2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

# Robust Minimum-time Constrained Control of Nonlinear Discrete-Time Systems: New Results

Gilberto Pin and Thomas Parisini

*Abstract*— **This work deals with the analysis and the design of robust minimum-time control laws for nonlinear discrete-time systems with possibly non robustly controllable target sets. Given a Lipschitz nonlinear transition map with bounded control inputs, in the paper we show that the reachability properties of the target set can be used to assess the existence of a robust positively controllable set which includes the target in its interior. This result is exploited to formulate a minimum-time control scheme with enhanced robustness properties able to ensure the ultimate boundedness of the state-trajectories in presence of bounded uncertainties even in the case in which the target set is not robustly positively controllable.**

## I. Introduction

The Minimum-time control problem consists in steering the state of a dynamic system from an initial state $x_0 \in \mathbb{R}^n$ to a given compact set $\Xi \subset \mathbb{R}^n$ (the so-called "target" set) in minimum time and in the presence of possible constraints (typically on the input variables).

The solution of the minimum-time problem is well-known in the case of linear systems with compact target sets (see [1], [2], [3], [4], [5], the survey paper [6] and the references therein), while further investigations are needed both to characterize the stability properties of nominal minimum-time control laws in a nonlinear setting and to design minimum-time controllers robust with respect to unmodelled nonlinearities and unknown external disturbances (see [7] and [8], [9] for some robust formulations based on dynamic programming and invariant-set theory for linear systems). Indeed, since the mathematical models available for the control design are often uncertain and the system may be affected by exogenous not measurable perturbations as well, in practice the synthesis of the control scheme is carried out with incomplete informations.

As for the linear case, it can be proven that if the target set is robustly positively controllable (i.e., the target set can be made robustly positively invariant by some control law verifying the input constraints (see [10]), then the nonlinear minimum-time control ensures the uniform boundedness of the closed-loop trajectories for a suitable set of initial states (see [11], [12], [6] as far as the linear case is concerned). Furthermore, we show that the ultimate boundedness property can be preserved even if the target set is not one-step robustly positively controllable, by suitably modifying the nominal minimum-time feedback law.

At the best of the authors' knowledge, the problem of guaranteeing the boundedness of the trajectories by minimum-time control with non controllable terminal sets has not yet been addressed in the current literature. By exploiting some ideas originally conceived by the authors in [13] in the context of Nonlinear Model Predictive Control (NMPC), in the present paper, a different design procedure is proposed to possibly address the typical conservatism of the conventional minimum-time methodologies.

In the NMPC framework, the terminal constraint is introduced with the aim of providing robust stability guarantees (see e.g. [14], [15], [16], [17] and [18] among the vast literature on the subject) and hence it is usually chosen as an arbitrary robust positively controllable set [19], [20], [21]. The inclusion of such a supplementary terminal condition introduces some conservatism and raises the additional issue of the recursive feasibility with respect to the new constraint (see [13], [22]). Conversely, in the minimum-time control setting, reaching the terminal set represents it by itself the objective of the control design. If the specified target set is not control-invariant, then, to achieve closed-loop robustness, the minimum-time control law is usually computed by imposing a different terminal constraint, chosen as an invariant subset of the nominal target set. In this case, the finite-time reachability of the target set, as well as the ultimate boundedness of the trajectories, can be guaranteed by set-theoretic arguments (see [5]). Nonetheless, the contraction of the target set represents a conservative provision for achieving the robust trajectory boundedness and the finite-time reach of the target in absence of uncertainties. In particular, the robust stability properties of the modified minimum-time problem with a restricted terminal region are achieved at the cost of a smaller feasible region, that is, a smaller capture basin.

Conversely, the minimum-time control scheme proposed in this paper allows to retain the original target set without restrictions. Moreover, when the target is robustly controllable, the devised methods guarantees the same robust performance (minimum reach-time and maximal admissible uncertainty for trajectory boundedness) of standard approaches.

## II. Basic Notations, Definitions and Specific Technical Results

In the following, the notation that will be used throughout the paper is introduced, together with the basic assumptions and the technical results that will be needed to state the main results. It is worth noting that, due to space limitations, all the proofs are omitted and the reader is referred to the Technical Report [23] for the details.

### A. Notations

Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$. Given a signal $s$, let $\mathbf{s}_{[t_1,t_2]}$ be a sequence defined from time $t_1$ to time $t_2$. In order to simplify the notation, when it is inferrable from the context, the subscript of the sequence is omitted. The set of discrete-time sequences of $s$ taking values in some subset $\Upsilon \subset \mathbb{R}^n$ is denoted by $\mathcal{M}_\Upsilon$. Moreover let us define $||\mathbf{s}|| \triangleq \sup_{k\geq 0}\{|s_k|\}$ and $||\mathbf{s}_{[t_1,t_2]}|| \triangleq \sup_{t_1 \leq k \leq t_2}\{|s_k|\}$, where $s_k$ denotes the value that the sequence $\mathbf{s}$ takes on in correspondence with the index $k$. Given a compact set $A \subseteq \mathbb{R}^n$, let $\partial A$ denote the boundary of $A$. Given a vector $x \in \mathbb{R}^n$, $\mathrm{d}(x, A) \triangleq \inf\{|\xi - x|, \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to $A$, while $\Phi(x, A) \triangleq \{-\mathrm{d}(x, \partial A)$ if $x \in A$, $\mathrm{d}(x, \partial A)$ if $x \notin A\}$ denotes the signed distance function. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, $\mathrm{dist}(A, B) \triangleq \inf\{\mathrm{d}(\zeta, A), \zeta \in B\}$ is the minimal set-to-set distance. The difference between two given sets

G. Pin is with Danieli Automation S.p.A, Udine, Italy (g.pin@dca.it); T. Parisini is with the Dept. of Electrical and Electronic Engineering at the Imperial College London, UK and also with the Dept. of Industrial and Information Engineering at the University of Trieste, Italy (t.parisini@paperplaza.net).

