2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Improvement of Moving Horizon Estimators via Direct Virtual Sensor techniques

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Abstract— **This paper presents a novel approach to the design of estimators for nonlinear systems. The proposed methodology is based on a combination of linear model-based Moving Horizon Estimation (MHE) and nonlinear Direct Virtual Sensor (DVS) techniques. Stability of the designed estimator is guaranteed and convex constraints on the variables to be estimated can be easily taken into account. Moreover, the optimality of the approach with respect to an "ideal" MHE (obtained by assuming exact knowledge of the system dynamics and of the global solution of the related nonlinear program) is analyzed. The approach is tested on a nonlinear mass-spring-damper system.**

I. INTRODUCTION

In this paper, we study the problem of estimating, at each time step t, a variable of interest v^t in a nonlinear discrete-time dynamical system. The variable v^t is assumed to be a nonlinear function of the system state x^t and input u^t, and it can be subject to constraints. Estimation problems for nonlinear systems are in general very difficult. The common approach is to obtain approximate solutions such as extended Kalman filters, [1], [2], [3], unscented Kalman filters, [4], ensemble filters, [5], or particle filters, [6], [7], [8]. However, no optimality properties are usually guaranteed by these approximations, even the stability of the estimation error is often not ensured. One of the few filtering techniques that are able to effectively cope with these issues is Moving Horizon Estimation (MHE) (see e.g. [9], [10]). In MHE, at each time step t an estimate $\hat{v}^t$ of v^t is computed, by solving a constrained optimization problem, which involves the simulation of a model of the system and the optimization of an estimate of the initial state some τ steps in the past. Such an optimization procedure is repeated at each time step, in a moving horizon (or receding horizon) fashion [9]. Interesting features of MHE are the possibility to treat in a quite straightforward way nonlinear models, and to include state and input constraints in the formulation. Moreover, by suitably designing the cost function of the underlying optimization problem, stability of the estimation error can be also guaranteed [9]. However, it has to be noted that MHE relies heavily on the knowledge of a system model, and this model may result to be inaccurate with consequent degradation of the estimation accuracy. Moreover, when nonlinear models are used, the resulting optimization problem may be not convex and finding a global minimum may thus involve a high computational complexity. On the other hand, when used with a linear model that has sufficiently good accuracy in a neighborhood of some system operating point, the MHE optimization problem is convex and it can be efficiently solved, and the resulting filter (referred to as "convex MHE" here) usually gives very good performance when the system state is close to such operating conditions.

In this paper, we propose a method that allows one to improve the performance of a convex MHE, when the system operating conditions are different from the ones pertaining to the linear model embedded in the MHE filter itself. At the same time, the method is able to exploit the good accuracy that the convex MHE achieves when the underlying linear model is accurate. The method can be applied to any convex MHE and it is based on the concept of Direct Virtual Sensor (DVS), i.e. a filtering algorithm derived directly from a finite number of measured data, without using a system model [11], [12]. We study here the features of this new technique, and we apply it in a simulation example, where the problem of estimating the state of a nonlinear mass-spring-damper system is considered.

II. PROBLEM FORMULATION

Consider a discrete-time, generally nonlinear system described in state-space form:

$$\begin{array}{l} x^{t+1} = F\left(x^t, \widetilde{u}^t, w^t\right) \\ \widetilde{y}^t = H_y\left(x^t, \widetilde{u}^t, w^t\right) \\ v^t = H_v\left(x^t, \widetilde{u}^t\right) \end{array} \tag{1}$$

where $t \in \mathbb{N}$, $x^t \in \mathbb{R}^{n_x}$ is the system state, $\widetilde{u}^t \in \mathbb{R}^{n_u}$ is the measured input, $\widetilde{y}^t \in \mathbb{R}^{n_y}$ is a measured output, $w^t \in \mathbb{R}^{n_w}$ is an unmeasured disturbance, and $v^t \in \mathbb{R}^{n_v}$ is an unmeasured variable of interest. Note that in (1) the disturbance w is a vector that includes both process disturbance and measurement noise.

In the following, a sequence of input values starting from time step t_1 up to time step t_2 will be denoted by $\tilde{U}_{t_1}^{t_2} = \{\tilde{u}^t\}_{t=t_1}^{t=t_2}$. Likewise, $\tilde{Y}_{t_1}^{t_2}$ and $W_{t_1}^{t_2}$ denote sequences of outputs and disturbances. The predicted trajectory of the state of system (1) at time step t obtained by starting from the state x^{t-j} at time step $t-j$ and by applying given sequences of inputs $\tilde{U}_{t-j}^{t-1}$ and disturbances W_{t-j}^{t-1} is indicated as $x(t, t-j, x^{t-j}, \tilde{U}_{t-j}^{t-1}, W_{t-j}^{t-1})$, while the disturbance-free predicted trajectory (i.e. $W_{t-j}^{t-1} = 0$) is denoted by $x(t, t-j, x^{t-j}, \tilde{U}_{t-j}^{t-1})$. The predicted output at time step t starting from the state x^{t-j} at time step $t-j$ and applying given sequences of inputs $\tilde{U}_{t-j}^{t}$ and W_{t-j}^{t} is denoted by

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This research has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n. PIOF-GA-2009-252284 - Marie Curie project "Innovative Control, Identification and Estimation Methodologies for Sustainable Energy Technologies."

