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Finite time control design for contPNs according to piecewise constant control actions

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***Abstract*—Advanced control design for continuous Petri nets requires the characterization of positive invariant and attractive regions in marking space. Based on such characterizations, piecewise constant control design is investigated. Sufficient conditions for control design in finite time are proposed. An algorithm for piecewise constant control design in minimal time is proposed as a consequence.**

I. INTRODUCTION

DEPENDABLE control for manufacturing systems leads to advanced methods and models. In this context, timed continuous Petri nets (contPNs) can be used in order to provide accurate representations of the behaviors of discrete event systems and to take benefit from the main advances in continuous systems control theory [6, 13]. Controllability and control actions for contPNs have been considered by adapting linear control theory to contPNs in order to take into account their specific properties [10, 19]. In particular the markings and flows are positive and mainly often bounded and contPNs are piecewise affine models [11]. Controllability domains have been defined for contPNs and conditions so that a contPN is controllable with bounded inputs have been provided [9, 15]. In addition, stationary markings resulting from a specific control action have been also characterized [14]. Then, optimal controls by mean of linear programming problems have been investigated. The basic idea is to maximize various linear cost functions that depend on control actions, initial marking, steady state and net parameters under linear constraints that specify the contPNs properties. Affine control laws [2], model predictive control [7] and piecewise-linear marking trajectories in minimal time [1] have been designed as a consequence.

Based on the characterization of positive invariant and attractive regions within finite time intervals, piecewise control design is investigated. Sufficient conditions to be checked so that a set of piecewise constant control actions moves the marking from a given initial value to the neighborhood of a desired value are proposed. An algorithm for piecewise constant control design in minimal time is proposed as a consequence.

II. CONPNS WITH CONTROL ACTIONS

A. Petri nets

A Petri net (PN) is defined as $<\boldsymbol{P}, \boldsymbol{T}, W_{PR}, W_{PO}>$ where $\boldsymbol{P} = \{P_i\}$ is a set of n places and $\boldsymbol{T} = \{T_j\}$ is a set of q transitions and $W = W_{PO} - W_{PR} \in (\mathbf{Z})^{n\times q}$ is the incidence matrix. $<PN, M_I>$ is a markied PN, M_I is the PN initial marking, $M = (m_i) \in (\mathbf{Z}^+)^n$ denotes the current marking vector and $R(PN, M_I)$ is the PN reachability set (i.e. the set of markings that are reachable from initial marking M_I) [6]. A Petri net is b-bounded if the marking m_i of each place P_i is bounded by b (i.e. for all $M \in R(W_{PO}, W_{PR}, M_I)$ and for all $P_i \in \boldsymbol{P}$, $m_i \leq b$). Each transition T_j fires according to the integer part of its enabling degree $enab_j(M)$:

$$enab_j(M) = \min \{m_k / w^{PR}_{kj} : P_k \in {}^\circ T_j\} \quad (1)$$

where ${}^\circ \boldsymbol{T_j}$ stands for the set of T_j upstream places. A firing sequence $\boldsymbol{\sigma}$ is defined as an ordered series of transitions that successively fire from initial marking M_I to marking M (i.e. $M_I\ [\boldsymbol{\sigma} > M)$. Such a sequence may be represented by the PN firing count vector $\sigma = (\sigma_j) \in (\mathbf{Z}^+)^q$, where σ_j is the number of T_j firings. PNs may have P-semiflows. A P-semiflow $y \in (\mathbf{Z}^+)^n$ is a non-zero solution of equation $y^T.W = 0$. Let us define $Y \in (\mathbf{Z}^+)^{n \times hp} = (y_1 \mid ...\mid y_{hp})$ as the matrix obtained according to all minimal P-semiflows y_i, $i = 1,...,hp$. Y is of rank hp and satisfies equation $Y^T.M = C$ with $C = Y^T.M_I$.

B. Controlled timed continuous Petri nets

Timed continuous PNs under infinite server semantic (contPNs) have been developed in order to provide continuous approximations of the discrete behaviors of PNs [6, 13]. The marking of each place is a continuous non negative real valued function of time and $M(t, M_I) \in (\boldsymbol{R}^+)^n$, $t \geq 0$ is the continuous marking trajectory that starts with M_I at $t = 0$. $X_{max} = \text{diag}(x_{max\,j}) \in (\boldsymbol{R}^+)^{q\times q}$ is the diagonal matrix of maximal firing speeds $x_{max\,j}$, $j = 1,\ldots q$ and $X(t, M_I) = (x_j(t, M_I)) \in (\boldsymbol{R}^+)^q$ is the firing speeds vector at time t in free regime that depends continuously on the marking of the places. The flow through the transition T_j is defined by (2):

$$x_j(t, M_I) = x_{max\,j}.enab_j(M(t, M_I)\ ) \quad (2)$$

Control actions may be introduced according to a reduction of the flow through the transitions [9]. Transitions in which a control action can be applied are called controllable and $\boldsymbol{T_c}$ is defined as the set of controllable transitions. Similarly $\boldsymbol{T_{nc}}$ is the set of uncontrollable transitions. For a contPN with q_c controllable transitions, let us assume, without any loss of generality that $\boldsymbol{T_c} = \{T_1, \ldots, T_{qc}\}$ ($\boldsymbol{T_c}$ can eventually be empty) and $\boldsymbol{T_{nc}} = \{T_{qc+1}, \ldots, T_q\}$ and define $Q_c = (I_{qc} \mid 0_{qc \times (q-qc)}) \in (\mathbf{Z}^+)^{qc \times q}$ and $Q_{nc} = (0_{(q-qc) \times qc} \mid I_{q-qc}) \in (\mathbf{Z}^+)^{(q-qc) \times q}$. The control actions are summed up in

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control vector $U(t) = (u_j(t)) \in (\boldsymbol{R}^+)^q$. As a consequence, the marking variation of contPNs is given by (3):

$$dM(t, M_I) / dt = W.(X(t, M_I)-U(t)) \qquad (3)$$

with $0 \leq Q_c.U(t) \leq Q_c.X(t, M_I)$ and $Q_{nc}.U(t) = 0$.