$A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, then the Pontryagin difference set $C$ is defined as $C = A \backsim B \triangleq \{x \in \mathbb{R}^n : x+\xi \in A, \forall \xi \in B\}$, while the Minkowski sum set is defined as $S = A \oplus B \triangleq \{x \in \mathbb{R}^n : x = \xi + \eta,\ \xi \in A,\ \eta \in B\}$. Given a vector $\eta \in \mathbb{R}^n$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball centered in $\eta$ and of radius $\rho$ is denoted as $\mathcal{B}^n(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^n : |\xi - \eta| \leq \rho\}$. The shorthand $\mathcal{B}^n(\rho)$ is used when the ball is centered in the origin. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\gamma(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ ($\mathcal{K}$-function) if it is continuous, zero at zero, and strictly increasing.

### B. Basic Assumptions and Definitions

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t\,,\ t \in \mathbb{Z}_{>0},\ x_0 = \bar{x} \tag{1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t$ the control vector, subject to the constraint

$$u_t \in U \subset \mathbb{R}^m, \tag{2}$$

with $U$ compact, and $d_t \in D \subset \mathbb{R}^n$, with $D$ compact, a bounded additive transition uncertainty vector.

In stating and proving the preliminary technical lemmas, and with the aim of simplifying the derivation of the main results, let the function $\hat{f}(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ verify the following assumption.

*Assumption 1 (Lipschitz):* The function $\hat{f}(x,u)$ is Lipschitz (L.) continuous w.r.t. $x \in \mathbb{R}^n$, uniformly in $u \in U$, with L. constant $L_{\hat{f}_x} \in \mathbb{R}_{\geq 0}$, that is, for all $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^n$

$$|\hat{f}(x,u) - \hat{f}(x',u)| \leq L_{\hat{f}_x} |x - x'|, \forall u \in U.$$

□

Moreover, to prove some results we also make use of the following assumption.

*Assumption 2 (Local Uniform Continuity w.r.t. $u$):* For any $x \in \mathbb{R}^n$ the function $\hat{f}(x,u)$ is uniformly continuous w.r.t. $u \in U$. That is, for any $u \in U$ and any $u' \in U$

$$|\hat{f}(x,u) - \hat{f}(x,u')| \leq \eta_u(|u - u'|), \forall x \in \mathbb{R}^n.$$

where $\eta_u(\cdot)$ is a $\mathcal{K}$-function. □

*Definition 2.1 (Controllability set to $\Xi$):* Given a map $\hat{f}(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ compact, and a set $\Xi \subset \mathbb{R}^n$, the (one-step) controllability set to $\Xi$, ($\mathcal{C}_1(\Xi)$) is given by

$$\mathcal{C}_1(\Xi) \triangleq \left\{x_0 \in \mathbb{R}^n | \exists\, u_{x_0} \in U : \hat{f}(x_0, u_{x_0}) \in \Xi\right\}. \tag{3}$$

□

*Definition 2.2 (Predecessor set of $\Xi$):* Given the map $\hat{g}(x) : \mathbb{R}^n \to \mathbb{R}^n$ and a set $\Xi \subset \mathbb{R}^n$, the (one-step) predecessor of $\Xi$, ($\mathcal{P}_1(\Xi)$) is given by

$$\mathcal{P}_1(\Xi) \triangleq \{x_0 \in \mathbb{R}^n | \hat{g}(x_0) \in \Xi\}. \tag{4}$$

□

*Definition 2.3 ($i$-steps Controllability Set to $\Xi$):* Given a map $\hat{f}(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ compact, and a set $\Xi \subset \mathbb{R}^n$, the $i$-steps controllability set to $\Xi$, ($\mathcal{C}_i(\Xi)$) is given by

$$\mathcal{C}_i(\Xi) \triangleq \left\{x_0 \in \mathbb{R}^n | \exists\, \mathbf{u}(x_0) \in U^i : \hat{x}(i, x, \mathbf{u}(x_0)) \in \Xi\right\}. \tag{5}$$

that is, $\mathcal{C}_i(\Xi)$ is the set of initial states $x_0 \in \mathbb{R}^n$ which can be driven into $\Xi$ by exactly $i$ feasible control actions. □

*Definition 2.4 ($i$-steps Predecessor of $\Xi$):* Given the map $\hat{g}(x) : \mathbb{R}^n \to \mathbb{R}^n$ and a set $\Xi \subset \mathbb{R}^n$, the $i$-steps predecessor of $\Xi$, ($\mathcal{P}_i(\Xi)$) is given by

$$\mathcal{P}_i(\Xi) \triangleq \left\{x_0 \in \mathbb{R}^n | \hat{g}^i(x_0) \in \Xi\right\}. \tag{6}$$

where $\hat{g}^i$ denotes the $i$-times composition of the map $\hat{g}$ with itself. □

*Definition 2.5 ($i$-steps Capture Basin to $\Xi$):* Given a map $\hat{f}(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ compact, and a set $\Xi \subset \mathbb{R}^n$, the $i$-steps capture basin to $\Xi$, ($\mathrm{Capt}_i(\Xi)$) is given by

$$\mathrm{Capt}_i(\Xi) \triangleq \bigcup_{j=1}^{i} \mathcal{C}_i(\Xi). \tag{7}$$

that is, $\mathrm{Capt}_i(\Xi)$ is the set of initial states $x_0 \in \mathbb{R}^n$ such that $\Xi$ is reached in at most $i$ steps (i.e., $\exists\, \mathbf{u}_{x_0} \in U^j : \hat{x}(j, x_0, \mathbf{u}_{x_0}) \in \Xi$ for at least one $j \in [1, \ldots, i-1]$, before possibly leaving $\Xi$. □

Moreover, the following property holds for controllability sets.

