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Gain-scheduled synchronization of uncertain parameter varying systems via relative H_∞ consensus

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Abstract— **The paper considers a problem of consensus-based synchronization of uncertain parameter varying multi-agent systems with Lipschitz-continuous nonlinearities. The objective is to construct simultaneously consensus and observer schedules for each agent to ensure an asymptotic synchronized behaviour of all agents. A gain-scheduling algorithm is proposed which solves this problem while maintaining a specified suboptimal H_∞ level of relative disagreement between the agents. The algorithm uses interpolation to ensure the continuity of the interconnection and observer gains. It preserves the H_∞ consensus properties of the interpolants.**

I. Introduction

It is widely recognized that multi-agent consensus problems have much similarity with problems of synchronization of complex dynamical networks [2], [5]. However, relatively few references have addressed behavior of consensus and synchronization algorithms in the presence of noise and uncertainty. A natural in this situation approach of distributed and decentralized H_∞ optimization has gained some attention recently [3], [2], [7], [12], [13]. In particular, in [12], [13] we have proposed an approach to the observer-based synchronization of complex dynamical networks of uncertain agents using an H_∞ consensus-based estimation methodology.

The approach in [12], [13] is based on optimization of the cost of relative H_∞ disagreement between agents. In comparison with the H_∞ tracking approach [3], [2], it leads to the synchronization mechanism where an H_∞-type consensus between the agents about their estimates of the reference plant is an essential prerequisite for synchronization. The approach is constructive, and allows one to obtain interconnection and observer gains to ensure synchronization, by solving an LMI optimization problem.

The objective of this paper is to extend this approach to systems which require a time-varying reference model for synchronization. Time variations of system coefficients pose an additional difficulty, since many standard robust control and filtering techniques developed for time-invariant systems are not directly transferable to time-varying systems. This is particularly true for techniques utilizing Linear Matrix Inequalities which were used in [12], [13]. In some cases, the issue can be resolved by focusing on a finite-horizon version of the problem [7]. However, this option is not always suitable when the aim is to achieve synchronization. In support of this point, we refer to problems of synchronization of nonlinear systems exhibiting chaotic behaviours, where the system must be continuously locked into synchronous operation, otherwise even small discrepancies between trajectories will cause the system to lose synchronization in a very short time.

Gain-scheduling techniques provide a powerful alternative to the finite-horizon analysis and synthesis of time-varying systems. The idea of this approach is to design a controller or filter as a function of the varying system parameters that are available for on-line monitoring and measurement.

One of the techniques extensively used in the controller design for parameter-varying systems involves scheduling controllers by interpolating the controllers designed for several operating points. Interpolation allows to avoid detrimental transients caused by controller switching. Yet, in general there is no guarantee that the system governed by an interpolated controller remains stable while traversing between operating points. This prompted the development of stability and robustness preserving interpolation techniques for gain scheduling [9], [10], [15].

This paper considers networks of uncertain dynamical systems-agents whose linear part is parameter-varying, and the nonlinear part is Lipschitz continuous. It develops an interpolation technique of gain-scheduled design to construct consensus-based synchronization protocols which enable such networks to synchronize while reaching a desired level of relative H_∞ consensus performance. The technical idea of our approach is based on the results in [12]. However, a direct application of the continuous gain interpolation methods to the synchronization scheme proposed in [12] proves to be difficult. Therefore in this paper the methodology in [12] is revisited to allow for such an application. This requires us to consider a somewhat different model for information processing by the agents. In contrast to [12] and similar to [11], in this paper we allow for the possibility to assign individual interconnection gains to each communication channel. Also, we consider a more general communication model, where links between the agents are subject to uncertain perturbations.

The vector dissipativity theory developed in [1] plays an instrumental role in the development of our approach. As an extension of the classic dissipativity theory [14], this theory enables the dissipativity properties of a large-scale system to be studied using vector storage functions and vector supply rates. Here, we use this methodology to construct parameter dependent vector Lyapunov functions to ensure robustness and H_∞ consensus performance of

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interpolated gain-scheduled synchronization protocols under consideration.

The approach undertaken in this paper is inspired by the robustness preserving interpolation technique from [8], [9], [15]. The common point between this paper and [8], [9], [15] is that we interpolate Lyapunov functions rather than controller gains. However, unlike the above publications, linear interpolation of Lyapunov functions is sufficient in our case to preserve synchronization and H_∞ consensus properties of the interpolants.

