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Quantized Consensus with Finite Data Rate under Directed Topologies

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Abstract— **This paper investigates the consensus problems in directed networks under communication constraints. Each agent has a real-valued state but can only exchange finitely many bits information with its neighbors at each time step. Based on dynamic difference encoding and decoding, a distributed algorithm is proposed to achieve quantized consensus asymptotically with as few as only one bit information exchange at each time step. The upper bound of asymptotic convergence rate and the group decision value are given. Furthermore, simulations are provided that demonstrate the effectiveness of the theoretical results.**

I. INTRODUCTION

Since the pioneering works of Vicsek et al. [1] and Jadbabaie et al. [2], the consensus problems attract a lot of attention from the control community. Olfati-Saber and Murray [3] solved the average consensus problems for a network of first-order integrators using directed graphs. From then on, quite a tremendous amount of interesting results [4], [5], [6], [7], [8] have been addressed, to name a few. Due to the fact that communication channels have a limited channel bandwidth, i.e., agents can only transmit a finite amount of information at each time step, message quantization should be considered.

The investigation of consensus under quantized communication started with [9], [10]. Kashyap et al. [9] and Nedic et al. [10] designed average consensus protocols under the assumption that each agent has integer-valued state. These protocols drive each agent to integer approximation of the average of the initial states. From then on, several results appeared recently that tackle this issue; there include [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. In [13] and [16], based on the dynamic encoding and decoding, Carli et al. designed a distributed protocol with error compensation and showed that the average consensus can be achieved asymptotically. Li et al. [20] adopted the same scheme and obtained stronger results: Consensus with the 1 bit quantizer.

This paper follows the work in [20] and investigates consensus problem for a more general class of networks; that is, information consensus under directed graphs. Dealing with directed and possibly unbalanced graphs, we need to take into account two difficulties: 1) that the Laplacian matrix, not being symmetric, might be not diagonalizable, 2) that the left eigenvector of the Laplacian matrix corresponding to the eigenvalue 1 is possibly different from $[1,1,\dots,1]^T$. Three lemmas are used to tackle with the difficulties. We adopt the coding/decoding schemes used in [20] with the discrete-time first-order direct networks under communication constrains. The control parameters are adjusted to ensure that as time progresses the state of each agent approaches a common group decision value. It is shown that if the network contains a spanning tree, the proposed algorithm guarantees quantized consensus with any given bits information exchange at each time step. An upper bound of asymptotic convergence rate and the group decision value are given. Different from the algorithm proposed in [20], the control parameters are chosen by solving a second-order equation to tackle with the difficulty caused by directed communication graphs and achieve average consensus in any rate communication. Finally, simulation results are provided to show the effectiveness of the proposed algorithm.

The paper is organized as follows. Section II introduces some notations and preliminaries on graph theory and gives a detailed description of the proposed protocol. In Section III, based on 3 lemmas, it is proved that quantized consensus can be achieved asymptotically with any data rate. We provide simulations results in Section IV and conclude the paper in Section V.

II. PROBLEM FORMULATION

This section introduces the networks of dynamic agents and presents the quantized consensus algorithm.

A. Graph Model

The information flow topology between agents is represented as a directed graph $\mathscr{G}=(\mathscr{V},\mathscr{E},\mathscr{A})$, where $\mathscr{V}=\{1,2,\dots,N\}$ is the set of vertices with i representing the ith agent, $\mathscr{E}\subset\{(i,j):i,j\in\mathscr{V}\}$ is the set of edges and $\mathscr{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathscr{G}$ satisfying $a_{ij}\geq 0,\forall i,j\in\mathscr{V}$. A directed edge denoted by the pair (j,i) represents a communication channel from j to i. The neighbors, N_i, of node i are all nodes that communicate to i, i.e., $N_i=\{j:a_{ij}\neq 0\}$. Also, $\deg_{\text{in}}(i)=\sum_{j=1}^N a_{ij}$ is called the in-degree of i, and the out-degree of i is defined as $\deg_{\text{out}}(i)=\sum_{j=1}^N a_{ji}$. $d=\max_i \deg_{\text{in}}(i)$ is called the degree of $\mathscr{G}$. The Laplacian matrix of $\mathscr{G}$ is defined as $\mathscr{L}=\mathscr{D}-\mathscr{A}$, where $\mathscr{D}=\text{diag}(\deg_{\text{in}}(1),\dots,\deg_{\text{in}}(N))$. A sequence of edges $(i_1,i_2),(i_2,i_3),\dots,(i_{k-1},i_k)$ is called a directed path from node i_1 to node i_k. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a directed tree formed by graph edges that connected all the nodes of the graph. We say that a graph contains a spanning tree if a subset of the edges forms a spanning tree.

