tilde{G} \end{bmatrix} = \tilde{S}_p\tilde{G}. \quad (32)$$

The same arguments apply also to the uncertainty Δ . Hence, by post-multiplying \tilde{S}_p and pre-multiplying its adjoint on (16) and (17), we see that the infinite-dimensional separator

$$\text{diag}[\tilde{S}_p, \tilde{S}_p]^* \tilde{\Theta} \text{diag}[\tilde{S}_p, \tilde{S}_p] = \left(\tilde{S}_p^* \tilde{\Theta}_{ij} \tilde{S}_p \right)_{i,j=1,2} \quad (33)$$

also satisfies the same robust stability conditions (16) and (17). The submatrix $\tilde{S}_p^* \tilde{\Theta}_{ij} \tilde{S}_p$ is nothing but the infinite matrix obtained by removing the first p rows and columns of $\tilde{\Theta}_{ij}$ (and then by shifting the result toward the left-upper direction to stay at the same position); note that $\tilde{\Theta}_{ij}$ is block Toeplitz in terms of submatrices in $\mathbf{R}^{Np \times Np}$, and the above p rows and columns removed correspond to only a fraction of the underlying block. Hence, we can repeat applying $\text{diag}[\tilde{S}_p, \tilde{S}_p]$ on the resulting “shifted infinite-dimensional separator” for N times, when the resulting shifted separator reverts to the original $\tilde{\Theta}$. We will thus have N distinguishable infinite-dimensional separators (including the original one) in this process, and their average also satisfies the same robust stability conditions since the conditions are affine with respect to the separator.

For example, if the original $\tilde{\Theta}$ corresponds to a static noncausal LPTV separator (i.e., $K = 0$ as in the band structure in Fig. 3 shown with solid lines), then we can see that the above averaged infinite-dimensional separator will have the band structure with respect to the size of $\mathbf{R}^{p \times p}$ (this in particular implies that it is block Toeplitz) shown in dash lines in the figure. In other words, the existence of a static noncausal LPTV separator satisfying the robust stability conditions under the lifting period $N = N_0$ ensures that of (dynamic) causal LTI FIR-separator with the factor $K = N_0 - 1$. This implies that even if we confine ourselves to *static* noncausal LPTV scaling in the lifted framework, it can be interpreted equivalently in the lifting-free framework and induces, in general, dynamic causal LTI scaling there (which is obviously more effective than static causal LTI scaling). Note that this observation is consistent with that in the preceding subsection with $\Theta(\zeta)$ taken to be the class of *static* causal LTI separators. More generally, we are readily led to the following result.

Theorem 6: A noncausal LPTV FIR-separator with the factor $K = K_0$ satisfying the robust stability conditions

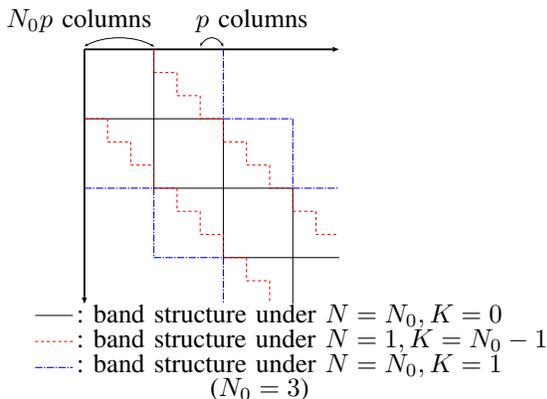


Fig. 3. Schematic picture of infinite matrix representation $\tilde{\Theta}_{ij}$

induces to the lifting-free framework a causal LTI FIR-separator with the factor $K = (K_0 + 1)N_0 - 1$, where N_0 is the underlying lifting period.