A control sequence $U(t)$, $t \geq 0$ that satisfies the preceding conditions for a given marking trajectory is named an bounded input control and the set BIC($\boldsymbol{T_c}$, M_I) of admissible control sequences $\{U(t),\ t \geq 0\}$ for initial marking M_I is defined as a consequence:

$$\text{BIC}(\boldsymbol{T_c}, M_I) = \{\{U(t), t \geq 0\}, \text{ such that } 0 \leq Q_c.U(t) \leq Q_c.X(t, M_I) \text{ and } Q_{nc}.U(t) = 0\} \qquad (4)$$

C. Regions for contPNs

For a contPN with P-semiflows represented by matrix Y, any reachable marking $M(t, M_I) \in (\boldsymbol{R}^+)^n$ satisfies $Y^T.M(t, M_I) = C$. So, linear dependencies between marking variables appear and the reachable set of contPNs is defined as a consequence. The reachable set, $R(contPN, M_I) \subset (\boldsymbol{R}^+)^n$, of a marked contPN, $< PN, X_{max}, M_I >$, is defined as the set of all reachable markings $M(t, M_I)$, $t \geq 0$, from a given initial marking M_I and for all matrices $X_{max} \in (\boldsymbol{R}^+)^{q \times q}$ of maximal firing speeds.

Switches occur in contPNs according to the function "min(.)" in the expression of the enabling degree (1). Let us define the critical place(s) for transition T_j at time t as the place(s) P_i such that $i = \text{argmin } \{m_k(t, M_I) / w^{PR}_{kj}, P_k \in {}^\circ T_j\}$. $R(contPN, M_I)$ can be partitioned in K reachable regions (r-regions) with $K \leq \Pi\{|{}^\circ T_j|, j = 1,...,q\}$: $R(contPN, M_I) = \mathbf{A_1} \cup \ldots \cup \mathbf{A_k}$. In the interior of any r-region $\mathbf{A_k}$, each transition has a single critical place. PN configurations [14] are used to define the r-regions. A configuration is a cover of $\boldsymbol{T}$ by its input arcs and assigns to each transition a single input place: $config(k) = \{(P_{i(k,j)}, T_j), j = 1,...q\}$, $k = 1,...,\Pi\{|{}^\circ T_j|, j = 1,...,q\}$, where $P_{i(k,j)} \in {}^\circ T_j$ is the single input place of transition T_j in configuration k ($i(k, j)$ stands for the index of the concerned place). R-regions of marked contPNs are defined as:

Definition: The reachable region (r-region) $\mathbf{A_k} \subset R(contPN, M_I)$, $k = 1,..., K$ of a marked contPN, $< PN, X_{max}, M_I >$, is defined for a given configuration $config(k)$, and for all matrices $X_{max} \in \boldsymbol{R}^+)^{qxq}$ as the set of all reachable markings $M(t, M_I)$, $t \geq 0$, that satisfy (1) $Y^T.M(t, M_I) = C$, (2) $\forall\ T_j \in \boldsymbol{T}$, $P_{i(k,j)}$ is the critical place of transition T_j for marking $M(t, M_I)$.

Each r-region $\mathbf{A_k}$ is characterized by a constraint matrix $A_k = (a^k_{ij}) \in (\boldsymbol{R}^+)^{q \times n}$, $k = 1,\ldots,K$, $i = 1,..., q$ and $j = 1,..., n$:

- $a^k_{ji(k,j)} = 1/w^{PR}_{i(k,j)j}$ for all $T_j \in T$,
- $a^k_{ji} = 0$ otherwise.

The constraint matrices A_k lead to a linear matrix inequality (LMI) that characterizes the r-regions according to proposition 1.

Proposition 1 [12]: Let us consider a contPN with K r-regions $\mathbf{A_k}$. Each r-region $\mathbf{A_k}$ is a polyhedral set characterized by the LMI $H_k.M \leq h_k$ with:

$$H_k = \begin{pmatrix} -I_n \\ A(k) \\ Y^T \\ -Y^T \end{pmatrix},\ h_k = \begin{pmatrix} 0 \\ 0 \\ C \\ -C \end{pmatrix} \text{ and } A(k) = \begin{pmatrix} A_k - A_1 \\ \cdots \\ A_k - A_{k-1} \\ A_k - A_{k+1} \\ \cdots \\ A_k - A_K \end{pmatrix}$$

and I_n is the identity matrix of size n.

The LMI $H_k.M \leq h_k$ is composed of $n + q(K\text{-}1) + 2hp$ inequalities to be satisfied by n variables. Let us notice that the set of inequalities $A(k) \leq 0$ include numerous trivial inequalities (i.e. $0 \leq 0$) and also several identical inequalities that can be removed for rapidity.