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B. Quantized Consensus Algorithm

In [6], a consensus algorithm was defined as

$$x_i(t+1) = x_i(t) + h\sum_{j\in N_i} a_{ij}\left(x_j(t) - x_i(t)\right), \quad t = 0,1,\ldots, i = 1,2,\ldots,N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the ith agent, and h is the control gain. Here we assume that the communication network is constituted of digital links. Thus only symbolic data can be exchanged between agents. The exchanged data is quantized by a $(2F+1)$-level uniform quantizer, which is given by

$$q_F(x) = \begin{cases} 0, & -1/2 < x < 1/2, \\ i, & (2i-1)/2 \le x < (2i+1)/2, \\ & i = 1,2,\ldots,F-1, \\ F, & x \ge (2F-1)/2, \\ -q_F(-x), & x \le -1/2. \end{cases} \tag{2}$$

In this paper, we adopt the coding/decoding schemes in [20] as follows. The difference encoder E_i of the ith agent has an internal state $y_i(t)$ with the dynamics

$$\begin{cases} y_i(0) = 0, \\ O_i(t) = q_F\left[\frac{1}{g(t-1)}\left(x_i(t) - y_i(t-1)\right)\right], \\ y_i(t) = g(t-1)O_i(t) + y_i(t-1), \end{cases} \tag{3}$$

where $g(t) > 0$ is the scaling function, and $O_i(t)$ is the output of E_i. For each edge $(i,j) \in \mathscr{E}$, the jth agent receives $O_i(t)$ and uses the following decoder to estimate $x_i(t)$:

$$\begin{cases} z_{ij}(0) = 0, \\ z_{ij}(t) = g(t-1)O_i(t) + z_{ij}(t-1), \end{cases} \tag{4}$$

where we define the decoder D_{ij} by $D_{ij}(O_i(t)) = z_{ij}(t)$.

In [20], the consensus algorithm is given by

$$x_i(t+1) = x_i(t) + h\sum_{j\in N_i} a_{ij}\left(z_{ji}(t) - y_i(t)\right). \tag{5}$$

By (3) and (4), we have

$$z_{ij}(t) = y_i(t), \quad j \in N_i, \tag{6}$$

i.e., the internal state $y_i(t)$ of encoder E_i equals to the estimates of $x_i(t)$ by its neighbors. Denote $X(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$, $Y(t) = [y_1(t), y_2(t) \ldots, y_N(t)]^T$, quantization error $e(t) = X(t) - Y(t)$, consensus error $c(t) = X(t) - \left(p^T X(0)\right)\mathbf{1}$, where the vector $p \in \mathbb{R}^{N\times 1}$ satisfies

$$p \ge 0, \quad p^T\mathbf{1} = 1 \quad \text{and} \quad p^T\mathscr{L} = \mathbf{0}^T, \tag{7}$$

and $\mathbf{1}$ and $\mathbf{0}$ denote column vectors with all ones and zeros, respectively. Then from (3), (4), (6) and (7), (5) can be written in a vector form as

$$\begin{cases} X(t+1) = (I - h\mathscr{L})X(t) + h\mathscr{L}e(t), \\ Y(t+1) = g(t)Q_F\left(\frac{X(t+1) - Y(t)}{g(t)}\right) + Y(t), \end{cases} \tag{8}$$

where $Q_F\left([x_1, x_2, \ldots x_N]^T\right) = [q_F(x_1), q_F(x_2), \ldots, q_F(x_N)]^T$.

III. MAIN RESULTS

Before moving on, we make the following assumptions: A1) $\mathscr{G}$ contains a spanning tree; A2) $\max_i |x_i(0)| \le C_x$, $\max_i |c_i(0)| \le C_c$, where C_x and C_c are known nonnegative constants; A3) $\mathscr{L}$ is diagonalizable; A4) $\mathscr{L}$ is not diagonalizable.

A. Supporting Lemmas

To tackle with the technical difficulties in the generalization from undirected to directed graphs, the following lemmas are needed to derive the main results.

Lemma 3.1: ([5]). If Assumption A1) holds, the graph Laplacian $\mathscr{L}$ has exactly one zero eigenvalue $\lambda_1(\mathscr{L})$, and all of the other eigenvalues $\lambda_2(\mathscr{L}), \lambda_3(\mathscr{L}), \ldots, \lambda_N(\mathscr{L})$ are in the open right half plane.