Notation: $\mathbf{R}^n$ denotes the real Euclidean n-dimensional vector space, with the norm $\|x\| \triangleq (x'x)^{1/2}$; the symbol $'$ denotes the transpose of a matrix or a vector. Given a real symmetric $k \times k$ matrix P, we let $\lambda_{\max}(P)$, $\lambda_{\min}(P)$, denote the largest and smallest eigenvalues of P, respectively. Also, for real symmetric X, Y, we write $Y > X$ ($Y \geq X$), when the matrix $Y - X$ is positive definite (positive semidefinite). $\mathrm{diag}[Q_1, \ldots, Q_N]$ denotes the block-diagonal matrix having $Q_1, \ldots, Q_N$ as its diagonal blocks. $\otimes$ is the Kronecker product of matrices. $\mathbf{1}_p$ is the vector in $\mathbf{R}^p$ with all unity components. We let $\|z\|_P \triangleq \sqrt{z'Pz}$. $L_2[0, \infty)$ will denote the Lebesgue space of $\mathbf{R}^k$-valued vector-functions $z(\cdot)$, defined on the time interval $[0, \infty)$, with the norm $\|z\|_2 \triangleq \left(\int_0^\infty \|z(t)\|^2 dt\right)^{1/2}$ and the inner product $\int_0^\infty z_1(t)' z_2(t) dt$.

II. Problem formulation

In this paper we will consider a directed graph topology $\mathbf{G} = (\mathbf{V}, \mathbf{E})$; $\mathbf{V}, \mathbf{E}$ are the set of vertices and the set of edges (i.e, the subset of the set $\mathbf{V} \times \mathbf{V}$), respectively. Without loss of generality, we let $\mathbf{V} = \{1, 2, \ldots, N\}$. It is assumed that the nodes of the graph $\mathbf{G}$ have no self-loops, i.e., $(i, i) \notin \mathbf{E}$.

For each $i \in \mathbf{V}$, let $\mathbf{V}_i = \{j : (j, i) \in \mathbf{E}\}$ be the set of nodes supplying information to node i, termed as the neighbourhood of node i. The cardinality of $\mathbf{V}_i$, known as the in-degree of node i, is denoted p_i. Also, q_i will denote the number of outgoing edges for node i, known as the out-degree of node i.

Consider a multi-agent system, consisting of a reference parameter varying nonlinear system

$$\dot{x} = A(\rho(t))x + B_1\phi(x), \quad x(0) = x_0, \tag{1}$$

and N uncertain parameter varying nonlinear dynamical agents,

$$\begin{aligned} \dot{x_i} &= A(\rho(t))x_i + B_1\phi(x_i) + u_i(t) + B_{2i}w_i(t), \quad (2) \\ x_i(0) &= x_{i0}, \end{aligned}$$

Here $x \in \mathbf{R}^n$ and $x_i \in \mathbf{R}^n$ are the state of the reference plant and the i-th agent, respectively, $u_i \in \mathbf{R}^n$ denotes the control input, $w_i \in \mathbf{R}^{r_i}$ is the disturbance, and $\rho(t)$ is the time-varying parameter, which is available to all agents. It is assumed that $\rho : [0, \infty) \to \Gamma \triangleq [\rho_{\min}, \rho_{\max}] \subset \mathbf{R}$ is a continuous function. The matrix-valued function $A(\cdot)$ is also assumed to be continuous on the interval Γ, while B_1 is a fixed matrix. Also, the function $\phi(x) : \mathbf{R}^n \to \mathbf{R}^l$ satisfies the global Lipschitz condition

$$\|\phi(x_1) - \phi(x_2)\|^2 \leq (x_1 - x_2)'R(x_1 - x_2), \quad \forall x_1, x_2 \in \mathbf{R}^n; \tag{3}$$

where $R = R' \geq 0$.

It is assumed that direct measurements of the reference (1) are not available. Instead each agent (2) receives broadcast signals from the reference plant and its neighbours which are corrupted by perturbations:

$$\begin{aligned} y_i &= C_{2i}x + D_{2i}w_i, \\ v_{ij} &= H_{ij}x_j + G_{ij}w_{ij}, \end{aligned} \tag{4}$$

here y_i, v_{ij} are the signals received by agent i from the reference plant and agent j, respectively, and w_{ij} is a disturbance affecting the information transmission from agent j to agent i. It is assumed that $w_i(\cdot), w_{ij}(\cdot) \in L_2[0, \infty)$, $i, j = 1, \ldots, N$. Also, we assume that $B_{2i}D_{2i}' = 0$, $E_{2i} = D_{2i}D_{2i}' > 0$, $F_{ij} = G_{ij}G_{ij}' > 0$ for all i.

In a general situation the reference may be unobservable from individual signals y_i, and the agents must use their neighbours' broadcast for feedback to achieve synchronization. This leads us to introduce the following protocol to interconnect the agents over the given graph $\mathbf{G}$:

$$\begin{aligned} u_i(t) &= L_i(\rho(t))(y_i - C_{2i}x_i) \\ &\quad + \sum_{j \in \mathbf{V}_i} K_{ij}(\rho(t))(v_{ij} - H_{ij}x_i), \end{aligned} \tag{5}$$

where $L_i(\cdot)$, $K_{ij}(\cdot)$ are matrix-valued gain functions to be constructed. The graph topology is assumed to be fixed, and the set of neighbours of agent i, $\mathbf{V}_i$ remains constant.