D. Piecewise constant control actions

The marking variation (3) of contPNs can be rewritten in each r-region $\mathbf{A_k}$. For all $M(t, M_I) \in \mathbf{A_k}$, equation (5) holds:

$$dM(t,M_I)/dt = W.(X_{max}.A_k.M(t,M_I)-U(t)) \qquad (5)$$

with $0 \leq Q_c.U(t) \leq Q_c.X_{max}.A_k.M(t, M_I)$ and $Q_{nc}.U(t) = 0$.

If the control actions are constant in r-region $\mathbf{A_k}$ (i.e. $U(t) = U_k$ for all $M(t, M_I) \in \mathbf{A_k}$), contPNs are piecewise-affine hybrid systems and for all $M(t, M_I) \in \mathbf{A_k}$, equation (5) leads to (6):

$$dM(t,M_I)/dt = W.X_{max}.A_k.M(t,M_I) - W.U_k \qquad (6)$$

with $0 \leq Q_c.U_k \leq Q_c.X_{max}.A_k.M(t, M_I)$ and $Q_{nc}.U_k = 0$.

A constant control vector U_k that satisfies the preceding conditions for a given marking trajectory included in $\mathbf{A_k}$ is named an admissible constant bounded input control and the sets CBIC($\mathbf{A_k}$, $\boldsymbol{T_c}$, M_I) of admissible constant control vectors for initial marking $M_I \in \mathbf{A_k}$ and CBIC($\mathbf{A_k}$, $\boldsymbol{T_c}$) of admissible constant control vectors in $\mathbf{A_k}$ for all initial marking $M_I \in \mathbf{A_k}$ are defined as a consequence:

$\text{CBIC}(\mathbf{A_k}, \boldsymbol{T_c}, M_I) = \{U_k \in (\mathbf{R}^+)^q$ such that $0 \leq Q_c.U_k \leq Q_c.X_{max}.A_k.M(t, M_I)$, $t \geq 0$ and $Q_{nc}.U_k = 0\}$

$\text{CBIC}(\mathbf{A_k}, \boldsymbol{T_c}) = \cup\ \{\text{CBIC}(\mathbf{A_k}, \boldsymbol{T_c}, M_I) : M_I \in \mathbf{A_k}\}$

For numerical issues, discrete time approximations of the continuous trajectories of contPNs will be used. First order approximations of equation (6) in discrete time with sampling period Δt result from direct or retrograde schemes. If the control actions are constant in r-region $\mathbf{A_k}$ and $U_k \in \text{CBIC}(\mathbf{A_k}, \boldsymbol{T_c}, M_I)$, the direct schema is given by (7):

$$M(t, M_I) = (A_{Dk})^t.M_I - \Sigma_k(t).U_k \qquad (7)$$

with $A_{Dk} = W.X_{max}.A_k.\Delta t.+I_n$. and $\Sigma_k(t) = \left(\sum_{j=1}^{t} (A_{Dk})^{j-1}\right).W.\Delta t$.

III. Positive invariance and Attraction in finite time

Positive invariance and attraction of polyhedral regions for linear and non-linear systems have been investigated according to the computation of LMIs and linear programming problems (LPPs) [3, 4, 5].

A. Positive invariance in finite time

Let us consider any region $\mathbf{A} \subseteq \mathbf{A_k}$. $\mathbf{A}$ is positive invariant for the contPN (3) in free of forced regime if and only if $M_I \in \mathbf{A}$ implies $M(t, M_I) \in \mathbf{A}$, for all $t \geq 0$. From a practical point of view, τ - positive invariance is introduced.

Definition: A region $\mathbf{A} \subseteq \mathbf{A_k}$ is τ -positive invariant for the contPN (3) in free of forced regime if and only if $M_I \in \mathbf{A}$ implies $M(t, M_I) \in \mathbf{A}$, for all $t = 0,..., \tau$.

LMIs can be used to check the τ - positive invariance of a given region A $\subseteq \mathbf{A_k}$.

Proposition 2: Let us consider a contPN with $\boldsymbol{T_c} = \{T_1, \ldots, T_{qc}\}$ and $\boldsymbol{T_{nc}} = \{T_{qc+1}, \ldots, T_q\}$ and K r-regions $\mathbf{A_k}$. Let us also consider an arbitrary region $\mathbf{A} \subseteq \mathbf{A_i}$ defined by $H.M \leq h$. The region $\mathbf{A}$ is τ - positive invariant for initial marking $M_I \in \mathbf{A}$ and system (7) under constant control actions U_i if and only M_I and U_i satisfy LMI (8):

$$\begin{pmatrix} H.A_{Di} & -H.\Sigma_i(1) \\ \vdots & \vdots \\ H.(A_{Di})^{\tau} & -H.\Sigma_i(\tau) \\ -Q_c.X_{max}.A_i & Q_c \\ \vdots & \vdots \\ -Q_c.X_{max}.A_i.(A_{Di})^{\tau-1} & Q_c.\left(I_q+X_{max}.A_i.\Sigma_i(\tau-1)\right) \\ 0 & -I_q \\ 0 & Q_{nc} \end{pmatrix} . \begin{pmatrix} M_I \\ U_i \end{pmatrix} \leq \begin{pmatrix} h \\ \vdots \\ h \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

Proof: the proof is a direct consequence of r-regions and polyhedral sets characterization (proposition 1), CBIC and τ - positive invariance definitions. The first group of inequalities (rows 1 to 3 of (8)) ensures that $M(t, M_I) \in \mathbf{A}$, $0 \leq t < \tau$. The second group of inequalities (rows 4 to 8 of (8)) ensures that the constant control actions are admissible CBIC according to $\boldsymbol{T_c}$ and $\boldsymbol{T_{nc}}$.