Denote

$$\rho_h = \max_{2\le i\le N} |1 - h\lambda_i(\mathscr{L})|. \tag{9}$$

Then we have the following lemma.

Lemma 3.2: If Assumption A1) holds and $h \in (0, 1/d)$, then $\rho_h < 1$ and

$$\lim_{h\to 0} \frac{1-\rho_h}{h} = \min_{2\le i\le N} \mathrm{Re}\left(\lambda_i(\mathscr{L})\right), \tag{10}$$

where d is the degree of the graph $\mathscr{G}$.

Proof: Fig. 1 is used to show the notations used in the proof. In complex plane, denote circle $\mathscr{H}$ and circle $\mathscr{I}$ for the circles centered at point $H(1,0)$ and point $I(d,0)$ intersect the real axis at point $C(2,0)$ and $L(2d,0)$, respectively. For any given $i = 2,3,\ldots,N$, let vector $\overrightarrow{OJ} = \lambda_i(\mathscr{L})$. The straight line OJ intersects circle $\mathscr{H}$ and circle $\mathscr{I}$ at point A and point M, respectively. Following from Gershgorin theorem and the definition of weighted matrix $\mathscr{A}$, we know that point J is located in circle $\mathscr{H}$ or falls on the circumference of the circle. Noting that $\angle OAC = \angle OML = \pi/2$, we have

$$\frac{OM}{OA} = \frac{OL}{OC} = \frac{2d}{2} = d.$$

where OM denotes the length of the vector $\overrightarrow{OM}$, and so do OA, OL and OC. This together with $h \in (0, 1/d)$ leads to $OG = hOJ < OA$, i.e., point G is located in circle $\mathscr{H}$, which implies that

$$|1 - h\lambda_i(\mathscr{L})| = HG < 1.$$

Let's denote by a_i and b_i the real and the imaginary part of λ_i, respectively. Noticing that $a_i > 0$ by Lemma 3.1, we have

$$\begin{aligned} \rho_h &= \max_{2\le i\le N} |1 - h(a_i + jb_i)| \\ &= \max_{2\le i\le N} \sqrt{1 - 2a_i h + \left(a_i^2 + b_i^2\right)h^2} \\ &= \max_{2\le i\le N} \{1 - a_i h + o(h)\} \\ &= 1 - h\min_{2\le i\le N} a_i + o(h), \quad h \to 0, \end{aligned}$$

which leads to (10). ■

Lemma 3.3: If Assumption A1) holds, the graph Laplacian can be decomposed as

$$\mathscr{L} = VDK \tag{11}$$

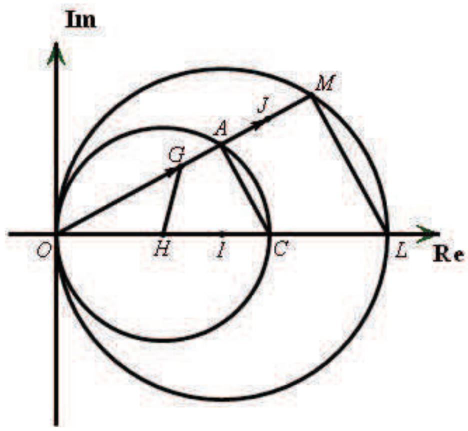

Fig. 1. Notations used in the proof for Lemma 3.2.

where $D \in \mathbb{C}^{N\times N}$ is the Jordan canonical form of $\mathscr{L}$ with the first diagonal entry equals to zero, $V = [\mathbf{1}, R] \in \mathbb{C}^{N\times N}$ and $K = \left[p, Q^T\right]^T \in \mathbb{C}^{N\times N}$ are nonsingular matrices. For any given $h \in (0, 1/d)$ and $r \in (\rho_h, 1)$, denote $J_1(h)$ as the submatrix of $I - hD$ formed by deleting the first row and the first column, $J_2(h,r) = r^{-1}J_1(h)$. Then there exist real numbers

$$C'_J(h,r) = \max_{i\in\mathbb{N}} \left\| [J_2(h,r)]^i \right\|_\infty, \tag{12}$$

and

$$C_J(h,r) = \lim_{n\to\infty} \sum_{i=0}^{n} \left\| [J_2(h,r)]^i \right\|_\infty. \tag{13}$$

Proof: Noting that $\mathscr{L}\mathbf{1} = \mathscr{L}^T p = \mathbf{0}$, (11) can be derived using the Jordan canonical form. By Lemma 3.2 and the definition of $J_2(h,r)$, all the eigenvalues of $J_2(h,r)$ are of modulus less or equal to $\frac{\rho_h}{r} < 1$. Thus, $J_2(h,r)$ is a convergent matrix, i.e., $[J_2(h,r)]^m \to 0$ as $m \to \infty$, which implies that $\|[J_2(h,r)]^m\|_\infty \to 0$ as $m \to \infty$. Then there exists real number $C'_j(h,r) \in \mathbb{R}_{\geq 0}$ satisfying (12).