The agents in (2) employ nonidentical matrices C_{2i}, D_{2i}; i.e., the agents have nonidentical measurement models. This distinguishes our model from other models used in the literature where all agents employ identical measurement models, and the leader is observable from agent's measurements; e.g., see [2]. Also, the controller gains $L_i(\cdot)$, $K_{ij}(\cdot)$ are regarded as the design parameters of the protocol.

Associated with the reference system (1), the set of agents (2) and the graph $\mathbf{G}$ is the disagreement function (cf. [6])

$$\Psi_{\mathbf{G}}(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j \in \mathbf{V}_i} \|x_j - x_i\|^2, \tag{6}$$

where $\mathbf{x} = [x_1' \ \ldots \ x_N']'$. Letting $e = x - x_i$ denote the synchronization error of agent i, and letting $\mathbf{e} = [e_1' \ \ldots \ e_N']'$, we note that $\Psi_{\mathbf{G}}(\mathbf{x}) = \Psi_{\mathbf{G}}(\mathbf{e})$.

We now define the synchronization problems under consideration in this paper. In the first problem we are concerned with achieving synchronization with a guaranteed H_∞ level of disagreement between the agents. In the second problem, a stronger version of disagreement performance is considered. It involves a penalty on the synchronization transient performance, additional to the penalty on the consensus transient performance.

Let $\mathbf{x}_0 = [x_{10}', \ldots, x_{N0}']'$, and

$$\begin{aligned} \|(\mathbf{x}_0, \mathbf{w}, \bar{\mathbf{w}})\|^2 &\triangleq \|\mathbf{1}_n \otimes x_0 - \mathbf{x}_0\|_P^2 \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\left(\|w_i\|_2^2 + \sum_{j=1}^{N}\|w_{ij}(\cdot)\|_2^2\right), \end{aligned}$$

where $P = P' > 0$ is a fixed matrix to be determined.

Definition 1: The problem of (weak) robust synchronization is to determine continuous feedback control and interconnection gain schedules $L_i(\rho)$, $K_i(\rho)$, $\rho \in \Gamma$, for the protocol (5) so that the following properties hold:

(i) In the absence of uncertain perturbations, the interconnection of unperturbed systems describing evolution of the synchronization error dynamics of each agent must be exponentially stable. That is,

$$\|e_i(t)\|^2 = \|x_i(t) - x(t)\|^2 \leq ce^{-\omega t}, \quad (\exists c, \omega > 0).$$

(ii) In the presence of uncertain perturbations, the following H_∞ consensus performance must be ensured

$$\sup_{\substack{x_{0i}\neq x_0, \\ (w_i, w_{ij}, j\in \mathbf{V}_i)_{i=1}^N \neq 0}} \frac{\int_0^\infty \Psi_{\mathbf{G}}(\mathbf{e}(t))dt}{\|(\mathbf{x}_0, \mathbf{w}, \bar{\mathbf{w}})\|^2} \leq \gamma^2. \quad (7)$$

Here, $\gamma > 0$ is a given constant.

(iii) All agents synchronize asymptotically,

$$\lim_{t\to\infty} \sum_{i=1}^N \|x(t) - x_i(t)\|^2 = 0. \quad (8)$$

The quantity on the left-hand-side of (7) defines the L_2 disagreement gain of the distributed observer.

Definition 2: Let $Q_1 > 0, \ldots, Q_N > 0$ be given matrices, and $Q \triangleq \mathrm{diag}[Q_1, \ldots, Q_N]$. The problem of strong robust synchronization is to find continuous feedback control and interconnection gain schedules $L_i(\rho)$, $K_i(\rho)$, $\rho \in \Gamma$, for the protocol (5) so that properties (i), (iii) of Definition 1 hold, along with the following property, which replaces (7):

$$\sup_{\substack{x_{0i}\neq x_0, \\ (w_i, w_{ij}, j\in \mathbf{V}_i)_{i=1}^N \neq 0}} \frac{\int_0^\infty (\mathbf{e}(t)'Q\mathbf{e}(t) + \Psi_{\mathbf{G}}(\mathbf{e}(t)))\,dt}{\|(\mathbf{x}_0, \mathbf{w}, \bar{\mathbf{w}})\|^2} \leq \gamma^2. \quad (9)$$

III. The Main Results

The derivation of the main result of the paper follows the general scheme of stability and robustness preserving interpolated gain-scheduling [8], [10], [15]. First, we revisit the results in [12] for a fixed parameter case and a more general class of agents under consideration in this paper. Next, a synchronization result will be established for a class of parameter-varying systems with a small-scale parameter variation. This extension will serve as the basis for the subsequent derivation of interpolated feedback schedules for a more general class of parameter-varying systems with bounded rate of parameter variations, which is the main result of this paper.