$y(t,t-j,x^{t-j},\tilde{U}^t_{t-j},W^t_{t-j})$, while $y(t,t-j,x^{t-j},\tilde{U}^t_{t-j})$ denotes the disturbance-free predicted output. Similarly, $v(t,t-j,x^{t-j},\tilde{U}^t_{t-j},W^{t-1}_{t-j})$ and $v(t,t-j,x^{t-j},\tilde{U}^t_{t-j})$ denote the predicted unmeasured variable v in the presence and in the absence of disturbances, respectively.

It is assumed that at any step t the disturbance w^t and the variable v^t are contained inside convex compact sets $\mathcal{W} \in \mathbb{R}^{n_w}$ and $\mathcal{V} \in \mathbb{R}^{n_v}$ respectively. The sets $\mathcal{W}$ and $\mathcal{V}$ are usually chosen on the basis of the available physical insight of the system. Finally, the system (1) is assumed to be uniformly observable (see e.g. [13]), i.e. for any two state values x^t_1, x^t_2 there exists a finite number of time steps N_o and a $\mathcal{K}$-function ζ such that, for any given sequence of inputs $\overline{U}_t^{t+N_o-1}$:

$$\zeta(\|x^t_1-x^t_2\|) \leq \sum_{j=0}^{N_o-1} \|y(t+j,t,x^t_1,\overline{U}_t^{t+j})-y(t+j,t,x^t_2,\overline{U}_t^{t+j})\|.$$

We recall that a continuous function $\zeta(z)$ is a $\mathcal{K}$-function if it is strictly monotone increasing, $\zeta(0)=0$, $\zeta(z)>0$ for any $z \neq 0$ and $\lim_{z\to\infty} = \zeta(z) = \infty$. In the described framework, the problem considered in this work can be stated as follows:

Problem 1: Find a function f ("estimator", "estimation algorithm" or "filter") that computes, at each time step t, an estimate $\widehat{v}^t \approx v^t$ such that $\widehat{v}^t \in \mathcal{V}$, whose estimation error $e^t = v^t - \widehat{v}^t$ is bounded in some norm and possibly minimal with respect to a suitable optimization criterion. ■

Obviously, if $v^t = x^t$ the described problem is equivalent to a state estimation problem, but in general one could be interested in estimating also other system variables. Due to the presence of constraints on the variable v, it is not easy to solve Problem 1 even in the case of a linear system (i.e. with linear functions F, H_y and H_v). Moreover, the presence of nonlinearities further increases the difficulty of Problem 1.

III. Moving Horizon Estimation

A. Nonlinear and convex Moving Horizon Estimators

Most of the design techniques employed in the literature to address Problem 1 rely on the knowledge of the system equations (1), of an initial estimate $\overline{x}^{t-j}$ of the system state at a suitably chosen time step $t-j$ and, finally, of given sequences of past measured input and output values, $\tilde{Y}^t_{t-\tau}$ and $\tilde{U}^t_{t-\tau}$ respectively, up to a finite number $\tau+1$ of past time steps. Among such design techniques, Moving Horizon Estimation (MHE) is widely recognized as one of the most promising due to its capability to take into account explicitly system nonlinearities and constraints. In MHE, a cost function of the following form is considered:

$$\begin{aligned} &J(\widehat{x}^{t-\tau},\tilde{U}^t_{t-\tau},W^t_{t-\tau},\tilde{Y}^t_{t-\tau},\overline{x}^{t-\tau}) = \\ &\sum_{j=0}^{\tau} L(e_y^{t-\tau+j},w^{t-\tau+j}) + \Phi(\overline{x}^{t-\tau},\widehat{x}^{t-\tau}), \end{aligned} \quad (2)$$

where the output error $e_y^{t-\tau+j}$, $j=0,\dots,\tau$ is defined as:

$$\begin{aligned} &e_y^{t-\tau+j} = \tilde{y}^{t-\tau+j} \\ &-y(t-\tau+j,t-\tau,\widehat{x}^{t-\tau},\tilde{U}^{t-\tau+j}_{t-\tau},W^{t-\tau+j}_{t-\tau}). \end{aligned} \quad (3)$$

In (2) the initial state guess $\overline{x}^{t-\tau}$ and the sequences $\tilde{Y}^t_{t-\tau}$, $\tilde{U}^t_{t-\tau}$ of measured outputs and inputs are known parameters in the optimization, while the initial state estimate $\widehat{x}^{t-\tau}$ and the disturbance sequence $W^t_{t-\tau}$ are optimization variables. The length $N=\tau+1$ of $\tilde{U}^t_{t-\tau}$ and of $\tilde{Y}^t_{t-\tau}$ is a design parameter, as well as the stage cost function $L(\cdot,\cdot)$ and the initial cost function $\Phi(\cdot,\cdot)$. Then, Problem 1 is cast in a numerical optimization framework, in which the following minimization problem has to be solved:

$$\min_{\widehat{x}^{t-\tau},W^t_{t-\tau}} J(\widehat{x}^{t-\tau},\tilde{U}^t_{t-\tau},W^t_{t-\tau},\tilde{Y}^t_{t-\tau},\overline{x}^{t-\tau}) \quad (4a)$$

subject to

$$\begin{aligned} &v(t-\tau+j,t-\tau,\widehat{x}^{t-\tau},\tilde{U}^{t-\tau+j}_{t-\tau},W^{t-\tau+j}_{t-\tau}) \in \mathcal{V}, \\ &\qquad \forall j \in [0,\tau] \\ &w(t-\tau+j) \in \mathcal{W},\ \forall j \in [0,\tau]. \end{aligned} \quad (4b)$$