If Assumption A3) holds, for any given $i \in \mathbb{N}$,

$$\left\| [J_2(h,r)]^i \right\|_\infty = \left(\frac{\rho_h}{r}\right)^i, \tag{14}$$

which leads to

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left\| [J_2(h,r)]^i \right\|_\infty = \lim_{n\to\infty} \sum_{i=0}^{n} \rho_h^i = \frac{r}{1-\rho_h}. \tag{15}$$

If Assumption A4) holds, (13) can be found in Example 7.9.3 in [21]. ∎

B. Convergence Analysis

The main theorems in this subsection are the generalizations to directed graphs of the work in [20], where only undirected graphs were considered.

Theorem 3.1: If Assumptions A1), A2) and A3) hold, for any given $h \in (0, 1/d)$ and $r \in (\rho_h, 1)$, let

$$M_1(h,r) = \frac{1+2hd}{2r} + \frac{2h^2d^2\|R\|_\infty\|Q\|_\infty}{r(r-\rho_h)}, \tag{16}$$

$$F_1(h,r) = \lfloor M_1(h,r) - 1/2 \rfloor + 1, \tag{17}$$

where R and Q are defined in Lemma 3.3, and for any given $F \geq F_1(h,r)$, let

$$g_0 > \max\left\{ \frac{(C_c r + 2hC_x d)(r-\rho_h)}{hd}, \frac{C_x}{F+1/2} \right\}. \tag{18}$$

Then under the protocol given by (3), (4) and (5) with the $(2F+1)$-level uniform quantizer (2) and the scaling function $g(t) = g_0 r^t$, the closed loop system (8) satisfies

$$\lim_{t\to\infty} x_i(t) = p^T X(0), \quad i = 1, 2, \ldots, N, \tag{19}$$

and $r_{asym} \leq r$, where

$$r_{asym} = \sup_{X(0)\neq\left(p^T X(0)\right)\mathbf{1}} \lim_{t\to\infty} \left(\frac{\left\|X(t) - \left(p^T X(0)\right)\mathbf{1}\right\|_\infty}{\left\|X(0) - \left(p^T X(0)\right)\mathbf{1}\right\|_\infty} \right)^{1/t}, \tag{20}$$

which is defined as convergence rate of maximal consensus error.

Proof: Noting that $\mathscr{L}^T p = \mathscr{L}\mathbf{1} = \mathbf{0}$, we rewrite (8) as

$$\begin{cases} c(t+1) = (I - h\mathscr{L})c(t) + h\mathscr{L}e(t), \\ e(t+1) = (I + h\mathscr{L})e(t) - h\mathscr{L}c(t) \\ \qquad\qquad -g(t)Q\left(\dfrac{(I+h\mathscr{L})e(t) - h\mathscr{L}c(t)}{g(t)}\right). \end{cases} \tag{21}$$

Let

$$\begin{cases} w(t) = \dfrac{1}{g(t)}c(t), \\ z(t) = \dfrac{1}{g(t)}e(t). \end{cases} \tag{22}$$

From (22) and $g(t) = g_0 r^t$, we have

$$\begin{cases} w(t+1) = r^{-1}(I - h\mathscr{L})w(t) + r^{-1}h\mathscr{L}z(t), \\ z(t+1) = r^{-1}\Delta(t), \end{cases} \tag{23}$$

where

$$\Delta(t) = (I+h\mathscr{L})z(t) - h\mathscr{L}w(t) - Q\left((I+h\mathscr{L})z(t) - h\mathscr{L}w(t)\right).$$