*Proposition 2.1:* Given two sets $\Xi_1 \subset \mathbb{R}^n$ and $\Xi_2 \subset \mathbb{R}^n$, then $\mathcal{C}_1(\Xi_1 \cup \Xi_2) = \mathcal{C}_1(\Xi_1) \cup \mathcal{C}_1(\Xi_2)$. □

*Definition 2.6 (RC, $d_1$-RC):*
- A compact set $\Xi \subset \mathbb{R}^n$ is Robustly Controllable (RC) under the map $\hat{f}(x,u)$, with $u \in U$, if $\Xi \subseteq \mathcal{C}_1(\mathrm{int}(\Xi))$.
- A compact set $\Xi \subset \mathbb{R}^n$ is Robustly Controllable in one-step w.r.t. additive perturbations $d \in \mathcal{B}^n(d_1)$ ($d_1$-RC) if ($\Xi \backsim \mathcal{B}^n(d_1)$) is not empty and $\Xi \subseteq \mathcal{C}_1(\Xi \backsim \mathcal{B}^n(d_1))$.

□

*Definition 2.7 (RPI):* A compact set $\Xi \subset \mathbb{R}^n$ is Robust Positively Invariant (RPI) under the map $\hat{g}(x)$, if $\Xi \subseteq \mathcal{P}_1(\mathrm{int}(\Xi))$. □

*Definition 2.8 (quasi-RPI):* A compact set $\Xi \subset \mathbb{R}^n$ is quasi-Robust Positively Invariant (RPI) under the map $\hat{g}(x)$ with maximum return-time $N$, if $\Xi \subseteq \bigcup_{i=1}^{N} \mathcal{P}_i(\mathrm{int}(\Xi))$ for some finite $N \in \mathbb{Z}_{>0}$; that is, for any $x_0 \in \bigcup_{i=1}^{N} \mathcal{P}_i(\mathrm{int}(\Xi))$, $\exists i \in \{1, \ldots, N\} : \hat{g}^i(x_0) \in \mathrm{int}(\Xi)$. □

### C. Main Technical Results

In the present subsection, some intermediate and one main results concerning the properties of robustly controllable sets under Lipschitz maps are given. These results are used extensively in the proof of the main contribution of the present work stated in Section IV. In stating and deriving the following lemmas, let the nominal system's transition map $\hat{f}$ verify the Lipschitz Assumption 1.

*Lemma 2.1 (Technical):* Given two compact sets $\Xi_1 \subset \mathbb{R}^n$, $\Xi_2 \subset \mathbb{R}^n$ and a positive scalar $d \in \mathbb{R}_{>0}$, if the following three conditions hold together: $i)$ $\Xi_1 \subseteq \mathcal{C}_1(\Xi_2)$, $ii)$ $\Xi_2 \subset \Xi_1$ and $iii)$ $\mathrm{dist}(\mathbb{R}^n \backslash \Xi_1, \Xi_2) \geq d$, then $\Xi_1$ is $d$-RC. □

*Lemma 2.2 (Technical):* Given a compact set $\Xi \subset \mathbb{R}^n$, assume that $\mathcal{C}_1(\Xi)$ is non-empty. Then, for any arbitrary $\eta \in \mathbb{R}_{\geq 0}$ it holds that:

$$\forall x \in \Xi \oplus \mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta\right),\ \exists u_x \in U\ :\ \hat{f}(x,u_x) \in \Xi \oplus \mathcal{B}^n(\eta). \tag{8}$$

□

*Lemma 2.3 (Technical):* Given a compact set $\Xi \subset \mathbb{R}^n$ and positive scalar $\rho \in \mathbb{R}_{>0}$, if $\Xi$ is $\rho$-RC, then $\mathcal{C}_1(\Xi)$ is $(L_{\hat{f}_x}^{-1}\rho)$-RC. □

The next result establishes the invariant properties of the $N$-steps controllability set $\mathcal{C}_N(\Xi)$ of a given $\rho$-RC set $\Xi$. Moreover, an inner (conservative) approximation of $\mathcal{C}_N(\Xi)$, containing $\Xi$ in its interior, is provided.

*Lemma 2.4 (Technical):* Given a compact set $\Xi \subset \mathbb{R}^n$, a finite integer $N \in \mathbb{Z}_{>0}$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, if $\Xi$ is $\rho$-RC, then[1]

i) $\mathcal{C}_N(\Xi)$ is $(L_{\hat{f}_x}^{-N}\rho)$-RC.

ii) $\mathcal{C}_N(\Xi) \supseteq \Xi \oplus \mathcal{B}^n\left(\dfrac{1-L_{\hat{f}_x}^{-N}}{L_{\hat{f}_x}-1}\rho\right)$

□

The following important result, that will play a key role in characterizing the robust stability properties of nonlinear minimum-time control laws, can now be stated.