Let us notice that proposition 2 can also be used to work out the set of CBIC control actions and also the set of initial markings such that the region $\mathbf{A}$ is τ - positive invariant. LPPs are used for that purpose.

B. Attraction in finite time

The region $\mathbf{A} \subset \mathrm{R}^n$ is attractive for the contPN (3) with initial marking M_I if and only if $\exists\, \Delta > 0$, $\forall\, \varepsilon > 0$, there is $\tau > 0$ such that $\inf\{\| M_I - M^* \| : M^* \in \mathbf{A}\} < \Delta$ implies $\inf\{\| M(t, M_I) - M^* \| : M^* \in \mathbf{A}\} < \varepsilon$, $t > \tau$. For engineering needs, it is more important to ensure attraction with finite attraction time τ [8]. A region $\mathbf{A} \subset \mathrm{R}^n$ is attractive with finite attraction time τ for the contPN (3) if and only if $\exists\, \Delta > 0$, $\forall\, M_I$ such that $\inf\{\| M_I - M^* \| : M^* \in \mathbf{A}\} < \Delta$ implies $\inf\{\| M(t, M_I) - M^* \| : M^* \in \mathbf{A}\} = 0$, $t \geq \tau$. The time τ is called the finite attraction time. For practical issues (τ, τ') – attractivity is introduced.

Definition: The region $\mathbf{A} \subset (\boldsymbol{R}^+)^n$ is (τ, τ') - attractive for the contPN (3) if and only if $\exists\, \Delta > 0$, $\forall\, M_I$ such that $\inf\{\| M_I - M^* \| : M^* \in \mathbf{A}\} < \Delta$ implies $\inf\{\| M(t, M_I) - M^* \| : M^* \in \mathbf{A}\} = 0$ for $\tau \leq t < \tau + \tau' + 1$.

LMIs can be used to check the (τ, τ') – attractivity of a given region A $\subseteq \mathbf{A_k}$.

Proposition 3: Let us consider a contPN with $\boldsymbol{T_c} = \{T_1,\ldots, T_{qc}\}$, $\boldsymbol{T_{nc}} = \{T_{qc+1}, \ldots, T_q\}$ and K r-regions $\mathbf{A_k}$. Let us also consider an arbitrary region $\mathbf{A} \subseteq \mathbf{A_i}$ defined by $H.M \leq h$. The region $\mathbf{A}$ is (τ, τ') - attractive for system (7) under constant control actions U_i if there exists a non empty set of initial markings $M_I \in \mathbf{A_i}$ such that M_I and U_i satisfy the LMI (9):

$$\begin{pmatrix} H_i & -H_i.\Sigma_i(0) \\ \vdots & \vdots \\ H_i.(A_{Di})^{\tau-1} & -H_i.\Sigma_i(\tau-1) \\ H.(A_{Di})^{\tau} & -H.\Sigma_i(\tau) \\ \vdots & \vdots \\ H.(A_{Di})^{k+1} & -H.\Sigma_i(k+1) \\ -Q_c.X_{max}.A_i & Q_c.\left(I_q+X_{max}.A_i.\Sigma_i(0)\right) \\ \vdots & \vdots \\ -Q_c.X_{max}.A_i.(A_{Di})^{k} & Q_c.\left(I_q+X_{max}.A_i.\Sigma_i(k)\right) \\ 0 & -I_q \\ 0 & Q_{nc} \end{pmatrix} . \begin{pmatrix} M_I \\ U_i \end{pmatrix} \leq \begin{pmatrix} h_i \\ \vdots \\ h_i \\ h \\ \vdots \\ h \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

with $k = \tau + \tau' - 1$.

Proof: the proof is a direct consequence of the definition of (τ, τ') – attractivity. The first group of inequalities (rows 1 to 3 of (9)) ensures that $M(t, M_I) \in \mathbf{A_i}$, $0 \leq t < \tau$. The second group of inequalities (rows 4 to 6 of (9)) ensures that $M(t, M_I) \in \mathbf{A}$, $\tau \leq t < \tau + \tau' + 1$. The last group of inequalities (rows 7 to 11 of (9)) ensures that the constant control actions are admissible CBIC according to $\boldsymbol{T_c}$ and $\boldsymbol{T_{nc}}$. Proposition 3 holds as a consequence.

Let us notice that proposition 3 can also be considered as a LPPs in order to work out the set of CBIC control actions such that the region $\mathbf{A}$ is (τ, τ') - attractive.

C. Control for r-regions switching in finite time

Proposition 3 and LMIs can be used to provide conditions on control actions such that the marking switches from r-region $\mathbf{A_i}$ to r-region $\mathbf{A_j}$ in a specific number τ of steps. Such a characterization is equivalent to $(\tau, 0)$ – attractivity of region $\mathbf{A_j}$.