A. Synchronization of fixed parameter systems

Let us fix $\rho \in \Gamma$ and consider the fixed parameter version of the uncertain reference system (1),

$$\dot{x} = A(\rho)x + B_1\phi(x) + \psi(t,x), \quad x(0) = x_0, \quad (10)$$

and the corresponding N uncertain fixed-parameter dynamical agents

$$\begin{aligned}\dot{x}_i &= A(\rho)x_i + B_1\phi(x_i) + \psi(t,x_i) \\ &\quad + u_i(t) + B_{2i}w_i(t), \qquad x_i(0) = x_{i0}. \quad (11)\end{aligned}$$

Compared with the system model of the agents given in (2), we have introduced an additional uncertainty term to the reference plant and the equations of agents' dynamics. The motivation for this will become clear later, when we will consider small parameter variations in the agents and the reference plant. Such small variations can be treated as an additional LFT-type uncertainty, capturing the mismatch between fixed system parameters used in the protocol design, and the true system parameters. It will be shown that the size of this mismatch can be characterized in terms of a uniform norm bound condition, such as

$$\|\psi(t,x) - \psi(t,x_i)\|^2 \leq \alpha^2\|e_i\|^2, \quad (12)$$

where $\alpha > 0$ is a constant.

Associated with the fixed parameter system (10), (11) is the fixed-parameter version of the protocol (5),

$$\begin{aligned}u_i(t) &= L_i(\rho)(y_i - C_{2i}x_i) \\ &\quad + \sum_{j\in\mathbf{V}_i} K_{ij}(\rho)(v_{ij} - H_{ij}x_i), \quad (13)\end{aligned}$$

where y_i, v_{ij} are defined in the same way as in (4). We now present a sufficient condition for the existence of a fixed parameter protocol (13) which ensures that the fixed parameter systems (11) achieve strong synchronization.

Given constants $\delta_i > 0$ and matrices $Q_i = Q_i' > 0$, $i = 1, \ldots, N$, consider the following coupled Linear Matrix Inequalities (LMIs) in scalar variables $\tau_i > 0$, $\theta_i > 0$ and matrix variables $X_i = X_i' > 0$:

$$\begin{bmatrix} S_i & \star & \star & \star & \star \\ B_2'X_i & -\gamma^2 I & \star & \star & \star \\ B_1'X_i & 0 & -\tau_i I & \star & \star \\ X_i & 0 & 0 & -\theta_i I & \star \\ \Xi_i' & 0 & 0 & 0 & -Z_i \end{bmatrix} < 0, \quad (14)$$

where

$$\begin{aligned}S_i &\triangleq X_i(A(\rho) + \delta_i I) + (A(\rho) + \delta_i I)'X_i + (p_i + q_i)I \\ &\quad - \gamma^2 C_{2i}'E_{2i}^{-1}C_{2i} - \gamma^2 \sum_{j\in\mathbf{V}_i} H_{ij}'F_{ij}^{-1}H_{ij} \\ &\quad + Q_i + \tau_i R + \theta_i\alpha^2 I, \\ \Xi_i &= \left[\gamma^2 H_{ij_1}'F_{ij_1}^{-1}H_{ij_1} - I \;\ldots\; \gamma^2 H_{ij_{p_i}}'F_{ij_{p_i}}^{-1}H_{ij_{p_i}} - I\right], \\ Z_i &= \mathrm{diag}\left[\frac{2\delta_{j_1}}{q_{j_1}+1}X_{j_1}, \;\ldots,\; \frac{2\delta_{j_{p_i}}}{q_{j_{p_i}}+1}X_{j_{p_i}}\right],\end{aligned}$$

$j_1, \ldots, j_{p_i}$ are the elements of the neighbourhood set $\mathbf{V}_i$.

Lemma 1: Let $\rho \in \Gamma$ be given and fixed. Suppose the graph $\mathbf{G}$, the matrices $Q_i = Q_i' > 0$ and the constants $\gamma, \delta_i > 0$ are such that the coupled LMIs (14) corresponding to the given ρ are feasible. Consider a collection of feasible triples $(\tau_{i,\rho}, \theta_{i,\rho}, X_{i,\rho})$, $i = 1, \ldots, N$, and define

$$K_{ij}(\rho) = \gamma^2 X_{i,\rho}^{-1}H_{ij}'F_{ij}^{-1}, \quad L_i(\rho) = \gamma^2 X_{i,\rho}^{-1}C_{2i}'E_{2i}^{-1}. \quad (15)$$

The network of fixed parameter agents (11), (13), (15) solves the fixed parameter version of the strong robust synchronization problem in Definition 2, with $\rho(t) \equiv \rho$. The matrix P in condition (9) corresponding to this solution is $P = \frac{1}{\gamma^2 N}\mathrm{diag}[X_{1,\rho}, \ldots, X_{N,\rho}]$.