*Theorem 2.1 ($N$-steps Reachability Implication):* Given a compact set $\Xi \subset \mathbb{R}^n$ and map $\hat{f}(x,u)$ verifying Assumption 1 and subject to (2), if the following inclusion holds for a finite integer $N \in \mathbb{Z}_{>0}$ and for a positive scalar $\rho_N \in \mathbb{R}_{>0}$:

$$\Xi \subseteq \mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\rho_N), \tag{9}$$

(i.e., $\Xi$ is reachable in at most $N$ steps from a set containing $\Xi$ in its interior under the nominal map $\hat{f}(x,u)$), then the set

$$\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N) \triangleq \mathcal{C}_N(\Xi) \cup \left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_{i+1}(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right)\right]\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right) \tag{10}$$

is $\overline{d}_N$-RC, with $\eta_i$ positive scalars depending on $\overline{d}_N$ according to the recursion

$$\overline{d}_N \triangleq \frac{L_{\hat{f}_x}-1}{L_{\hat{f}_x}^N - 1}\rho_N\,, \quad \eta_0 = \rho_N - \overline{d}_N\,, \tag{11}$$

and

$$\eta_i \ = L_{\hat{f}_x}^{-1}\eta_{i-1} - \overline{d}_N,\ \forall i \in \{1,\dots N-2\}.$$

□

The reader can refer to Figure 1 for a schematization of the sets involved in the statement Theorem 2.1.

*Remark 2.1:* Note that, if the target set $\Xi$ is not $\rho$-RC, then the region $(\mathrm{Capt}_N(\Xi)\backslash\mathcal{C}_N(\Xi))$ may not be empty. In this case, for any initial condition in $(\mathrm{Capt}_N(\Xi)\backslash\mathcal{C}_N(\Xi))$, the state cannot be driven to $\Xi$ in exact $N$ steps, but $\Xi$ can be reached for some $i$ with $i < N$ from the capture basin. Therefore, condition (9) (reachability in at most $N$ steps) is less restrictive than requiring the exact $N$-steps controllability of $\Xi$. □

[1]The very special case $L_{\hat{f}_x} = 1$ can be trivially addressed by a few suitable modifications of the results of the paper and is omitted for the sake of clarity.

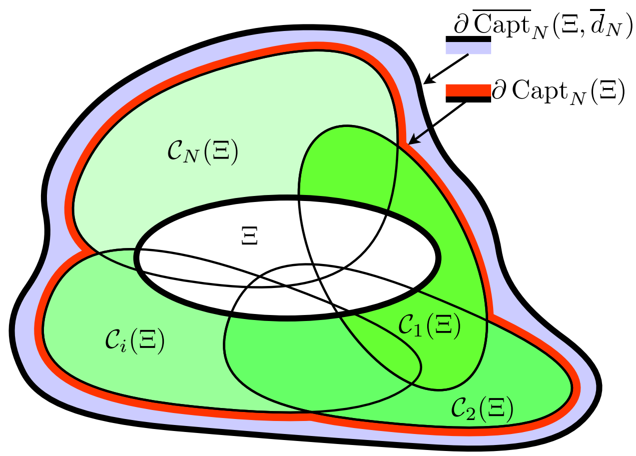


Fig. 1. Scheme of the sets involved in the statement and proof of Theorem 2.1. $\mathrm{Capt}_N(\Xi)$ denotes the $N$-steps capture basin of $\Xi$, while $\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N)$ is an extension of the nominal capture basin (see (10)) that can be proven to be robust positively controllable. That is, there exists a control law, compliant with the input constraints, that renders $\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N)$ RPI.

## III. Problem Statement

The minimum-time control problem for discrete-time systems not affected by uncertainties has the following well known formulation: given an initial state $x_0 \in \mathbb{R}^n$ and a target set $\Xi \subset \mathbb{R}^n$, find a sequence of control actions $\mathbf{u} \in \mathcal{M}_U$ which minimizes the time $T_{MT}(x_0|\Xi)$ such that $\hat{x}(x_0, T_{MT}, \mathbf{u}) \in \Xi$. In the following, we will denote as $T^o_{MT}(x_0|\Xi)$ the minimum (optimal) reach time. The above formulation is commonly referred to as the open-loop approach to the minimum-time problem, that is, an optimal control sequence is determined on the basis of the specific given initial state, relying on a nominal model of the controlled system. In the linear framework, it is well-known that the minimum-time problem admits a feedback solution, that is, it is possible to determine a control function $u = \kappa_{MT}(x|\Xi)$ such that $T_{MT}(\cdot|\Xi)$ is minimized for any possible initial state. We point out the minimum-time control law $\kappa_{MT}(\cdot|\Xi)$ is not, in general, unique. Therefore, for the sake of the present discussion, the notation $\kappa_{MT}(x|\Xi)$ will denote an arbitrary selection among the possible minimum-time control actions for the state $x$.

Of course, in nominal conditions, open-loop and feedback formulations are equivalent in the sense that a feedback solution is optimal if and only if for any initial states $x_0$ the control sequence $\mathbf{u}$ produced by the control $u = \kappa_{MT}(x,\Xi)$ along the systems'trajectory is optimal in the open-loop sense. On the other hand, the feedback approach allows also to embed in the design of the controller some a priori information on the disturbances/uncertainties, yielding to minimum-time control laws with enhanced robustness properties (see [5]). However, for a generic nonlinear system, it is very difficult to obtain an explicit minimum-time control function, even in the nominal case. Moreover, in practice, the search for a minimum-time open-loop sequence is performed over a compact set of sequences of finite length, assuming a specified upper bound $N \in \mathbb{Z}_{>0}$.

A viable solution to alleviate the lack of robustness of open-loop approaches consists in solving repeated finite-time optimizations problem along the system trajectory, using the current state measurement as initial condition. In the sequel, we will devise sufficient conditions (related, in particular, to the controllability and reachability properties of the target set) under which the RH approach guarantees the recursive feasibility of the optimization (that is, the robust postive invariance or the quasi-invariance of the feasible region) and the boundedness of the closed-loop state trajectories.