Proposition 4: Let us consider a contPN with $\boldsymbol{T_c} = \{T_1,\ldots, T_{qc}\}$, $\boldsymbol{T_{nc}} = \{T_{qc+1}, \ldots, T_q\}$ and K r-regions $\mathbf{A_k}$. The region $\mathbf{A_j}$ is $(\tau, 0)$ - attractive for the system (7) from initial markings $M_I \in \mathbf{A_i}$ and for control actions U_i if and only if M_I and U_i satisfy the LMI (10):

$$\begin{pmatrix} H_i.A_{Di} & -H_i.\Sigma_i(1) \\ \vdots & \vdots \\ H_i.(A_{Di})^{\tau-1} & -H_i.\Sigma_i(\tau-1) \\ H_j.(A_{Di})^{\tau} & -H_j.\Sigma_i(\tau) \\ -Q_c.X_{max}.A_i & Q_c.\left(I_q+X_{max}.A_i.\Sigma_i(0)\right) \\ \vdots & \vdots \\ -Q_c.X_{max}.A_i.(A_{Di})^{\tau-1} & Q_c.\left(I_q+X_{max}.A_i.\Sigma_i(\tau-1)\right) \\ 0 & -I_q \\ 0 & Q_{nc} \end{pmatrix}.\begin{pmatrix} M_I \\ U_i \end{pmatrix} \le \begin{pmatrix} h_i \\ \vdots \\ h_i \\ h_j \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (10)$$

Proof: the proof is also a consequence of the definition of (τ, τ') – attractivity. The first group of inequalities (rows 1 to 3 of (10)) ensures that $M(t, M_I) \in \mathbf{A_i}$, $1 \le t < \tau$. The second group of inequalities (row 4 of (10)) ensures that $M(\tau, M_I) \in \mathbf{A_j}$, (i.e. the marking switches from region $\mathbf{A_i}$ to $\mathbf{A_j}$). The last group of inequalities (rows 6 to 10 of (10)) ensures that the constant control actions are admissible CBIC according to $\boldsymbol{T_c}$ and $\boldsymbol{T_{nc}}$. Proposition 4 holds as a consequence.

Let us notice that proposition 4 can also be considered as a LPPs in order to work out the set of CBIC control actions and the set of initial markings such that the region $\mathbf{A_j}$ is (τ, 0) - attractive.

IV. Piecewise Constant Controls Actions for Finite Time Control Design

The results of the section III can be used in order to design piecewise constant control actions suitable to drive the marking vector from an initial marking M_I to the neighborhood of a desired marking M_d according to a sequence $S = s(1)...s(L)$ of $L \le K$ r-regions to be crossed in finite times $\tau_1, \tau_2, ..., \tau_L$. The i^{th} r-region in sequence S will be referred as $s(i)$. It is assumed that S satisfies:

(1) $M_I \in s(1)$ and $M_d \in s(L)$

(2) two successive r-regions in S have nonempty intersections: $s(i) \cap s(i+1) \neq \emptyset$ for $i = 1,..., L$-1,

(3) each r-region does not appear more than once in S: $s(i) \neq s(j)$ for $i \neq j$.

For any initial marking M_I and desired marking M_d, such sequences S exist as long as the *R(contPN, M_I)* is a convex set that is partitioned according to the r-regions. Without any loss of generality one can label the r-regions according to the sequence S: $S = \mathbf{A_1} ... \mathbf{A_L}$ and $s(i) = \mathbf{A_i}$, i =1,...,L. Let us also consider a polyhedral neighborhood $\mathbf{A} \subset \mathbf{A_L}$ of the desired marking M_d (i.e. $M_d \in \mathbf{A}$) defined by $H.M \le h$. The piecewise constant control actions that drive the marking vector from M_I to region $\mathbf{A}$ in finite time and according to S can be worked out with proposition 5:

Proposition 5: Let us consider a contPN with $\boldsymbol{T_c} = \{T_1,\dots, T_{qc}\}$, $\boldsymbol{T_{nc}} = \{T_{qc+1}, \dots, T_q\}$, K r-regions $\mathbf{A_k}$, an initial marking M_I and a desired marking M_d. Let us also consider a sequence of r-regions $S = s(1)...s(L) = \mathbf{A_1} ... \mathbf{A_L}$, a set of finite times $\tau = \{\tau_1, \tau_2, ..., \tau_L\}$ and a region $\mathbf{A} \subset \mathbf{A_L}$ defined by $H.M \le h$, such that $M_I \in \mathbf{A_1}$ and $M_d \in \mathbf{A}$. There exists a set of piecewise constant control actions $\{U_i$, $i = 1,\dots L\}$ that drive the marking of the system (7) from M_I to $\mathbf{A}$ in finite time $\tau_1 + \tau_2 + ... + \tau_L$ if U_i, $i = 1,\dots L$, satisfy the LMI (11):

$$\begin{pmatrix} T(1,1) & 0 & \cdots & 0 \\ T(2,1) & T(2,2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ T(L,1) & T(L,2) & \cdots & T(L,L) \\ T(L+1,1) & T(L+1,2) & & T(L+1,L) \end{pmatrix}.\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_L \end{pmatrix} \le \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_L \\ Y_{L+1} \end{pmatrix} \qquad (11)$$

with:

$$T(L+1,j) = H.(A_{Di})^{\tau_i}.\pi(i,j).\Sigma_j(\tau_j), \qquad j < L+1$$
$$T(L+1,L) = -H.\Sigma_i(\tau_i)$$
$$Y_{L+1} = h - H.(A_{Di})^{\tau_i}.\pi(i,0).M_I$$