The main idea of the proof is to show that the interconnected system describing dynamics of the synchronization errors associated with the reference (10) and the multi-agent system (11) has the properties of vector dissipativity [1] with respect to the vector storage function $[V_1(e_1)\ \ldots\ V_N(e_N)]'$, $V_i(e_i) = e_i' X_{i,\rho} e_i$, and a suitably defined vector of supply rates. An N-component large scale system

$$\begin{aligned} \dot{e}_i &= f_i(e_i) + g(e_i)\bar{w}_i + H_i(e_1, e_2, \ldots, e_N), \\ z_i &= h_i(e_i) + q_i(x_i)\bar{w}_i, \quad t \geq 0, \end{aligned} \tag{16}$$

is vector dissipative with respect to a vector of supply rates $(S_1(\mathbf{z}, \bar{w}_1), \ldots, S_N(\mathbf{z}, \bar{w}_N))'$ and a vector of continuous non-negative-definite storage functions $V(\mathbf{e}) = [V_1(e_1)\ \ldots\ V_N(e_N)]'$ if there exists a so-called class $\mathscr{W}$ function $\pi \in \mathbf{R}_+^N \to \mathbf{R}^N$, $\pi(0) = 0$, such that the system $\dot{r} = \pi(r)$ is Lyapunov stable and the following vector dissipation inequality holds for all $t > t_0 \geq 0$

$$\begin{aligned} V_i(x_i(t)) - V_i(x_i(t_0)) \leq &\int_{t_0}^{t} S_i(y(\tau), \xi(\tau)) d\tau \\ &+ \int_{t_0}^{t} \pi_i(V_1(x_1), \ldots, V_N(x_N)) d\tau. \end{aligned}$$

The large-scale system (16) is exponentially vector dissipative if the system $\dot{r} = \pi(r)$ is asymptotically stable.

The proofs of our results use linear functions $\pi(r) = \Pi r$, where Π is an essentially nonnegative matrix. Such linear functions are class $\mathscr{W}$ functions [1]. In the special case of linear function Π, the above vector dissipation inequality is equivalent to the vector differential inequality

$$\dot{V}_i(x_i(t)) \leq S_i(y(t), \xi(t)) + [\Pi V(x(t))]_i. \tag{17}$$

Also in this special case, the property of exponential dissipativity requires that π must be a Hurwitz matrix.

B. Synchronization under small parameter variations

The fixed parameter protocol (13) can be used for synchronization of the parameter varying multi-agent system (1), (2), provided parameter variations are sufficiently small, i.e., $\rho(t) \approx \rho_0\ \forall t \geq 0$. Small variations of the matrix $A(\cdot)$ can be treated as parameter mismatch disturbances, and the parameter varying systems (2) and (1) can be regarded as a perturbation of corresponding fixed-parameter systems. To formalize this idea, let us fix $\rho_0 \in \Gamma$, and define

$$\psi(t, x) = (A(\rho(t)) - A(\rho_0))x.$$

Then, the parameter varying multi-agent system (1), (2) can be rewritten in the form of a collection of uncertain systems (10), (11). The robust synchronization protocol of the form (13) can now be designed based on the representation (10), (11) using Lemma 1. Provided the variations of the matrix $A(\rho(t))$ about $A(\rho_0)$ are small enough to satisfy (12), it will guarantee strong robust synchronization of the agents (11) to the plant (10), in the presence of both the disturbances $w_i(\cdot)$ and $w_{ij}(\cdot)$, and the parameter variation in $A(\cdot)$. In order to proceed in this direction, a bound of the form (12) must be established. Note that when $\rho(\cdot) \equiv \rho_0$, then $\psi \equiv 0$ and the uncertainty constraints (12) are trivially satisfied. Also note that since $A(\rho)$ is continuous on the compact set Γ, then there always exists a constant $\alpha > 0$ such that (12) holds. We conclude this discussion by formally stating the above observation about the synchronization of the system (1), (2) under small parameter variations as a lemma.

Lemma 2: Let a fixed $\rho_0 \in \Gamma$ be given. Suppose that

$$(A(\rho(t)) - A(\rho_0))'(A(\rho(t)) - A(\rho_0)) \leq \alpha^2 I \quad \forall t \geq 0. \tag{18}$$

Suppose the graph $\mathbf{G}$ and the constants $\gamma > 0$ and $\delta_i > 0$ are such that the coupled LMIs (14) with $\rho = \rho_0$ are feasible, and let $(\tau_{i,\rho_0}, \theta_{i,\rho_0}, X_{i,\rho_0})$, be a corresponding collection of feasible triples, $i = 1, \ldots, N$. Then the network of agents (11) equipped with the protocols (13), (15), where $\rho = \rho_0$, solves the problem of strong robust synchronization in Definition 2. The matrix P in conditions (7) and (8) corresponding to this solution is $P = \frac{1}{\gamma^2 N}\mathrm{diag}[X_{1,\rho_0}, \ldots, X_{N,\rho_0}]$.

C. Interpolation of synchronization protocols

Robustness of the protocol (13), (15) is due to the assumption that variations of $A(\rho(\cdot))$ are sufficiently small. Even though we alluded in the previous section that an α can be always found to satisfy (18) globally on Γ, this may lead to an excessively conservative synchronization scheme, or even a failure to satisfy the conditions in Lemma 1. On the other hand, using smaller $\alpha > 0$ may result in the property (18) not holding globally. However it holds locally for any choice of α, since the function $A(\cdot)$ is continuous on the compact set Γ. This observation underlies our approach.