The paragraph proves by induction that if a $(2F+1)$-level uniform quantizer with $F \geq F_1(h,r)$ is applied, the quantizer will never be saturated. When $t = 0$, from $Y(0) = 0$, we get

$$\begin{aligned} &(I+h\mathscr{L})z(0) - h\mathscr{L}w(0) \\ =&\frac{1}{g_0}(I+h\mathscr{L})(X(0) - Y(0)) + \frac{1}{g_0}h\mathscr{L}\left(X(0) - \left(p^T X(0)\right)\mathbf{1}\right) \\ =&\frac{1}{g_0}(I+h\mathscr{L})X(0) - \frac{1}{g_0}h\mathscr{L}X(0) = \frac{1}{g_0}X(0). \end{aligned}$$

This together with (18) gives

$$\|(I+h\mathscr{L})z(0) - h\mathscr{L}w(0)\|_\infty = \frac{1}{g_0}\|x(0)\|_\infty \leq \frac{1}{g_0}C_x < F + \frac{1}{2},$$

i.e., the quantizer is unsaturated. For any given nonnegative integer k, suppose that when $t = 0, 1, \ldots, k$, the quantizer is not saturated, i.e.,

$$\sup_{0\leq t\leq k} \|\Delta(t)\|_\infty \leq \frac{1}{2},$$

which together with (23) gives

$$\sup_{1\leq t\leq k+1} \|z(t)\|_\infty \leq \frac{1}{2r}. \tag{24}$$

Following Lemma 3.3, let

$$\begin{aligned}\overline{w}(t) &= Kw(t) = r^{-1}(K - hK\mathscr{L})w(t) + hK\mathscr{L}z(t)\\ &= r^{-1}(K - hDK)w(t) + hDKz(t)\end{aligned}$$

and decompose $\overline{w}(t) = \begin{bmatrix}\overline{w}_1(t) & \overline{w}_2^T(t)\end{bmatrix}^T$ with a scalar $\overline{w}_1(t)$. Considering that $X(t+1) = X(t) - h\mathscr{L}Y(t)$, we have $p^T X(t+1) = p^T X(t), t = 0, 1, \ldots$, which leads to

$$p^T c(t) = p^T X(t) - p^T \left(p^T X(0)\right)\mathbf{1} = p^T X(t) - p^T X(0) = 0$$

and

$$\overline{w}_1(t) = p^T w(t) = 0, \quad t = 0, 1, \ldots.$$

On the other hand,

$$\begin{aligned}\overline{w}_2(t+1) =& J_2(h,r)\overline{w}_2(t) + r^{-1}hQ\mathscr{L}z(t)\\ =& [J_2(h,r)]^{t+1}\,\overline{w}_2(0) + r^{-1}h\,[J_2(h,r)]^t\,Q\mathscr{L}z(0)\\ &+ r^{-1}h\sum_{i=0}^{t-1}[J_2(h,r)]^i\,Q\mathscr{L}z(t-i).\end{aligned}$$

Thus, when $t = k+1$, noting that $w(t) = R\overline{w}_2(t)$ and $\overline{w}_2(t) = Qw(t)$, we have

$$\begin{aligned}w(k+1) =& R\,[J_2(h,r)]^{k+1}\,Qw(0) + r^{-1}hR\,[J_2(h,r)]^k\,Q\mathscr{L}z(0)\\ &+ r^{-1}hR\sum_{i=0}^{k-1}[J_2(h,r)]^i\,Q\mathscr{L}z(k-i).\end{aligned} \tag{25}$$

Now we estimate the three items on the right hand side of (25), separately. For the first item, we have

$$\begin{aligned}&\left\|R\,[J_2(h,r)]^{k+1}\,Qw(0)\right\|_\infty\\ \le& \|R\|_\infty\left\|[J_2(h,r)]^{k+1}\right\|_\infty\|Q\|_\infty\|w(0)\|_\infty\\ \le& \frac{\|R\|_\infty\|Q\|_\infty C_c}{g_0}\left(\frac{\rho_h}{r}\right)^{k+1}\\ \le& \frac{\|R\|_\infty\|Q\|_\infty C_c}{g_0}\left(\frac{\rho_h}{r}\right)^{k}.\end{aligned} \tag{26}$$

For the second item, we have

$$\begin{aligned}&\left\|r^{-1}hR\,[J_2(h,r)]^k\,Q\mathscr{L}z(0)\right\|_\infty\\ \le& r^{-1}h\,\|R\|_\infty\|Q\|_\infty\left\|[J_2(h,r)]^k\right\|_\infty\|\mathscr{L}\|_\infty\|z(0)\|_\infty\\ \le& 2r^{-1}hd\,\|R\|_\infty\|Q\|_\infty\frac{C_x}{g_0}\left(\frac{\rho_h}{r}\right)^k\\ =& \frac{2hd\,\|R\|_\infty\|Q\|_\infty C_x}{g_0 r}\left(\frac{\rho_h}{r}\right)^k.\end{aligned} \tag{27}$$