*Problem 3.1 (Nominal Minimum-Time Control (open-loop)):* Given a compact admissible set $U \subset \mathbb{R}^m$ for the input of the system (1), a compact target set $\Xi \subset \mathbb{R}^n$, a finite integer $N \in \mathbb{Z}_{>0}$ and the nominal state-transition map $\hat{f}(x,u)$ of the system, at each time $t \in \mathbb{Z}_{\geq 0}$ determine a sequence $\mathbf{u}^o_{[t,t+N-1]} = \{u^o_t, u^o_{t+1}, \dots, u^o_{t+N-1}\}$, in correspondence of the current state measurement $x_t$, such that:

$$\mathbf{u}^o_{[t,t+N-1]} = \arg\min_{\mathbf{u}_{[t,t+N-1]} \in \Upsilon(x_t|\Xi)} \Big\{ T_{MT}(x_t, \mathbf{u}_{[t,t+N-1]}|\Xi, N) \Big\},$$

where $\Upsilon(x_t|\Xi)$ is the set of feasible control sequences for the current state $x_t$:

$$\Upsilon(x_t|\Xi) \triangleq \Big\{ \mathbf{w}_{[t,t+N-1]} \in U^N | \\ \exists \tau \in \{1,\dots,N\} : \hat{x}(\tau, x_t, \mathbf{w}_{[t,t+\tau-1]}) \in \Xi \Big\},$$

and

$$T_{MT}(x_t, \mathbf{u}_{[t,t+N-1]}|\Xi, N) \\ \triangleq \min \Big\{ \tau \in \{1,\dots,N\} : \hat{x}(\tau, x_t, \mathbf{u}_{[t,t+\tau-1]}) \in \Xi \Big\}, \quad (12)$$

Finally, apply to the plant the first element of $\mathbf{u}^o_{[t,t+N-1]}$ by setting $u_t = u^o_t$. □

This problem can be solved, in the discrete-time framework, by checking the feasibility of the target set constraint in (12). The feasibility check approach consists in embedding Problem 3.1 into a family of input-constrained minimum-distance problems as follows:

$$J^o_{MD}(x_t|\Xi, \tau) = \min_{\mathbf{u}_{[t,t+\tau-1]} \in U^\tau} \Phi\Big(\hat{x}(\tau, x_t, \mathbf{u}_{[t,t+\tau-1]}), \Xi\Big). \quad (13)$$

parametrized by the integer $\tau \in \mathbb{Z}_{>0}$. For a given $\tau$, the minimizer is a fixed-length sequence $\mathbf{u}_{t,t+\tau-1}$ belonging to the compact set $U^\tau$.

The feasible region for the original problem 3.1 is the basin of capture $\mathrm{Capt}_N(\Xi)$, which verifies

$$\mathrm{Capt}_N(\Xi) \\ = \{x_t \in \mathbb{R}^n | \quad \exists \overline{\tau} \in \{1,\dots,N\} : J^o_{MD}(x_t, \Xi, \overline{\tau}) \leq 0\}.$$

Then, assuming that $x_t \in \mathrm{Capt}_N(\Xi)$, at each time $t \geq 0$ the optimal time $T^o_{MT}(x_t|\Xi)$ is determined as the minimum among $\tau \in \{1,\dots,N\}$ for which problem (13) yields to

$$J^o_{MD}(x_t|\Xi, \tau) \leq 0, \quad (14)$$

that is

$$T^o_{MT}(x_t|\Xi) = \min\{\tau \in \{1,\dots,N\} : J^o_{MD}(x_t|\Xi,\tau) \leq 0\}$$

Once the minimum-time $T^o_{MT}(x_t|\Xi)$ has been determined, we can take as a solution any control sequence which may steer the state to $\Xi$ in $T^o_{MT}(x_t|\Xi)$ steps. A simple choice is

$$\mathbf{u}^o_{[t,t+T^o_{MT}(x_t|\Xi)-1]} \\ = \arg\min_{\mathbf{u}_{[t,t+T^o_{MT}(x_t|\Xi)-1]} \in U^{T^o_{MT}(x_t|\Xi)}} \Phi\Big(\hat{x}(T^o_{MT}(x_t|\Xi), x_t, \mathbf{u}_{[t,t+T^o_{MT}(x_t|\Xi)-1]}), \Xi\Big) \quad (15)$$

It is important to determine those conditions under which, starting from $x_0 \in \mathrm{Capt}_N(\Xi)$, the trajectories remain in the feasible set, in order to guarantee the solvability of the optimization for any time $t > 0$.

## IV. Recursive Feasibility under the Nonlinear Minimum-time Control with a Robust Positively Controllable Target Set

In case the target set $\Xi$ is robust positively controllable (i.e., there exists an admissible control law which renders $\Xi$ RPI), then the $N-$steps capture basin $\mathrm{Capt}_N(\Xi)$ coincides with the $N$-steps controllability set of $\Xi$. Moreover $\mathrm{Capt}_N(\Xi)$ is RPI under the minimum-time control, as formally stated by the following theorem.

*Theorem 4.1 ($\Xi$ $\rho$-RC $\to$ $\mathrm{Capt}_N(\Xi)$ RPI):* Given a $\rho-$RC compact target set $\Xi \subset \mathbb{R}^n$, then the Nominal nonlinear Minimum-Time Control $\kappa_{MT}(x_t)$ guarantees the boundedness of the closed-loop trajectories within the set $\mathrm{Capt}_N(\Xi)$, for any initial condition $x_0 \in \mathrm{Capt}_N(\Xi)$ and for any admissible uncertainty realization $\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$, with $d_N = L^{-N}_{\hat{f}_x} \rho_N$.