The above theorem suggests that searching for the conventional causal LTI scaling for robustness analysis could possibly be replaced somehow by that for (possibly, only static) noncausal LPTV scaling. In fact, this promising possible ability of noncausal LPTV scaling is exactly what has been suggested in Theorem 1 in the preceding study [9] and motivated the present study. In this respect, it is easy to confirm that the assertion of the above theorem is equivalent to that in the earlier theorem mentioned above, restated with the words of the infinite matrix framework. In the earlier proof, however, the underlying mechanism that leads the assertion has not been very clear and intuitively comprehensible. In contrast, we believe that the mechanism has been made very clear with the infinite matrix framework developed in the present paper.

Remark 1: The above theorem holds even for $K_0 = \infty$. On the other hand, if we consider the special case of *causal* LPTV scaling and $K_0 = 0$ (i.e., static causal LPTV scaling), then $\tilde{\Theta}_{ij}$ reduces to an infinite block diagonal matrix with N matrices in $\mathbf{R}^{p \times p}$ appearing on the diagonal in a cyclic fashion. Hence, its shifted versions with \tilde{S}_p retain the same form, and so does their average. In fact, it is immediately seen that all the matrices on the diagonal of the average are common, which implies that it is nothing but the infinite matrix representation of a *static* causal LTI separator. This implies that static *causal* LPTV separators have no ability to induce dynamic causal LTI scaling in the lifting-free framework; within the class of *static* separators, such an ability is specific to (strictly) *noncausal* ones. Note that the assertion of the theorem does not exclude such a degenerate case with static causal LPTV separators.

An interesting open question about noncausal LPTV scaling is to characterize the class of causal LTI separators in the lifting-free framework that can equivalently be handled in the lifting-based framework with *static* noncausal LPTV separators. Here, let us consider the class $\Theta_{N_0-1}(\zeta)$ of causal LTI FIR-separators with the factor $K = N_0 - 1$. Note because of the lack of the converse assertion of Theorem 6 that even if there exists an infinite-dimensional separator associated with $\Theta(\zeta) \in \Theta_{N_0-1}(\zeta)$ (see the associated band structure in Fig. 3 with dash lines) and satisfying the robust stability conditions, it does not necessarily lead to the existence of an infinite-dimensional separator corresponding to a *static* noncausal LPTV separator and satisfying the same conditions. Even though such a converse assertion could possibly be justified somehow under some appropriate assumptions (such as N being large enough), and it is important to extend the study toward such a direction, the present lack of the direct converse assertion can also be attributed simply to the fact that the aforementioned band structure (about causal LTI separators) can never be covered by that for *static* noncausal LPTV separators shown with solid lines. However, the former can be covered instead with the band structure for noncausal LPTV separators with $K = 1$ shown with dash-dot lines. This immediately implies that dealing with (dynamic) noncausal LPTV FIR-separators

with $K = 1$ under the lifting period N_0 is at least as effective as causal LTI FIR-separators with $\Theta_{N_0-1}(\zeta)$. That is, such noncausal LPTV separators can definitely induce $\Theta_{N_0-1}(\zeta)$.

We finally remark that the above arguments are closely related to those in our recent study [13], but the present arguments are much more straightforward through the use of the infinite matrix framework. We believe that these arguments will be a basis for tackling the open question to have more definite answers.

VI. CONCLUSION

Motivated by the study on clarifying further relationship between the conventional lifting-free scaling and lifting-based noncausal LPTV scaling approaches based on separator-type robust stability theorems, this paper gave the infinite matrix representation counterparts of the robust stability conditions in these theorems. This led to a unified framework for dealing with these two approaches through the idea of infinite-dimensional separators. Explicit forms of the infinite-dimensional separators were given, and were shown to have different types of block Toeplitz (band) structure in these two approaches. The arguments also showed how noncausal and time-varying nature as well as frequency dependence introduced into scaling are reflected on the (band) structure. It was then demonstrated that the difference in the structure provides us with a very clear and intuitive interpretation on the difference in the two scaling approaches, and that some relationship between these approaches can be understood with the infinite matrix framework in a very comprehensible way. The benefit of developing the infinite matrix framework is thus clear, and this framework is believed to be very useful in further theoretical studies.

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