If a solution $(\widehat{x}^{t-\tau*},W^{t*}_{t-\tau})$ to (4) is found, the estimate $\widehat{v}^{\text{MHE}}$ is computed as the predicted value of v starting from the optimized initial state guess $\widehat{x}^{t-\tau*}$ and applying the optimal sequence $W^{t*}_{t-\tau}$ and the measured sequence $\tilde{U}^t_{t-\tau}$:

$$\widehat{v}^{t,\text{MHE}} = v(t,t-\tau,\widehat{x}^{t-\tau*},\tilde{U}^t_{t-\tau},W^{t*}_{t-\tau}). \quad (5)$$

Finally, problem (4) is solved at each time step after having updated the sequences $\tilde{Y}^t_{t-\tau}$ and $\tilde{U}^t_{t-\tau}$ with new measurements, according to the following Moving Horizon algorithm:

Algorithm 1: Moving Horizon Estimation

1) At time step t, update the sequences $\tilde{Y}^t_{t-\tau}$ and $\tilde{U}^t_{t-\tau}$ with the measured variables $\tilde{y}^t$, $\tilde{u}^t$;
2) update the initial state guess as $\overline{x}^{t-\tau} = x(t-\tau,t-\tau-1,\widehat{x}^{t-\tau-1*},\tilde{U}^{t-\tau}_{t-\tau-1},W^{t-\tau*}_{t-\tau-1})$, where $W^{t-\tau*}_{t-\tau-1}$ is part of the the optimal disturbance sequence $W^{t-1*}_{t-\tau-1}$, computed at time step $t-1$, and $\widehat{x}^{t-\tau-1*}$ is the optimal initial state computed at time step $t-1$;
3) solve the optimization problem (4) to compute $\widehat{x}^{t-\tau*}$ and $W^{t*}_{t-\tau}$;
4) compute the estimate $\widehat{v}^{t,\text{MHE}}$ given by (5);
5) repeat the procedure from step 1) by setting $t=t+1$.

The resulting MHE, when the system (1) is nonlinear, is named here "nonlinear MHE". If the dynamical system underlying the optimization problem (4) is linear and functions $L(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are chosen to be convex, the optimization problem to be solved results to be convex with respect both to the optimization variables $\widehat{x}^{t-\tau}$, $W^t_{t-\tau}$ and to the parameters $\overline{x}^{t-\tau}$, $\tilde{U}^t_{t-\tau}$, $\tilde{Y}^t_{t-\tau}$. In this case, the resulting MHE is named here "convex MHE".

Although nonlinear MHE is potentially a powerful approach whose diffusion is increasing, some issues are still open.

One of these problems is that the nonlinear program (NLP) (4) is in general non-convex. In this case, finding the global minimum of (4) may be extremely hard, whereas local minima of this function, which are easier to be found, may lead to poor estimates and/or "jumps" in the estimated variable between two subsequent time steps.

A second relevant drawback is that, in many practical applications, the online implementation of a nonlinear MHE can not be performed, since it requires to solve the NLP (4) at each sampling time, and this task cannot be performed online if the sampling time is too small.

Finally, another important problem, shared by the MHE approach with all the other model-based design methods (e.g. Extended Kalman Filter) is that the system (1) in most practical situations is not known.

On the other hand, in the case of a convex MHE the optimization problem (4) can be solved efficiently. Now, in many practical cases some information on the dynamics of the system is available, as well as on the values of the related physical parameters, and a linearized system model, in some nominal operating condition $(\overline{x}, \overline{u}, \overline{w})$ of interest, can be easily obtained. In these cases, convex MHEs are able to provide a quite accurate estimate and to also take into account constraints on the variable to be estimated.

The novel idea of this paper is to exploit the advantages of a convex MHE designed by using a linear system model, thus obtaining a good estimate of $\widehat{v}^{t,MHE} \simeq v^t$ at least when the system operates in a neighborhood of $(\overline{x}, \overline{u}, \overline{w})$, and to then improve the obtained filter by means of a so-called Direct Virtual Sensor (DVS) approach, based on Nonlinear Set Membership (NSM) function approximation theory. In particular, the DVS-NSM approach can be used to derive an approximation $\widehat{\Delta}^{NSM,t}$ of the residue signal $\Delta^t = v^t - \widehat{v}^{t,MHE}$ directly from the data measured in a preliminary experiment, in which v_t is also measured (and, consequently, a measure of Δ^t is available). Then, the value of $\widehat{\Delta}^{NSM,t}$ is used to improve the estimation accuracy in those operating conditions in which the convex MHE gives poor results.

Before introducing the DVS-NSM approach, it is now useful to make some more considerations on the structure of a stable MHE, either nonlinear or convex, and on the regularity properties of a convex MHE.