Moreover, the closed-loop trajectories starting at time $t = 0$ from any point $x(0) = x_0 \in \mathrm{Capt}_N(\Xi)$ are ultimately bounded in the compact set

$$\Upsilon_N(\Xi, d_N) \triangleq \Xi \oplus \mathcal{B}\left(\frac{L^N_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} d_N\right) \subseteq \mathrm{Capt}_N(\Xi), \quad (16)$$

which is reached in finite time, for any possible realization of the uncertainties ($\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$), that is, $x(t, x_0, \mathbf{u}_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) \in \Upsilon_N(\Xi, ||\mathbf{d}_{[0,t-1]}||) \subseteq \Upsilon_N(\Xi, d_N)$, $\forall x_0 \in \mathrm{Capt}_N(\Xi)$, $\forall t \geq N$, $\forall \mathbf{d}_{[0,t-1]}$ : $||\mathbf{d}_{[0,t-1]}|| \leq d_N$. □

Now, our analysis is extended to the case in which $\Xi$ is not one step robust positively controllable. In this regard, the result of Theorem 2.1, obtained by set-invariance theoretic analysis, implies the existence of a (possibly non unique) control law which renders the set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ a $\overline{d}_N$-RPI set, with $\overline{d}_N$ defined in (11). However, in the presence of uncertainties, the set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ is such that $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N) \supset \mathrm{Capt}_N(\Xi)$. Recalling that the feasible region for Problem 3.1 is as small as $\mathrm{Capt}_N(\Xi)$, in the set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N) \backslash \mathrm{Capt}_N(\Xi)$ the finite-time RH problem does not admit a solution (i.e., the feasibility check (14) fails). Therefore, we seek for a backup control law to be applied in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ when the minimum-time problem is not solvable, but capable to keep the trajectories bounded in the extended $N$-steps capture basin $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$. Notably, by Theorem 2.1 we have established the existence of a control law, compliant with the input constraints, capable to achieve this task. We are now on the way to show how such a robust control law can be obtained.

## V. Robust Nonlinear Minimum-time Control Laws with Non-Robust Positively Controllable Target Sets

In the following, we are going to describe a modified minimum-time control scheme, which is referred to as Robustified Nonlinear Minimum-Time Control (RNMT), that guarantees the quasi-invariance of the feasible region, despite bounded uncertainties and with mild assumptions on the target set $\Xi$. The RNMT control, computed online according to Procedure 5.1 below, consists in a control scheme that switches between the regular minimum-time control and a backup control action when unfeasibility occurs during transient; hence, in nominal conditions, the RNMT corresponds to the convential open-loop minimum-time control, being the feasible region invariant in this case. Conversley, in perturbed conditions, the backup control action is taken from a buffer in which a time-optimal control sequence had been saved after the most recent feasible optimization. The key point of this procedure is that feasbility is recovered before buffer overrun occurs. As long as the system's state enters the feasible region, an optimal solution is computed and the buffer is reinitialized with a new sequence, that will be used to cope with future unfeasibility occurrences.

The above RNMT scheme is formalized in the following procedure which shows the actions to be performed by the controller.

*Procedure 5.1 (RNMT):* Let the controller be equipped with two buffers: $i)$ $\mathbf{u}^b \in \mathbb{R}^m \times N$, used to store a sequence of $N$ control actions; $ii)$ $T^b \in \mathbb{Z}$, that stores the discrete-time instant in which the sequence stored in $\mathbf{u}^b$ had been computed. Moreover, let us denote as $\leftarrow$ a data assignment operation. Given the buffer (memory array) $\mathbf{u}^b$, let $u^b(i)$ represent the $i$-th element of the array, with $i \in \{1, \ldots, N\}$.

**Initialization**

Assuming that, at time instant $t = 0$, the initial condition verifies $x_0 \in \mathrm{Capt}_N(\Xi)$, solve the nominal minimum-time Problem 3.1 obtaining an optimal control sequence $\mathbf{u}^o_{0,N-1}$;
store $\mathbf{u}^b \leftarrow \mathbf{u}^o_{0,N-1}$;
store $T^b \leftarrow 0$;
apply $u_o = u^b(1)$ to the plant.

**On-line Control Computation**

```
for t ∈ Z_{>0} :
    given x_t, perform the feasibility test (14) for τ ∈ {1,...,N};
    if exists at least one τ for which J^o_{MD}(x_t|Ξ,τ) ≤ 0, then :
        compute u^o_{t,t+N-1} with (15) ;
        overwrite the buffer u^b ← u^o_{t,t+N-1};
        set T^b ← t;
    end if;
    apply u_t = u^b(t − T^b + 1) to the plant;
end for;
```

□

The following theorem formally states the recursive feasibility property (that is, the quasi-invariance of the feasible region) of the RNMT scheme for bounded additive uncertainties.

*Theorem 5.1 (Quasi-invariance of the feasible set):* Given a compact target set $\Xi \subset \mathbb{R}^n$ (possibly not robustly controllable) such that $\Xi \subseteq \mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\rho_N)$, then, for any initial condition $x_0 \in \mathrm{Capt}_N(\Xi)$, the RNMT control $u_t = \kappa_{RNMT}(t, x_t)$ guarantees that the closed-loop system's trajectory is ultimately contained in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ for any admissible uncertainty realization $\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$, with $\overline{d}_N$ given by (11). Moreover, the compact set $\Upsilon_N(\Xi, \overline{d}_N) \subseteq \mathrm{Capt}_N(\Xi)$ (defined in (17) below) is reached in finite-time from $x_0$ and is quasi-RPI in closed-loop, for any possible realization of the uncertainties. □

$$\Upsilon_N(\Xi, \overline{d}_N) \triangleq \Xi \oplus \mathcal{B}^n \left( \max_{j \in \{1,\ldots,N\}} \left\{ \frac{L^j_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} \overline{d}_N \right\} \right) \quad (17)$$

Notice that the ultimate confinement property in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$, together with the quasi-robust positive invariance of the compact set $\Upsilon_N(\Xi, \overline{d}_N)$, they by themselves do not imply the ultimate boundedness of the trajectories, since $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ can be unbounded. The boundedness in a compact subset of $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ can be proven by invoking the further Assumption 2 and by exploiting the the presence of input contraints.