Our design method can be summarized as follows. First a collection of constants $\alpha_k > 0$ and 'grid points' $\Gamma_0 = \{\rho_k, k = 1, \ldots, M\}$ is selected so that for any $\rho \in \Gamma$ there exists at least one point ρ_k with the property

$$(A(\rho) - A(\rho_k))'(A(\rho) - A(\rho_k)) \leq \alpha_k^2 I. \tag{19}$$

Let U_k be the largest connected neighbourhood of ρ_k consisting of all $\rho \in \Gamma$ for which (19) holds. It is natural to assume that $\Gamma \subseteq \cup_{k=1}^{M} U_k$. Next, using Lemma 1, we compute the synchronization protocol (13) for the uncertain parameter-varying agents plants (11) for each ρ_k. The robustness of this protocol stated in Lemma 1 yields that the conclusion of Lemma 2 holds under the condition $\{\rho(t), t > 0\} \subset U_k$. Along with the condition $\Gamma \subseteq \cup_{k=1}^{M} U_k$, this establishes an analog to the stability covering condition in [8].

For the above procedure to be possible to carry out, there must exist α_k and Γ_0 such that the LMIs (14) with these parameters are feasible. We assume that this requirement is satisfied. Then, for each fixed $\rho \in \Gamma$, a synchronization protocol can be assigned to the system (2) by scheduling one of the fixed-parameter protocols (13) corresponding to an index $k \in \{k : \rho \in U_k\}$. However, if this selection of protocols is applied to the parameter-varying system (1), its gains may become discontinuous at the time instant when the trajectory of the parameter $\rho(t)$ exits the set U_k and enters the set U_{k+1}. Such discontinuities are undesirable in many practical situations, since they result in transients which usually have an adverse effect on system performance. We therefore propose a *continuous* interpolation of the fixed-parameter synchronization protocols which preserves, along with the synchronization properties of the nominal system, the property of interpolants to guarantee a specified level of the relative H_∞ disagreement between the agents.

Consider and arbitrary fixed $\rho \in \Gamma$, and the collection of constants $\alpha_k > 0$, $k = 1, \ldots, M$, and grid points Γ_0 discussed above. Select k such that $\rho \in U_k$, and let $(\tau_{i,\rho_k}, \theta_{i,\rho_k}, X_{i,\rho_k})$, $i = 1, \ldots, N$, be feasible triples of the LMIs (14). It is straightforward to show that the pairs $(\tau_{i,\rho_k}, X_{i,\rho_k})$ are feasible solutions to the following coupled LMIs in $X_i = X_i' > 0$, $\tau_i > 0$, $i = 1, \ldots, N$:

$$\begin{bmatrix} \bar{S}_i(\rho) & \star & \star & \star \\ B_2' X_i & -\gamma^2 I & \star & \star \\ B_1' X_i & 0 & -\tau_i I & \star \\ \Xi_i' & 0 & 0 & -Z_i \end{bmatrix} < 0, \qquad (20)$$

where the matrices Z_i and Ξ_i are the same as in (14), and

$$\begin{aligned} \bar{S}_i(\rho) \triangleq\ & X_i(A(\rho) + \delta_i I) + (A(\rho) + \delta_i I)' X_i + (p_i + q_i) I \\ & - \gamma^2 C_{2i}' E_{2i}^{-1} C_{2i} - \gamma^2 \sum_{j \in \mathbf{V}_i} H_{ij}' F_{ij}^{-1} H_{ij} + Q_i + \tau_i R. \end{aligned}$$

Next, suppose $\rho \in U_k \cap U_{k+1}$. Then we conclude that both $(\tau_{i,\rho_k}, X_{i,\rho_k})$, and $(\tau_{i,\rho_{k+1}}, X_{i,\rho_{k+1}})$ are feasible solutions to the LMIs (20). This allows us to construct interpolated feasible solutions to (20) as follows. For a $\lambda \in [0,1]$, define

$$\begin{aligned} X_{i,\lambda} &= \lambda X_{i,\rho_k} + (1-\lambda) X_{i,\rho_{k+1}}, \\ \tau_{i,\lambda} &= \lambda \tau_{i,\rho_k} + (1-\lambda) \tau_{i,\rho_{k+1}}. \end{aligned} \qquad (21)$$

It is straightforward to verify that $\tau_{i,\lambda}, X_{i,\lambda}$ in (21) satisfy the LMIs (20).

Using the above fact, we can now define a collection of interpolated gains for the protocols (5), as follows. Suppose the collection of positive constants α_k and the grid points Γ_0 has the following properties:

$$(A(\rho) - A(\rho_k))'(A(\rho) - A(\rho_k)) \le \alpha_k^2 I, \quad \rho_k \le \rho < \bar{\rho}_k, \qquad (22)$$

$$(A(\rho) - A(\rho_{k+1}))'(A(\rho) - A(\rho_{k+1})) \le \alpha_{k+1}^2 I, \quad \underline{\rho}_{k+1} < \rho \le \rho_{k+1}, \qquad (23)$$

where $\rho_k < \underline{\rho}_{k+1} < \bar{\rho}_k < \rho_{k+1}$. In particular, this implies that $(\underline{\rho}_{k+1}, \bar{\rho}_k) \subset U_k \cap U_{k+1}$.