B. Structural properties of stable MHE estimators

Once the design parameters N, $L(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ have been chosen, the MHE Algorithm 1 can be regarded to as a function f^{MHE} whose arguments are the initial state guess $\overline{x}^{t-\tau}$ and the measured sequences $\tilde{Y}^t_{t-\tau}$ and $\tilde{U}^t_{t-\tau}$:

$$\widehat{v}^{t,\mathrm{MHE}} = f^{\mathrm{MHE}}(\tilde{Y}^t_{t-\tau}, \tilde{U}^t_{t-\tau}, \overline{x}^{t-\tau}). \tag{6}$$

Moreover, from step 2) of the MHE Algorithm 1 it can be noted that at each time step t the initial state guess $\overline{x}^{t-\tau}$ is a function of the sequences $\tilde{Y}^{t-1}_{t-\tau-1}$ and $\tilde{U}^{t-1}_{t-\tau-1}$ and of the initial state guess $\overline{x}^{t-\tau-1}$, i.e.:

$$\overline{x}^{t-\tau} = g(\tilde{Y}^{t-1}_{t-\tau-1}, \tilde{U}^{t-1}_{t-\tau-1}, \overline{x}^{t-\tau-1}).$$

Then, the estimate (6) can be also expressed as

$$\widehat{v}^{t,\mathrm{MHE}} = f^{\mathrm{MHE}}(\tilde{Y}^t_{t-\tau-1}, \tilde{U}^t_{t-\tau-1}, \overline{x}^{t-\tau-1}), \tag{7}$$

where, with a slight abuse of notation,

$$f^{\mathrm{MHE}}(\tilde{Y}^t_{t-\tau-1}, \tilde{U}^t_{t-\tau-1}, \overline{x}^{t-\tau-1}) = \\ f^{\mathrm{MHE}}(\tilde{Y}^t_{t-\tau}, \tilde{U}^t_{t-\tau}, g(\tilde{Y}^{t-1}_{t-\tau-1}, \tilde{U}^{t-1}_{t-\tau-1}, \overline{x}^{t-\tau-1})).$$

Thus, assuming that the MHE algorithm (6) is set up at time step $t_0+\tau$ with an initial state guess $\overline{x}^{t_0}$, the estimate $\widehat{v}^{t,MHE}$ at the generic time t can be also expressed as

$$\widehat{v}^{t,\mathrm{MHE}} = f^{\mathrm{MHE}}(\tilde{Y}^t_{t_0}, \tilde{U}^t_{t_0}, \overline{x}^{t_0}), \tag{8}$$

where f^{MHE} is the function given by the recursive application of (7). A relatively large literature exists (see e.g. [10] and the references therein) regarding the study of sufficient conditions on the system (1), on the constraint sets V, W and on the design parameters N, $L(\cdot,\cdot)$, $\Phi(\cdot,\cdot)$ so that the estimator is asymptotically stable. If f^{MHE} is asymptotically stable, then for any (small) $\mu > 0$ there exists a sufficiently large number of time steps m such that, for any two initial state guesses $\overline{x}^{t_0}_1$, $\overline{x}^{t_0}_2$, it holds that

$$\|f^{\mathrm{MHE}}(\tilde{Y}^{t_0+m}_{t_0}, \tilde{U}^{t_0+m}_{t_0}, \overline{x}^{t_0}_1) - f^{\mathrm{MHE}}(\tilde{Y}^{t_0+m}_{t_0}, \tilde{U}^{t_0+m}_{t_0}, \overline{x}^{t_0}_2)\| \\ \leq \mu.$$

That is, the effect of the initial condition $\overline{x}^{t_0}$ tends to fade away, see [13]. Thus, with arbitrarily good precision, after a suitable (possibly large) number m of time steps it can be considered that the estimate $\widehat{v}^{t_0+m}$ depends only on the sequences $\tilde{Y}^{t_0+m}_{t_0}$, $\tilde{U}^{t_0+m}_{t_0}$ and not on the initial state $\overline{x}^{t_0}$. Then, in general a stable MHE algorithm can be expressed as a Nonlinear Finite Impulse Response (NFIR) estimator f^{MHE}_o plus a "small" truncation error e^t_{trunc}:

$$f^{\mathrm{MHE}}(\tilde{Y}^t_{t_0}, \tilde{U}^t_{t_0}, \overline{x}^{t_0}) = f^{\mathrm{MHE}}_o(\tilde{Y}^t_{t-m}, \tilde{U}^t_{t-m}) + e^t_{trunc}. \tag{9}$$

Furthermore, convex MHEs also enjoy a continuity property. In fact, problem (4) can be regarded to as a parametric optimization problem $\mathcal{P}(s,\theta)$, with optimization variable $s = (\widehat{x}^{t-\tau}, W^t_{t-\tau})$ and parameters $\theta = (\tilde{U}^t_{t-\tau}, \tilde{Y}^t_{t-\tau}, \overline{x}^{t-\tau})$. Now, in the context of multi-parametric programming it has been shown that the optimizer $s^*(\theta) = \arg\min \mathcal{P}(s,\theta)$ is a continuous (in general non-differentiable) function of θ [14]. Thus, being the system linear, the estimate $\widehat{v}^{t,MHE}$ results to be a continuous function of $\overline{x}^{t-\tau}$, $\tilde{U}^t_{t-\tau}$, $\tilde{Y}^t_{t-\tau}$. The parameter $\overline{x}^{t-\tau}$ is, on its turn, a continuous function of $\overline{x}^{t-\tau-1}$, $\tilde{U}^{t-1}_{t-\tau-1}$, $\tilde{Y}^{t-1}_{t-\tau-1}$ and so on backward in time. Therefore, it turns out that, in the case of convex MHEs, the function $f^{\mathrm{MHE}}(\tilde{Y}^t_{t_0}, \tilde{U}^t_{t_0}, \overline{x}^{t_0})$ and the NFIR filter $f^{\mathrm{MHE}}_o(\tilde{Y}^t_{t-m}, \tilde{U}^t_{t-m})$ (9) are continuous with respect to their arguments.