*Corollary 5.1 (Ultimate boundedness):* If the nominal transition function of the system, $\hat{f}$, verifies, in addition to Assumptions of Theorem 5.1, the further Assumption 2, ($\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(\overline{d}_N)}$), then the closed-loop trajectories under the RNMT control are ultimately bounded in a compact set $\Lambda_N(\Xi, \overline{d}_N, \overline{u})$ (defined in (18) below) for any initial condition $x_0 \in \mathrm{Capt}_N(\Xi)$, that is: $x(t, x_0, \mathbf{u}_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) \in \Lambda_N(\Xi, \overline{d}_N, \overline{u})$, $\forall x_0 \in \mathrm{Capt}_N(\Xi, \overline{d}_N)$, $\forall t \geq N$, $\forall \mathbf{d}_{[0,t-1]} : ||\mathbf{d}_{[0,t-1]}|| \leq \overline{d}_N$. □

$$\Lambda_N(\Xi, \overline{d}_N, \overline{u}) \triangleq \overline{\mathrm{Capt}}_N(\Xi) \cap \Upsilon_N(\Xi, \overline{d}_N, \overline{u}) \oplus \mathcal{B}^n \left( \max_{j \in \{1,\ldots,N\}} \left\{ \frac{L^j_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} (\eta_u(\overline{u}) + \overline{d}_N) \right\} \right). \quad (18)$$

## VI. SIMULATION RESULTS

To show the effectiveness of the method, we apply the robustified nonlinear minimum-time control to the following discrete-time open-loop unstable system:

$$\begin{cases} x_{(1)_{t+1}} &= x_{(1)_t} \left[ 1.1 + 0.4 \,\mathrm{sign}(\mathrm{x}_{(1)_t}) u_t \right] \\ &\quad + (x_{(2)_t}^2 + 2)^{-1} u_t + d_{(1)_t} \\ x_{(2)_{t+1}} &= 0.94\, x_{(2)_t} - x_{(2)_t} u_t + d_{(2)_t} \end{cases}, t \in \mathbb{Z}_{\geq 0}. \quad (19)$$

subjected to the input constraint $|u_t| < 2$. The subscripts $(i)$, $i \in \{1, 2\}$ in (19) denote the $i$-th component of $x_t \in \mathbb{R}^2$, while $d_t \in \mathbb{R}^2$ is a bounded exogenous disturbance. First, we prove that the nominal transition function of the system is Lipschitz continuous with respect to the state variables.

It can be easily shown that the nonlinear transition function $\hat{f}(x, u) : \mathbb{R}^2 \times [-R, R] \to \mathbb{R}^2$, with $\hat{f}(x, u) = \left( \hat{f}_{(1)}(x_{(1)}, x_{(2)}, u), \hat{f}_{(2)}(x_{(2)}, u) \right)$ given by

$$\begin{aligned} \hat{f}_{(1)}(x_{(1)}, u) &= x_{(1)} \left[ 1.1 + 0.4 \,\mathrm{sign}(\mathrm{x}_{(1)}) u \right] \\ &\qquad + (x_{(2)}^2 + 2)^{-1} u, \\ \hat{f}_{(2)}(x_{(2)}, u) &= 0.94\, x_{(2)} - x_{(2)}\, u, \end{aligned} \quad (20)$$

is Lipschitz continuous in $x$, uniformly for $u \in [-2, 2]$.

Figure 2 shows some sample closed-loop trajectories in nominal conditions (i.e., $d_t = 0, \forall t \in \mathbb{Z}_{\geq 0}$) for $N = 5$ and with target set

$$\Xi_1 = \left\{ x \in \mathbb{R}^2 : x^\top \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} x \leq 1 \right\}.$$

The RNMT strategy, in the nominal case, steers the state into the target set in minimum-time. Being $\Xi_1$ robustly controllable, then the inclusion $\Xi_1 \subset \mathrm{Capt}_1(\Xi_1)$ holds; therefore, Theorem 4.1 can be used to assert the robust positive invariance of the feasible region.

To complete the analysis, nominal trajectories obtained with a non robust positively controllable target $\Xi_2$, given

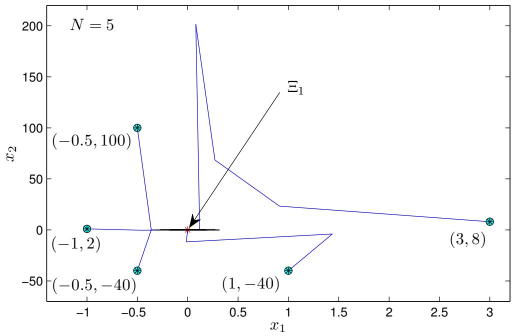


Fig. 2. Sample closed-loop trajectories under th RMNT control in nominal conditions

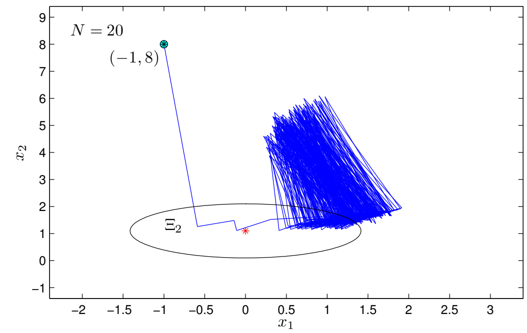


Fig. 4. Sample closed-loop trajectory under th RMNT control with bounded perturbations. The trajectory remains bounded in perturbed conditions despite a non robust positively controllable target set has been used

by

$$\Xi_2 = \left\{ x \in \mathbb{R}^2 : \left( x - \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} \right)^\top \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \left( x - \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} \right) \leq 1 \right\}$$

are shown in Figure 3, for $N = 20$.