Similarly, for the last item, by (24) we have

$$\begin{aligned}&\left\|r^{-1}hR\sum_{i=0}^{k-1}[J_2(h,r)]^i\,Q\mathscr{L}z(k-i)\right\|_\infty\\ \le& r^{-1}h2d\frac{1}{2r}\|R\|_\infty\sum_{i=0}^{k-1}\left\|[J_2(h,r)]^i\right\|_\infty\|Q\|_\infty\\ \le& \frac{hd}{r^2}\|R\|_\infty\|Q\|_\infty\frac{1-\left(\frac{\rho_h}{r}\right)^k}{1-\frac{\rho_h}{r}}\\ =& \frac{hd\,\|R\|_\infty\|Q\|_\infty}{r(r-\rho_h)}\left[1-\left(\frac{\rho_h}{r}\right)^k\right].\end{aligned} \tag{28}$$

Then by $r \in (\rho_h, 1)$, (18) and (26)-(28), we get

$$\|w(k+1)\|_\infty \le \frac{hd\,\|R\|_\infty\|Q\|_\infty}{r(r-\rho_h)}, \tag{29}$$

which together with (16), (17) and (24) gives

$$\begin{aligned}&\|(I+h\mathscr{L})z(k+1) - h\mathscr{L}w(k+1)\|_\infty\\ \le& \|I+h\mathscr{L}\|_\infty\|z(k+1)\|_\infty + h\|\mathscr{L}\|_\infty\|w(k+1)\|_\infty\\ \le& \frac{1+2hd}{2r} + 2hd\frac{hd\,\|R\|_\infty\|Q\|_\infty}{r(r-\rho_h)}\\ =& \frac{1+2hd}{2r} + \frac{2h^2d^2\,\|R\|_\infty\|Q\|_\infty}{r(r-\rho_h)}\\ =& M_1(h,r) < F_1(h,r) + \frac{1}{2}\\ \le& F + \frac{1}{2},\end{aligned} \tag{30}$$

i.e., the quantizer is unsaturated when $t = k+1$. Thus, by induction, the applied $(2F+1)$-level uniform quantizer will never be saturated. By the definition of $w(t)$ and (29), (19) holds.

From $c(k+1) = g_0 r^{k+1} w(k+1)$ and (25), similar to (26)-(28), we have

$$\begin{aligned}\frac{\|c(k+1)\|_\infty}{\|c(0)\|_\infty} \le& \|R\|_\infty\|Q\|_\infty\rho_h^{k+1}\\ &+ \frac{2hd\,\|R\|_\infty\|Q\|_\infty\rho_h^k C_x}{\|c(0)\|_\infty}\\ &+ \frac{g_0 hd\,\|R\|_\infty\|Q\|_\infty}{(r-\rho_h)\|c(0)\|_\infty}\left(r^k - \rho_h^k\right).\end{aligned}$$

This together with $r \in (\rho_h, 1)$ gives

$$\begin{aligned}&\lim_{k\to\infty}\left(\frac{\|c(k+1)\|_\infty}{\|c(0)\|_\infty}\right)^{\frac{1}{k+1}}\\ =& \lim_{k\to\infty}\exp\left\{\frac{1}{k+1}\ln\left(\frac{\|c(k+1)\|_\infty}{\|c(0)\|_\infty}\right)\right\}\\ \le& \lim_{k\to\infty}\exp\left\{\frac{1}{k+1}\left[\ln\left(\frac{g_0 hd\,\|R\|_\infty\|Q\|_\infty}{(r-\rho_h)\|c(0)\|_\infty}r^k\right) + o(1)\right]\right\}\\ =& \lim_{k\to\infty}\exp\left\{\frac{1}{k+1}\left[k\ln r + o(1)\right]\right\} = r, \quad \forall c(0) \ne 0,\end{aligned} \tag{31}$$

i.e., $r_{asym} \le r$. ■

Remark 3.1: In [4], the asymptotic convergence factor was defined as

$$r_{asym} = \sup_{X(0)\ne\left(p^T X(0)\right)\mathbf{1}}\lim_{t\to\infty}\left(\frac{\left\|X(t)-\left(p^T X(0)\right)\mathbf{1}\right\|_2}{\left\|X(0)-\left(p^T X(0)\right)\mathbf{1}\right\|_2}\right)^{1/t}.$$

In this paper, we adopt $\|\cdot\|_\infty$ vector norm rather than $\|\cdot\|_2$ vector norm, because for finite-dimensional real or complex vector spaces, all vector norms are equivalent. Intuitively speaking, we consider the maximal consensus error at each time step, neglecting the number of agents.