IV. Improvement of MHE via the Direct Virtual Sensors approach

In this Section, the case $v^t \in \mathbb{R}$ and $\mathcal{V} = [\underline{v}, \overline{v}]$ is considered for simplicity of notation. Suppose that a linear model of the system (1) is available, either obtained by linearization of the involved physical laws or identified from experimental data. Let

$$\widehat{f}^{\mathrm{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) \\ \widetilde{\boldsymbol{\varphi}}^t_{t-m} \doteq (\tilde{Y}^t_{t-m}, \tilde{U}^t_{t-m})$$

be a convex MHE designed on the basis of this linear model, and let us define the following *residue function*:

$$\Delta(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) \doteq f^{\mathrm{MHE}}_o(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) - \widehat{f}^{\mathrm{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m})$$

where f^{MHE}_o is the unknown nonlinear MHE estimator defined in (9).

The approach proposed in this paper is to identify, directly from a set of data generated by the system (1), a NFIR approximation $\widehat{\Delta}(\widetilde{\boldsymbol{\varphi}}^t_{t-m})$ of $\Delta(\widetilde{\boldsymbol{\varphi}}^t_{t-m})$ enjoying suitable optimality properties, and to obtain an Improved Moving Horizon Estimator (IMHE) $\widehat{f}^{\mathrm{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) + \widehat{\Delta}(\widetilde{\boldsymbol{\varphi}}^t_{t-m})$, allowing us to get accurate estimates even when the system (1) is not operating in linearity conditions. Such an IMHE estimator is also called Direct Virtual Sensor (DVS), see [12]. The following problem is thus considered in the next Section:

Problem 2: Given a set of data

$$D = \{\widetilde{u}^t, \widetilde{y}^t, \widetilde{v}^t, t = 1, 2, \ldots, L\} \tag{10}$$

where $\widetilde{v}^t \doteq v^t + \xi^t$ is the measured value of v^t corrupted by the noise ξ^t, find an estimator of the form

$$\widehat{v}^t = \widehat{f}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) = \widehat{f}^{\text{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) + \widehat{\Delta}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) \tag{11}$$

with estimation error minimal with respect to a suitable criterion. ■

The estimator $\widehat{f}$ is selected within the following set of Lipschitz continuous functions

$$\begin{aligned}\mathcal{F}(\gamma, m) \doteq \Big\{ \widehat{f}^{\text{MHE}} + g : \Big| g(\boldsymbol{\varphi}^t_{t-m}) - g(\widehat{\boldsymbol{\varphi}}^t_{t-m}) \Big| \\ \leq \gamma \left\| \boldsymbol{\varphi}^t_{t-m} - \widehat{\boldsymbol{\varphi}}^t_{t-m} \right\|_\infty, \forall \boldsymbol{\varphi}^t_{t-m}, \widehat{\boldsymbol{\varphi}}^t_{t-m} \in \Phi \Big\}\end{aligned} \tag{12}$$

where $\|\cdot\|_\infty$ is the ℓ_∞ norm, $\gamma \geq 0$ is the Lipschitz constant, and the regressor domain Φ is a bounded convex subset of $\mathbb{R}^{(m+1)(n_y+n_u)}$. The motivation for considering this set is to ensure the estimator stability.

Let us define the estimator f_o as the best approximation within the set $\mathcal{F}(\gamma, m)$ of the MHE estimator f_o^{MHE} in (9):

$$f_o \doteq \arg \min_{f \in \mathcal{F}(\gamma, m)} \left\| f_o^{\text{MHE}} - f \right\|_\infty$$

where $\|f\|_\infty \doteq \operatorname{ess\,sup}_{\boldsymbol{\varphi} \in \Phi} |f(\boldsymbol{\varphi})|$ is the L_∞ functional norm. Note that, if the MHE optimization problem (4) is convex in both the optimization variables and the parameters, then $f_o^{\text{MHE}} \in \mathcal{F}(\gamma, m)$ (see e.g. [14]) and, consequently, $f_o = f_o^{\text{MHE}}$. In other words, f_o is an "ideal" MHE, i.e. a Lipschitz continuous MHE, obtained by assuming that an exact model of the system dynamics is available, and that the global optimum of the MHE optimization problem can be computed. Here, f_o is assumed to be unknown, and an optimal approximation of it of the form (11) is looked for.