For every $\rho \in \Gamma$, select $k, k+1$ such that $\rho \in [\rho_k, \rho_{k+1}]$, and define $\lambda = \dfrac{\bar{\rho}_k - \rho}{\bar{\rho}_k - \underline{\rho}_{k+1}}$, and

$$X_{i,\rho} = \begin{cases} X_{i,\rho_k}, & \rho \in [\rho_k, \underline{\rho}_{k+1}], \\ X_{i,\lambda}, & \rho \in [\underline{\rho}_{k+1}, \bar{\rho}_k], \\ X_{i,\rho_{k+1}}, & \rho \in [\bar{\rho}_k, \rho_{k+1}], \end{cases} \qquad (24)$$

$$\tau_{i,\rho} = \begin{cases} \tau_{i,\rho_k}, & \rho \in [\rho_k, \underline{\rho}_{k+1}], \\ \tau_{i,\lambda}, & \rho \in [\underline{\rho}_{k+1}, \bar{\rho}_k], \\ \tau_{i,\rho_{k+1}}, & \rho \in [\bar{\rho}_k, \rho_{k+1}], \end{cases} \qquad (25)$$

$$K_{ij}(\rho) = \gamma^2 X_{i,\rho}^{-1} H_{ij}' F_{ij}^{-1}, \quad L_i(\rho) = \gamma^2 X_{i,\rho}^{-1} C_{2i}' E_{2i}^{-1}. \qquad (26)$$

The functions K_{ij}, L_i are continuous on Γ, since $X_{i,\lambda} > 0$ for all $\lambda \in [0,1]$.

The following theorem is the main result of the paper. It shows that the interpolated synchronization protocol (5), with the gains $K_{ij}(\cdot)$, $L_i(\cdot)$, defined in (26), provides robust synchronization of the system (2) to the reference (1). Let $\bar{\Gamma}$ be the set consisting of all the corner points $\underline{\rho}_{k+1}$, $\bar{\rho}_k$, which lie inside Γ. Without loss of generality, we assume that $\rho(0) \notin \bar{\Gamma}$.

Theorem 1: Suppose

$$\sup_{t \ge 0} |\dot{\rho}| \le \min_i \left\{ \lambda_{\min}(Q_i) \left[\sup_k \frac{\| X_{i,\rho_{k+1}} - X_{i,\rho_k} \|}{\bar{\rho}_k - \underline{\rho}_{k+1}} \right]^{-1} \right\}. \qquad (27)$$

Then the network of agents (2) equipped with the protocols (5), (26), solves the robust synchronization problem in Definition 1. The matrix P in condition (7) corresponding to this solution is $P = \frac{1}{\gamma^2 N} \mathrm{diag}[X_{1,\rho(0)}, \ldots, X_{N,\rho(0)}]$.

Corollary 1: Suppose there exists $\eta \in [0,1)$ such that

$$\sup_{t \ge 0} |\dot{\rho}| \le \eta \min_i \left\{ \lambda_{\min}(Q_i) \left[\sup_k \frac{\| X_{i,\rho_{k+1}} - X_{i,\rho_k} \|}{\bar{\rho}_k - \underline{\rho}_{k+1}} \right]^{-1} \right\}. \qquad (28)$$

Then the network of agents (2) equipped with the protocols (5), (26), solves the strong robust synchronization problem in Definition 2, with Q replaced with $(1-\eta)Q$. The matrix P is defined in the same way as in Theorem 1.

IV. Example

To illustrate the results of the paper, consider a problem of synchronization of a set of 2nd-order linear parameter varying systems to a nonlinear Lorenz system. The reference system is described by the equations

$$\begin{aligned} \dot{x}^{(1)} &= -\sigma x^{(1)} + \sigma x^{(2)}, \\ \dot{x}^{(2)} &= r x^{(1)} - x^{(2)} - \rho(t) x^{(1)}, \\ \dot{\rho} &= -b\rho(t) + x^{(1)} x^{(2)}. \end{aligned} \qquad (29)$$

It is known to exhibit chaotic dynamics when $\sigma = 10$, $r = 28$, and $b = 8/3$. Although the system is nonlinear, the first two equations can be regarded as a parameter varying system, with $\rho(t)$ interpreted as a scheduling variable. This subsystem is of the form (1), and will be considered as a reference system in this example. This reference system is linear, so $\phi(x) \equiv 0$, and $R = 0$.