$$T(i,j) = \begin{pmatrix} -H_i.(A_{Di})^0.\pi(i,j).\Sigma_j(\tau_j) \\ \vdots \\ -H_i.(A_{Di})^{\tau_i-1}.\pi(i,j).\Sigma_j(\tau_j) \\ Q_c.X_{max}.A_i.(A_{Di})^0.\pi(i,j).\Sigma_j(\tau_j) \\ \vdots \\ Q_c.X_{max}.A_i.(A_{Di})^{\tau_i-1}.\pi(i,j).\Sigma_j(\tau_j) \\ 0 \\ 0 \end{pmatrix}, \quad i = 1,\dots,L, \qquad i > j$$

$$Y_i = \begin{pmatrix} h_i - H_i.(A_{Di})^0.\pi(i,0).M_I \\ \vdots \\ h_i - H_i.(A_{Di})^{\tau_i-1}.\pi(i,0).M_I \\ Q_c.X_{max}.A_i.(A_{Di})^0.\pi(i,0).M_I \\ \vdots \\ Q_c.X_{max}.A_i.(A_{Di})^{\tau_i-1}.\pi(i,0).M_I \\ 0 \\ 0 \end{pmatrix}, \quad i = 1,\dots,L$$

$$T(i,i) = \begin{pmatrix} -H_i.\Sigma_i(0) \\ \vdots \\ -H_i.\Sigma_i(\tau_i-1) \\ Q_c+Q_c.X_{max}.A_i.\Sigma_i(0) \\ \vdots \\ Q_c+Q_c.X_{max}.A_i.\Sigma_i(\tau_i-1) \\ -I_q \\ Q_{nc} \end{pmatrix}, \quad \pi(i,j) = \left(\prod_{k=j+1}^{i-1}\left(A_{D(i-k)}\right)^{\tau_{i-k}}\right)$$

Proof: The marking trajectory is detailed in any region $s(i) = \mathbf{A_i}$, i =1,...L, according to the following equation:

$$\begin{aligned} &M(\tau_1+\dots+\tau_{i-1}+t, M_I) \\ &= (A_{Di})^t.\left(\prod_{k=1}^{i-1}\left(A_{D(i-k)}\right)^{\tau_{i-k}}\right).M_I \\ &- (A_{Di})^t.\sum_{j=1}^{i-1}\left(\prod_{k=j+1}^{i-1}\left(A_{D(i-k)}\right)^{\tau_{i-k}}\right).\Sigma_j(\tau_j).U_j - \Sigma_i(t).U_i, \\ &t = 0,\dots,\tau_i - 1 \end{aligned}$$

Then, the proof results directly from the application of the results established for regions switching and (τ, τ') – attractivity.

Remarks:

1. The computation of equation (11) provides all sequences of CBIC control actions that drive the marking vector from M_I to $\mathbf{A}$ according to the sequence S and to the set τ. The solutions are described as a set of L regions in $(\mathbf{R}^+)^q$.
2. Let us notice that the proposition 5 can also be used to work out simultaneously the region of initial marking in

region $s(1)$ = $\mathbf{A_1}$ and the sequences of CBIC control actions that drive the marking vector in $\mathbf{A}$. In that case the LMI (12) is considered:

$$\begin{pmatrix} T(1,0) & T(1,1) & & \cdots & 0 \\ T(2,0) & T(2,1) & & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ T(L,0) & T(L,1) & & \cdots & T(L,L) \\ T(L+1,0) & T(L+1,1) & & \cdots & T(L+1,L) \end{pmatrix} . \begin{pmatrix} M_I \\ U_1 \\ U_2 \\ \vdots \\ U_L \end{pmatrix} \leq \begin{pmatrix} Y'_1 \\ Y'_2 \\ \vdots \\ Y'_L \\ Y'_{L+1} \end{pmatrix} \quad (12)$$

with $T(L+1,0) = H.(A_{Di})^{\tau_i}.\pi(i,0),\ Y_{L+1} = h$, and:

$$T(i,0) = \begin{pmatrix} H_i.(A_{Di})^0.\pi(i,0) \\ \vdots \\ H_i.(A_{Di})^{\tau_i-1}.\pi(i,0) \\ Q_c.X_{max}.A_i.(A_{Di})^0.\pi(i,0) \\ \vdots \\ Q_c.X_{max}.A_i.(A_{Di})^{\tau_i-1}.\pi(i,0) \\ 0 \\ 0 \end{pmatrix}, Y'_i = \begin{pmatrix} h_i \\ \vdots \\ h_i \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \ i = 1, \ldots, L$$

3. The proposition 5 can be combined with an optimization procedure in order to work out the control actions that drive the marking vector from M_I to $\mathbf{A}$ according to the sequence S in minimal time (figure 4). The idea is to start the computation by considering only the first line of equation (11) with $\tau_1 = 1$. Then τ_1 increases as long as no solution is found to reach $\mathbf{A_2}$ from M_I in finite time τ_1. When a non empty region of solution is found, the restriction LMI(2) of equation (11), given by (13) is considered with set $\tau(2) = \{\tau_1, 1\}$, and the computation starts again.

$$LMI(2){:}\ \ \begin{pmatrix} T(1,1) & 0 & \cdots & 0 \\ T(2,1) & T(2,2) & \ddots & \vdots \end{pmatrix} . \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \leq \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \quad (13)$$

The same optimization continues with the restrictions LMI(2), LMI(3),..., LMI(L) and sets of finite times $\tau(2) = \{\tau_1, 1\}$, $\tau(3) = \{\tau_1, \tau_2, 1\}$, …, $\tau(L) = \{\tau_1, \tau_2, \ldots, 1\}$. A failing test $\tau_i \leq \tau_{max\ i}$ can be also considered to limit the computation time. In case the algorithm fails in region $\mathbf{A_i}$ (i.e. $\tau_i > \tau_{max\ i}$), τ_{i-1} is increased and LMI(i-1) is considered again. This algorithm leads to all sequences of piecewise constant control actions that drive the marking vector from M_I to $\mathbf{A}$ according to the sequence S in minimal time.