To this aim, consider that the estimation error of an estimator $\widehat{f}$ of the form (11) is bounded as

$$\begin{aligned}\left| v^t - \widehat{f}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \right| = \left| e^t_o + f_o\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) - \widehat{f}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \right| \leq \\ \leq |e^t_o| + \left| f_o\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) - \widehat{f}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \right|\end{aligned} \tag{13}$$

where $e^t_o \doteq v^t - f_o\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)$ is the estimation error of f_o and $|f_o\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) - \widehat{f}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)|$ is the bias between the estimator $\widehat{f}$ and f_o. Since e_o does not depend on $\widehat{f}$, the aim is to reduce the bias.

Clearly, this bias is not known, since f_o is not known. In order to derive a bound on it, some assumptions on e^t_o and $\xi^t \doteq \widetilde{v}^t - v^t$ (the measurement error on v^t, see Problem 2) are required. Here, e^t_o is assumed to be bounded as

$$\left| e^t_o \right| \leq \delta_o, \ \forall t$$

for some $\delta_o \geq 0$. Note that, being $f_o \in \mathcal{F}(\gamma, m)$, this assumption is satisfied if the MHE (8) is asymptotically stable. Assuming that ξ^t is also bounded, we have that the noise

$$d^t \doteq \widetilde{v}^t - f_o\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) = \xi^t + e^t_o$$

is bounded as

$$\left| d^t \right| \leq \varepsilon, \ \forall t$$

for some $\varepsilon \geq 0$.

Note that the values of the bound ε on noise and the Lipschitz constant γ can be suitably chosen by means of the validation procedure in [15]. The value of δ_o is not required for the design of the optimal DVS presented in the following.

On the basis of the above assumptions, the Feasible Estimators Set is now defined.

Definition 1: Feasible Estimators Set:

$$FES \doteq \left\{ f \in \mathcal{F}(m, \gamma) : \left| \widetilde{v}^t - f\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \right| \leq \varepsilon, t \in [1, L] \right\}.$$

■

According to this definition, FES is the smallest set guaranteed to contain f_o. The tightest bound on the bias in (13) is thus given by $\sup_{f \in FES} \left| f\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) - \widehat{f}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \right|$, leading to the following definition of worst-case estimation error.

Definition 2: Worst-case estimation error of a DVS $\widehat{f}$:

$$ED\left(\widehat{f}, t\right) \doteq \delta_o + \sup_{f \in FES} \left| f\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) - \widehat{f}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \right|. \tag{14}$$

■

Looking for a DVS that minimizes this error, leads to the following optimality concept.

Definition 3: A DVS f^* is optimal if

$$ED\left(f^*\right) = \inf_f ED\left(f\right), \quad \forall t.$$

■

Let us now define the DVS

$$\widehat{v}^t = f_c\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)$$

where

$$f_c\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) \doteq \widehat{f}^{\text{MHE}} + \frac{1}{2}\left[\overline{\Delta}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) + \underline{\Delta}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)\right] \tag{15}$$

$$\begin{aligned}\overline{\Delta}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) &\doteq \min\left[\overline{v} - \widehat{f}^{\text{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}), \overline{\Lambda}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)\right] \\ \underline{\Delta}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) &\doteq \max\left[\underline{v} - \widehat{f}^{\text{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}), \underline{\Lambda}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)\right] \\ \overline{\Lambda}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) &\doteq \min_{k \in [1, L]}\left(\Delta\widetilde{v}^k + \varepsilon + \gamma\left\|\widetilde{\boldsymbol{\varphi}}^t_{t-m} - \widetilde{\boldsymbol{\varphi}}^k_{k-m}\right\|_\infty\right) \\ \underline{\Lambda}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) &\doteq \max_{k \in [1, L]}\left(\Delta\widetilde{v}^k - \varepsilon - \gamma\left\|\widetilde{\boldsymbol{\varphi}}^t_{t-m} - \widetilde{\boldsymbol{\varphi}}^k_{k-m}\right\|_\infty\right)\end{aligned} \tag{16}$$

and $\Delta\widetilde{v}^k \doteq \widetilde{v}^k - \widehat{f}^{\text{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m})$.

Theorem 1: i) The DVS f_c is optimal.
ii) The following tight bounds on v^t hold:

$$\begin{aligned}\widehat{f}^{\text{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) + \underline{\Delta}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) - \delta_o \leq v^t \\ \leq \widehat{f}^{\text{MHE}}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) + \overline{\Delta}(\widetilde{\boldsymbol{\varphi}}^t_{t-m}) + \delta_o.\end{aligned}$$

iii) The worst-case estimation error of f_c is given by

$$ED\left(f_c, t\right) = \delta_o + \frac{1}{2}\left[\overline{\Delta}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right) - \underline{\Delta}\left(\widetilde{\boldsymbol{\varphi}}^t_{t-m}\right)\right]. \tag{17}$$

iv) The constraints are satisfied:

$$\underline{v} \leq f_c\left(\widetilde{\boldsymbol{\varphi}}_{t-m}^t\right) \leq \overline{v}. \tag{18}$$