In accordance with the aforementioned choice of the reference system, consider 2nd-order agents whose dynamics are described by equations of the form

$$\begin{aligned} \dot{x}_i^{(1)} &= -\sigma x_i^{(1)} + \sigma x_i^{(2)} + u_i^{(1)} + B_{2i}^{(1)} w_i, \\ \dot{x}_i^{(2)} &= (r - \rho(t)) x_i^{(1)} - x_i^{(2)} + u_i^{(2)} + B_{2i}^{(2)} w_i. \end{aligned} \qquad (30)$$

These dynamics are governed by the signal $\rho(t)$ generated by the reference system (29). The system (30) can be rewritten in the form of equations (2) with

$$A(\rho) = \begin{bmatrix} -\sigma & \sigma \\ (r-\rho) & -1 \end{bmatrix}, \quad B_{2i} = \begin{bmatrix} B_{2i}^{(1)} \\ B_{2i}^{(2)} \end{bmatrix}, \quad u_i = \begin{bmatrix} u_i^{(1)} \\ u_i^{(2)} \end{bmatrix}.$$

As before, w_i, $i = 1, \dots, N$, describe L_2-integrable perturbations. In this example, we randomly chose $B_{2i} = [0.0806\ 0.0232]'$ for all agents.

It is easy to verify using simulations that if $u_i \equiv 0$, the systems (30) do not synchronize to the reference system. In fact, when $w_i \equiv 0$, some trajectories of the system converge to the origin. Therefore, a feedback control is necessary to achieve synchronization. We now show that synchronization can be achieved by means of a protocol of the form (5).

The design of such a protocol (5) for this example proceeds as follows. First, a simple ring structure of the network was chosen, so that agent i can receive information from agent $i-1$ and can forward its state to agent $i+1$. That is, $\mathbf{V}_i = \{i-1\}$ for $i = 2, \dots$, and $\mathbf{V}_1 = \{N\}$. In this example, we let $N = 5$.

Since the framework of the paper allows for synchronization via imperfect measurements and imperfect communication, we let $D_{2i} = 0.01$, $G_{i,i-1} = 0.2$ and randomly selected a set of matrices C_{2i}, $H_{i,i-1}$ to provide each agent with partial measurements of the reference plant and the partial information about the states of its neighbours. Also, the Lorenz system (29) was simulated on the interval $[0, 10]$, with the initial conditions $[0.3\ 0.3\ 20]'$, to determine the range of $\rho(t)$. It was found that on this time interval, $\Gamma = [8.3874, 42.7367]$ and $|\dot{\rho}| \leq 264.0108$.

Next, 11 equally spaced grid points were chosen as the set Γ_0, $\rho_k = 8.3874 + 3.4349(k-1)$, $k = 1, \dots, 11$. Then, the LMI optimization problem $\min \gamma^2$ subject to the LMI (14) was solved at each grid point, with $\delta = 0.01$, $\alpha_k^2 = 12.0359$, and $Q = 1000I$. These parameters were chosen to ensure that conditions (22), (23) hold with $\bar{\rho}_k = \rho_{k+1}$ and $\underline{\rho}_{k+1} = \rho_k$, and also to ensure the satisfaction of the rate bound condition (27). Clearly, $\rho(0) = 20 \notin \Gamma_0$.

We have verified that all of the conditions of Corollary 1 Theorem 1 are satisfied in this example including (28) which holds with $\eta = 0.8684$. The upper bound on the H_∞ disagreement gain, guaranteed by the interpolated gain-scheduled synchronization protocol constructed in this example will be $\max_k \gamma_k^2 = 1.7501$.

To verify synchronization properties of the constructed protocol u_i, the interconnected system (30), (5) was simulated on the time interval $[0, 10]$, and error dynamics were plotted versus time. As an example, the plot of $x^{(1)} - x_i^{(1)}$ versus time is shown in Figure 1. It was observed in our simulations that all synchronization errors converged to 0, as predicted.

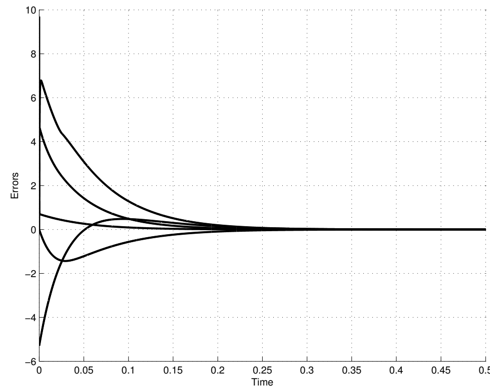

Fig. 1. The synchronization errors $x^{(1)} - x_i^{(1)}$ versus time.

V. Conclusion

We have revisited the previous results on robust consensus-based synchronization of uncertain multi-agent systems to extend them to the class of parameter varying agents. In addition, a more general model than that in [12] has been considered which involves communications via imperfect channels. The paper has extended the gain-scheduling via interpolation technique to the class of synchronization problems for large-scale systems consisting of parameter varying agents with a Lipschitz continuous nonlinearity. Our gain-scheduled synchronization algorithm involves solving a series of synchronization problems for the fixed parameter system considered at several operating points, and a linear interpolation of the vector storage functions constructed at each of the operating points.

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