V. EXAMPLE

Let us consider the marked contPN in figure 1 with 2 P-semiflows $Y = ((0\ 0\ 0\ 1\ 1)^T \mid (1\ 1\ 2\ 1\ 0)^T) \in (\mathbf{Z}^+)^{5 \times 2}$, $C = (4\ 5)^T$, maximal firing rates $\mu = (1, 1, 1, 1)^T$, and $\boldsymbol{T_c} = \{T_1, T_2, T_3, T_4\}$. Three r-regions $\mathbf{A_1}$ to $\mathbf{A_3}$ exist for this contPN.

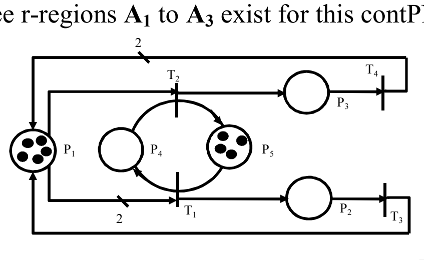

Figure 1: An example of contPN with $M_I = (5\ 0\ 0\ 0\ 4)^T$

Let us also consider X_{max} = diag(μ) with $\mu = (1, 1, 1, 1)^T$, $\Delta t = 0.05$, $M_I = (4\ 0.3\ 0.1\ 0.5\ 3.5)^T \in \mathbf{A_1}$ and $M_d = (2\ 0.5\ 0.75\ 1\ 3)^T \in \mathbf{A_1}$. In that simple case the sequence $S = \mathbf{A_1}$ is considered. Let also consider a neighborhood of M_d according to the LMI:

$$\begin{aligned} I_n.M &\leq (\varepsilon + 1).M_d \\ -I_n.M &\leq (\varepsilon - 1).M_d \end{aligned} \quad (14)$$

with $\varepsilon = 0.25$. The computation of equation (11) for optimal finite time $\tau_1 = 25$ (τ_1 is obtained according to the optimization algorithm described in section III) leads to the region of constant control actions depicted in figure 2. Considering the particular constant action $U_1 = (0.69, 0.19, 0.02, 0.1)^T$, the marking trajectory in figure 3 is obtained. The computation of the LMI given by equation (12) also provides the set of initial markings in r-region $\mathbf{A_1}$ so that there exist CBIC control actions to reach $\mathbf{A}$ in finite time $\tau_1 = 25$ (figure 3, grey color).

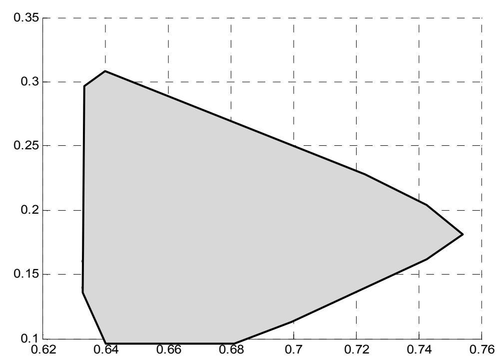

Figure 2: Control design in finite time in a single region: region of CBIC control actions in plan (u_1, u_2) to drive $M(t, M_I)$ from M_I to M_d in $\mathbf{A_1}$.

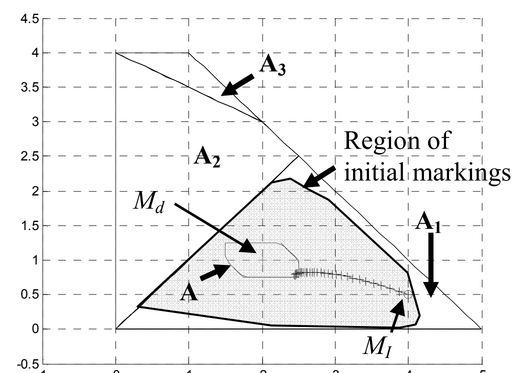

Figure 3: Control design in finite time in a single region: marking trajectory in plan (m_1, m_4) to drive $M(t, M_I)$ from M_I to M_d in $\mathbf{A_1}$.

For the same contPN with $\mu = (5, 1, 8, 8)^T$ and same set $\boldsymbol{T_c}$, let us now consider $M_I = (2\ 1\ 0.5\ 1\ 3)^T \in \mathbf{A_1}$ and $M_d = (1\ 0.1\ 0.1\ 3.7\ 0.3)^T \in \mathbf{A_3}$. In that case the sequence $S = \mathbf{A_1}.\mathbf{A_2}.\mathbf{A_3}$ is considered. Let also consider a neighborhood of M_d according to (14) with $\varepsilon = 0.5$. The computation of the LMI given by equation (11) with the set of finite times $\tau = \{\tau_1, \tau_2, \tau_3\} = \{10, 80, 20\}$ leads to a non empty region of solutions. Considering the particular constant action $U_1 = (1.54, 0.40, 1.88, 0.89)^T$ in $\mathbf{A_1}$, $U_2 = (2.43, 0.53, 5, 2)^T$, in $\mathbf{A_2}$ and $U_3 = (1.07, 0.31, 0.09, 0.06)^T$ in $\mathbf{A_3}$, the marking trajectory in figure 4 is obtained. The application of the LMI given by equation (12) also provides the set of initial markings in r-region $\mathbf{A_1}$ so that **A** is reached according to S and τ. This region (figure 4, in grey color) coincides with $\mathbf{A_1}$.