Proof. See [16]. ∎

V. SIMULATION EXAMPLE

Consider the equations of motion of a nonlinear mass-spring-damper system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\beta(x_1(t))x_2(t) + \kappa(x_1(t))x_1(t) + u(t) \\ y(t) &= x_1(t) + w(t) \end{aligned} \tag{19}$$

where $u(t)$ is the input force in N, $w(t)$, $|w(t)| \leq 0.025$ is a uniformly distributed measurement noise, $x_1(t)$ is the mass position in m, $x_2(t)$ is the speed in m/s, and

$$\begin{aligned} \kappa(x_1) &= a_0 \exp(a_1 x_1) + a_2 \\ \beta(x_1) &= a_0 \exp(a_3 x_1) + a_2. \end{aligned} \tag{20}$$

The parameter values are $a_0 = 0.7, a_1 = -1, a_2 = 0.3, a_3 = -2$. Moreover, the speed $x_2(t)$ is mechanically saturated between ± 1 m/s:

$$x_2(t) \in \mathbb{X} = [-1, 1], \forall t. \tag{21}$$

The origin is a globally asymptotically stable fixed point and the system is input-to-state stable, so that experiments can be carried out in open-loop.

A first experiment has been performed, assuming that all the states can be measured, to identify a second-order, discrete time LTI model of the system (19). This model is of the form

$$\begin{aligned} \left[\begin{array}{c} x_1^{t+1} \\ x_2^{t+1} \end{array}\right] &= A \left[\begin{array}{c} x_1^t \\ x_2^t \end{array}\right] + Bu^t \\ y^t &= C \left[\begin{array}{c} x_1^t \\ x_2^t \end{array}\right] + w^t, \end{aligned}$$

with sampling time $t_s = 0.05$ s. A uniformly distributed random input $u(t)$ with amplitude 0.4 N, plus a sequence of zero-mean square wave signals and sinusoids of increasing amplitudes, from 0.4 N to 0.6 N, has been injected for 1000 s to the system. A uniform random noise $r^t : |r^t| \leq 0.025$ has been added to all of the measured quantities, i.e. $\tilde{x}^t$, $\tilde{u}^t$, $\tilde{y}^t$. The model matrices A, B and C, identified via least squares, are

$$A = \left[\begin{array}{cc} 0.988 & 0.043 \\ 0.927 & -0.48 \end{array}\right], \; B = \left[\begin{array}{c} 0.009 \\ 0.053 \end{array}\right] \; C = \left[\begin{array}{cc} 1 & 0 \end{array}\right]. \tag{22}$$

It can be noted that the matrices (22) are close to the linearized and discretized equations of the system (19) at $x \simeq 0$. Therefore, a good accuracy of the MHE filter in linear operating conditions is expected. The cost function used in the MHE has been designed according to (2), with $N = 3$ and

$$\begin{aligned} L(e_y, w) &= Qe_y^2 + Rw^2 \\ \Phi(\hat{x}, \overline{x}) &= (\hat{x} - \overline{x})^T Q_x(\hat{x} - \overline{x}), \end{aligned} \tag{23}$$

where

$$Q = 10, \; R = 1, \; Q_x = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0.01 \end{array}\right] \tag{24}$$

where $(\hat{x} - \overline{x})^T$ indicates the transpose of $(\hat{x} - \overline{x})$. Moreover, the constraint (21) has been included in the optimization problem (4), which results to be a quadratic program.

A second 1000-s-long experiment has been carried out to collect the data $\tilde{x}^t$, $\hat{x}^{t,\mathrm{MHE}}$, and the related values of the regressor $\widetilde{\boldsymbol{\varphi}}_{t-m}^t$, with $m = 6$. A uniform random measurement noise $r^t : |r^t| \leq 0.025$ has been added to all the measured quantities. The values of N, Q, R, Q_x, m, have been tuned by trial and error procedures in order to achieve the best performance of the MHE filter and of the related DVS correction. For each state x_1^t and x_2^t, an IMHE estimator of the form (15) has been designed. The related parameters γ_1, ε_1 and γ_2, ε_2 have been estimated according to the guidelines given in [15]. In particular, the values $\gamma_1 = 10^{-7}, \varepsilon_1 = 0.07, \gamma_2 = 4.37$ and $\varepsilon_2 = 0.07$ have been chosen. Moreover, the regressor $\widetilde{\boldsymbol{\varphi}}_{t-m}^t$ has been scaled in order to adapt to the properties of the collected data (see [15] for more details). Note that the quite low value of γ_1 indicates that the estimation errors on this variables have low variability with respect to $\widetilde{\boldsymbol{\varphi}}_{t-m}^t$. This is reasonable, since the first variable is directly measured and the related estimation error is practically negligible and due to the measurement noise only. On the other hand, a higher estimation error occurs for the second state variable, so that the DVS technique presented in this paper is actually able to provide a significant improvement.

The designed MHE estimator and its improved version, IMHE, have been tested in a third experiment, performed by injecting square waves of varying amplitudes to the system. Also in this experiment, all the measured quantities have been corrupted by a uniform random measurement noise r^t : $|r^t| \leq 0.025$. Note that the considered noise amplitude is quite large, corresponding to a noise-to-signal ratio of about 10% in average.

TABLE I

SIMULATION EXAMPLE. BIAS OF THE MHE AND IMHE FILTERS WITH DIFFERENT INPUT SQUARE WAVES.