If the Laplacian $\mathscr{L}$ is not diagonalizable, similar to the proof of Theorem 3.1, we have the following results.

Theorem 3.2: If Assumptions A1), A2) and A4) hold, for any given $h \in (0,1/d)$ and $r \in (\rho_h, 1)$, let

$$M_2(h,r) = \frac{1+2hd}{2r} + \frac{2h^2d^2\|R\|_\infty\|Q\|_\infty}{r(r-\rho_h)}C_J(h,r), \quad (32)$$

$$F_2(h,r) = \lfloor M_2(h,r) - 1/2 \rfloor + 1, \quad (33)$$

where $C_J(h,r)$ is defined in Lemma 3.3. For any given $F \geq F_2(h,r)$, let

$$g_0' > \max\left\{ \frac{(C_c r + 2hC_x d)(r-\rho_h)}{hd}, \frac{C_x}{F+1/2} \right\}. \quad (34)$$

Then under the protocol given by (3), (4) and (5) with the $(2F+1)$-level uniform quantizer (2) and the scaling function $g(t) = g_0' r^t$, the closed loop system (8) satisfies (19) with $r_{asym} \leq r$.

C. Any Rate Consensus

For any given $h \in (0,1/d)$, ρ_h is the convergence rate of the system (1) in the case of perfect communication in [4]. In Theorem 3.1, the number of quantization levels $F(h,r) \to \infty$ as $r \to \rho_h$. Fortunately, the information consensus can be achieved with limited communication data rate at the cost of slower convergence.

If Assumptions A1), A2) and A3) hold, for any given $F \in \mathbb{N}^*$, let

$$\Omega_F = \left\{ (\alpha,\beta) \,|\, \alpha \in (0,1/d), \beta \in (\rho_\alpha, 1), M_1(\alpha,\beta) < F + \frac{1}{2} \right\}, \quad (35)$$

where ρ_α is defined by (9) and $M_1(\alpha,\beta)$ is defined by (16). Obviously, for any $(h,r) \in \Omega_F$, under the protocol given by (3), (4) and (5) with the $(2F+1)$-level uniform quantizer (2), the closed loop system (8) satisfying (19).

Theorem 3.3: Suppose Assumptions A1)-A3) hold. For any given $F \in \mathbb{N}^*$, Ω_F is nonempty.

Proof: By the definition of ρ_α and Lemma 3.2, $\rho_\alpha \to 1$ and $\frac{\alpha^2}{1-\rho_\alpha} \to 0$ as $\alpha \to 0$, which together with $\beta \in (\rho_\alpha, 1)$ gives that

$$\lim_{\alpha\to 0} M_1(\alpha,\beta) = \lim_{\alpha\to 0}\left(\frac{1+2\alpha d}{2} + \frac{2\alpha^2 d^2 \|R\|_\infty \|Q\|_\infty}{1-\rho_\alpha} \right) = \frac{1}{2}.$$

Thus, for any given $F \in \mathbb{N}^*$, there exist $\alpha \in (0,1/d)$ and $\beta^* \in (\rho_{\alpha^*}, 1)$ such that

$$M_1(\alpha^*, \beta^*) < F + \frac{1}{2}.$$

■

The following algorithm generates h and r which are fitting for any number of quantization levels.

Theorem 3.4: For any given $F \in \mathbb{N}^*$, choose a constant $h^* \in (0,1/d)$ satisfying

$$2d^2 \|R\|_\infty \|Q\|_\infty (h^*)^2 + (1-\rho_{h^*})dh^* - (1-\rho_{h^*})F < 0. \quad (36)$$

(ii) Choose a constant $r^* \in (r_F, 1)$, where r_F denotes the larger solution of the following second-order equation

$$\begin{aligned} &(2F+1)x^2 - [(1+2h^*d) + \rho_{h^*}(2F+1)]x + \\ &\left[\rho_{h^*}(1+2h^*d) - 4d^2\|R\|_\infty \|Q\|_\infty (h^*)^2\right] = 0. \end{aligned} \quad (37)$$