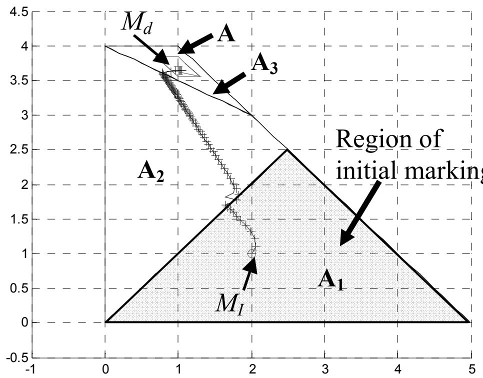

Figure 4: Control design in finite time with sequence $S = \mathbf{A_1}.\mathbf{A_2}.\mathbf{A_3}$: marking trajectory in plan (m_1, m_4) to drive $M(t, M_I)$ from M_I to M_d in $\mathbf{A_3}$.

IV. CONCLUSIONS

Positive invariant and attractive regions in finite time have been characterized for contPNs with piecewise constant control actions. Linear matrix inequalities are used to that purpose. The computation of such LMIs with LPP not only checks the invariance or attractivity of a given polyhedral region, but also provides the set of admissible, bounded input, constant control actions. Piecewise constant controls in finite and minimal time result as a consequence.

In future works we will continue the investigation of control problems for contPNs and also the approximation of stochastic Petri nets by contPNs with piecewise-constant maximal firing speeds.

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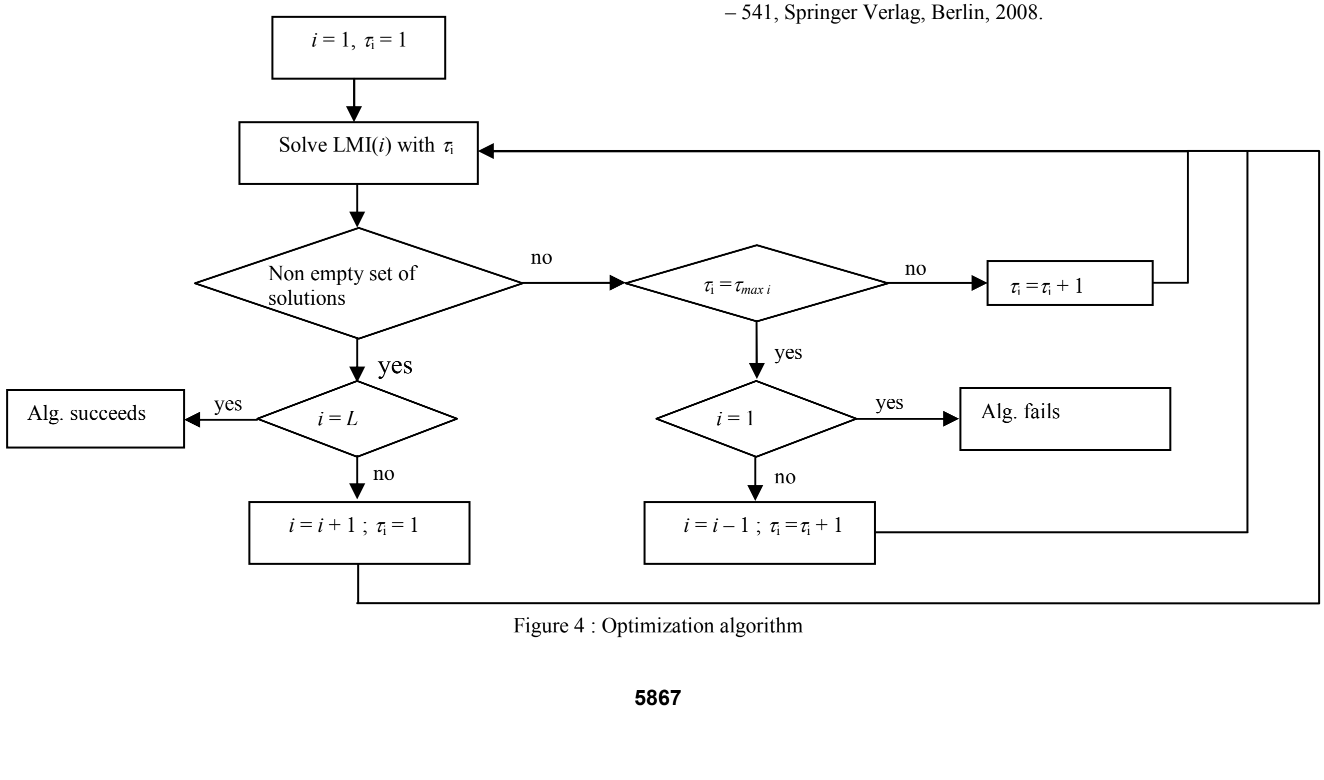

Figure 4 : Optimization algorithm