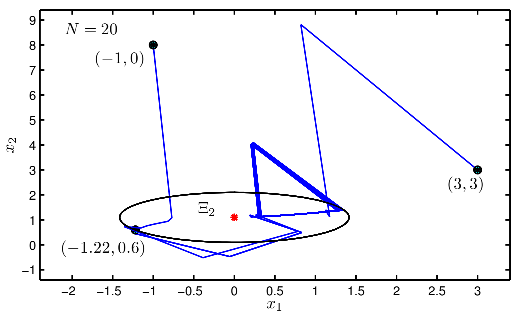


Fig. 3. Sample closed-loop trajectories under the RMNT control in nominal conditions, with a non robustly controllable target set $\Xi_2$. From the initial point $x_0 = (-1.22, 0.6) \in \Xi_2$ the state cannot be kept inside $\Xi_2$ with the available control input. The trajectories asymptotically reach a triangle-shaped limit-cycle condition which temporarily leaves the target set

The trajectories departing from all the considered initialization points asymptotically reach a triangle-shaped limit-cycle condition, that temporarily exits from the target set.

Even in this case, the RNMT can face exogenous perturbations, as shown in Figure 4 , where a bounded disturbance ($||d_t|| \leq \sqrt{2}, \forall t \in \{0, \ldots, 100\}$) has been simulated. Notably, the quasi-invariance of the feasibile region has been guaranteed and the closed-loop trajectoris have been maintined in a neighbourhood of the limit-cycle reached in nominal conditin.

## References


[1] V. Kucera, "The structure and properties of time-optimal discrete linear control," *IEEE Transactions on Automatic Control*, vol. 16, no. 4, pp. 375 – 377, 1971.

[2] S. S. Keerthi and E. G. Gilbert, "Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints," *IEEE Transactions on Automatic Control*, vol. 32, no. 5, pp. 432–435, 1987.

[3] D. Mayne and W. Schroeder, "Robust time-optimal control of constrained linear systems," *Automatica*, vol. 33, no. 12, pp. 2103–2118, 1997.

[4] Z. Gao, "On discrete time optimal control: a closed-form solution," in *Proc. American Control Conference*, 2004, pp. 52–58.

[5] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Boston: Birkhauser, 2008.

[6] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[7] D. Bertsekas and I. Rhodes, "On the min-max reachability of target set and target tubes," *Automatica*, vol. 7, pp. 233–247, 1971.

[8] S. Rakovic and D. Mayne, "Robust time-optimal obstacle avoidance problem for constrained discrete-time systems," in *Proc. of the 44th IEEE Conference on Decision and Control, Seville*, Dec 2005.

[9] S. Raković, E. Kerrigan, D. Mayne, and J. Lygeros, "Reachability analysis of discrete-time systems with disturbances," *IEEE Trans. on Automatic Control*, vol. 51, no. 4, pp. 546–561, 2006.

[10] I. Kolmanovsky and E. Gilbert, "Theory and computation of invariant sets for discrete-time linear systems," *Math.Probl. in Engineering*, vol. 35, no. 11, pp. 1747–1767, 1999.

[11] F. Blanchini, "Minimum-time control for uncertain discrete-time linear systems," in *Proceedings of the 31st IEEE Conference on Decision and Control*, Tucson, 1992, pp. 2629–2634.

[12] ——, "Ultimate boundedness control for discrete time uncertain systems via set-induced lyapunov functions," *IEEE Trans. on Automatic Control*, vol. 39, pp. 428–433, 1994.

[13] G. Pin and T. Parisini, "Extended recursively feasible model predictive control of nonlinear discrete-time systems," in *Proc. American Control Conference*, Baltimore, 2010, pp. 5483–5488.

[14] D. Q. Mayne and H. Michalska, "An implementable receding horizon controller for stabilization of nonlinear systems," in *Proc. of the 32nd Conf. on Decision and Control*, Honolulu, 1990.

[15] D. Mayne, J. Rawlings, C. Rao, and P. Skokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[16] D. Limón, T. Alamo, and E. F. Camacho, "Stability analysis of systems with bounded additive uncertainties based on invariant sets: Stability and feasibility of mpc," in *Proc. American Control Conference*, 2002.

[17] L. Magni, D. M. Raimondo, and R. Scattolini, "Regional input-to-state stability for nonlinear model predictive control," *IEEE Trans. on Automatic Control*, vol. 51, no. 9, 2006.

[18] D. Limón, T. Alamo, F. Salas, and E. F. Camacho, "Input to state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties," *Automatica*, vol. 42, pp. 797–803, 2006.

[19] E. Kerrigan and J.M.Maciejowski, "Robust feasibility in model predictive control: Necessary and sufficient conditions," in *Proc. of the IEEE Conf. on Decision and Control*, 2001, pp. 728–733.

[20] G. Pin and T. Parisini, "Set invariance under controlled nonlinear dynamics with application to robust RH control," in *Proc. of the IEEE Conf. on Decision and Control*, Cancun, 2008, pp. 4073 – 4078.

[21] G. Pin, D. Raimondo, L. Magni, and T. Parisini, "Robust model predictive control of nonlinear systems with bounded and state-dependent uncertainties," *IEEE Trans. on Automatic Control*, vol. 54, no. 7, pp. 1681–1687, 2009.

[22] E. Kerrigan and J.M.Maciejowski, "Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control," in *Proc.IEEE Conf. Decision and Control*, 2000.

[23] G. Pin and T. Parisini, "Robust minimum-time constrained control of nonlinear discrete-time systems: New results," *DI3 Technical Report. Available at* `http://control.units.it`, no. DI3.2011.1, 2011.