Then we have

$$M_1(h^*, r^*) \in \Omega_F(h,r). \quad (38)$$

Proof: From (10) and the definition of ρ_{h^*}, we can easily get h^* satisfying (36) and $0 < h^* < 1/d$. Consider now the following quadratic function

$$\begin{aligned} f(r) =& (2F+1)r^2 - [(1+2h^*d) + \rho_{h^*}(2F+1)]r \\ &+ \left[\rho_{h^*}(1+2h^*d) - 4d^2\|R\|_\infty \|Q\|_\infty (h^*)^2\right]. \end{aligned}$$

Note that

$$f(\rho_{h^*}) = -4d^2\|R\|_\infty \|Q\|_\infty (h^*)^2 < 0 \quad (39)$$

and

$$-\frac{1}{2}f(1) = 2d^2\|R\|_\infty \|Q\|_\infty (h^*)^2 + (1-\rho_{h^*})dh^* - (1-\rho_{h^*})F.$$

This together with (36) gives

$$f(1) > 0. \quad (40)$$

By (39), (40) and the definition of r_F, for any given $r^* \in (r_F, 1) \subset (\rho_{h^*}, 1)$, we have $f(r^*) > 0$, which leads to

$$\frac{1+2h^*d}{2r^*} + \frac{2(h^*)^2 d^2 \|R\|_\infty \|Q\|_\infty}{r^*(r^*-\rho_{h^*})} < F + \frac{1}{2},$$

i.e., (38) holds. ■

IV. SIMULATION RESULTS

This section provides two simulation examples to illustrate the proposed protocol and show its effectiveness. Consider now a random geometric graph generated by choosing $N = 20$ points at random in the unit square, and then placing a directed edge between each pair of points at distance less than 0.3. The entries of $\mathscr{A}$ are randomly chosen from 1, 2 and 3. In the case we have that $d = 12$ and $\|R\|_\infty \|Q\|_\infty = 60.342$. where $\max_{1\leq i\leq N}|\lambda_i| = 11.552$. In Fig. 2, the parameters h and r are 0.08 and 0.751, respectively. It is shown that the consensus is achieved with a 2-level uniform quantizer within 25 steps. Notice that $r = 0.751$ is relatively near by $\rho_h = 0.750$, which is the convergence rate under perfect communication channel. In Fig. 3, a larger $r = 0.9$ is used and one bit consensus comes true at the cost of slower convergence.

V. CONCLUSIONS

Taking into account the general case of directed information exchange, this paper has studied the quantized consensus problem for directed networks of discrete-time first-order agents under a bounded number of bits communication. Based on dynamic difference encoding and decoding, the proposed algorithm achieves quantized consensus asymptotically with as few as only one bit information exchange at each time step. The upper bound of convergence rate is defined as the maximal consensus error and depends on the number of quantization levels and the topology of networks. It is shown that faster convergence requires more bits of information to exchange with both theoretical and experimental results.

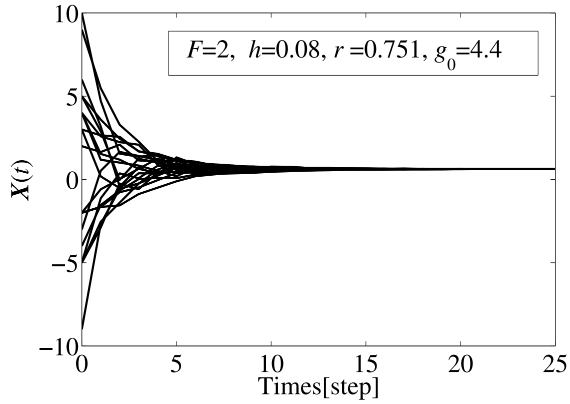

Fig. 2. Curves of states with $F = 2$ and $r = 0.751$

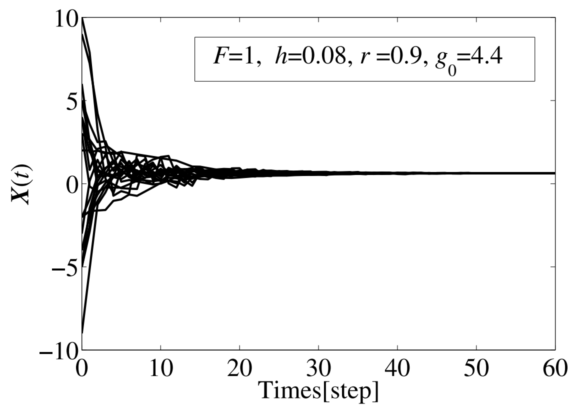

Fig. 3. one bit consensus with $r = 0.9$

VI. ACKNOWLEDGMENTS

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