Input ampl. (N)	0.5	1	2	2.5
MHE, x_1^t (m)	$1.0\cdot10^{-3}$	$7.0\cdot10^{-3}$	$3.2\cdot10^{-2}$	$4.0\cdot10^{-2}$
IMHE, x_1^t (m)	$-1.7\cdot10^{-3}$	$6.3\cdot10^{-4}$	$8.5\cdot10^{-4}$	$1.0\cdot10^{-2}$
MHE, x_2^t (m/s)	$1.6\cdot10^{-2}$	$9.0\cdot10^{-2}$	$4.5\cdot10^{-1}$	$5.5\cdot10^{-1}$
IMHE, x_2^t (m/s)	$-3.8\cdot10^{-3}$	$2.0\cdot10^{-3}$	$-1.6\cdot10^{-2}$	$-3.4\cdot10^{-3}$

TABLE II

SIMULATION EXAMPLE. RMSE OF THE MHE AND IMHE FILTERS WITH DIFFERENT INPUT SQUARE WAVES.

Input ampl. (N)	0.5	1	2	2.5
MHE, x_1^t (m)	$8.0\cdot10^{-3}$	$1.1\cdot10^{-2}$	$4.1\cdot10^{-2}$	$5.0\cdot10^{-2}$
IMHE, x_1^t (m)	$8.0\cdot10^{-3}$	$8.5\cdot10^{-3}$	$9.9\cdot10^{-3}$	$2.9\cdot10^{-3}$
MHE, x_2^t (m/s)	$2.4\cdot10^{-2}$	$1.0\cdot10^{-1}$	$5.4\cdot10^{-1}$	$6.6\cdot10^{-1}$
IMHE, x_2^t (m/s)	$1.5\cdot10^{-2}$	$2.3\cdot10^{-2}$	$9.1\cdot10^{-2}$	$5.9\cdot10^{-2}$

A first square wave with amplitude equal to 0.5 N has been used to test the estimators in linear system operating conditions. The following square waves, with growing amplitudes up to 2.5 N, have been used to test the filters when nonlinearities are gradually predominant. The obtained results, in terms of bias and Root Mean Square Error (RMSE), are reported in Tables I and II, respectively. The estimate of the first state

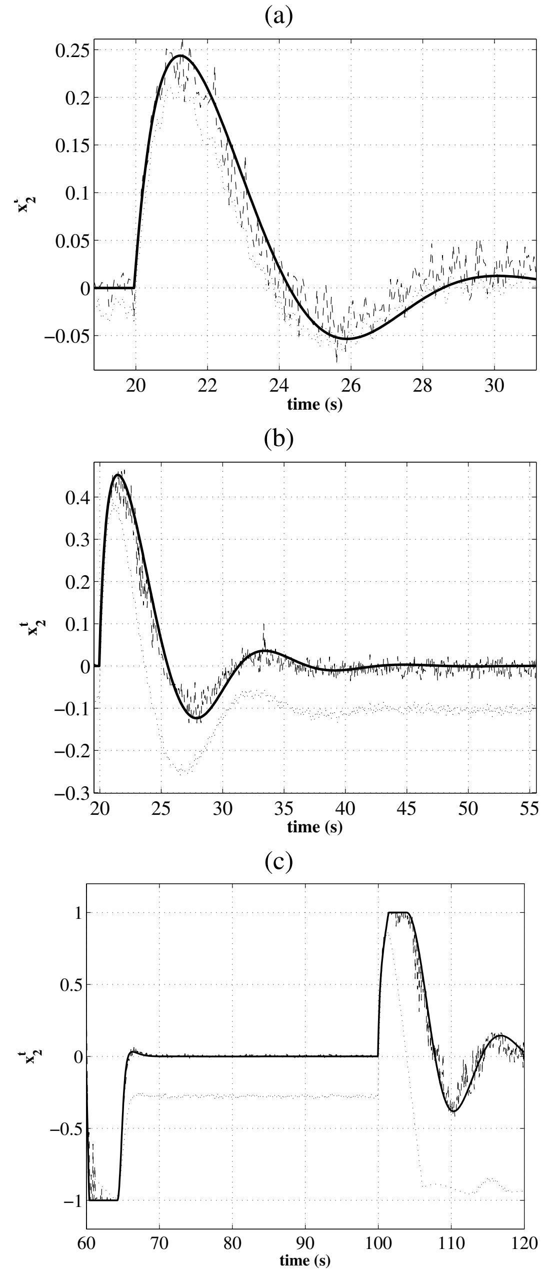

Fig. 1. Simulation example. Courses of the speed x_2^t (solid line) and of its estimates $\hat{x}_2^{t,\text{MHE}}$ (dotted line), $\hat{x}_2^{t,\text{IMHE}}$ (dashed line) with a square wave input with amplitude equal to (a) 0.5, (b) 1 and (c) 2.5

variable is of little interest, since it is directly measured and both estimators achieve very good accuracy. As regards the second state variable, it can be noted that the MHE and IMHE filters give quite similar results in linear operating conditions, as expected, while in nonlinear conditions the IMHE is able to achieve a significant improvement with respect to the MHE. As an example, the time course of x_2^t and of its estimates $\hat{x}_2^{t,\text{MHE}}$, $\hat{x}_2^{t,\text{IMHE}}$ provided by the MHE and IMHE filters, respectively, are shown in Fig. 1(a)-(c). The different performance of the MHE in linear operating conditions (Fig. 1(a)) and in nonlinear ones (Fig. 1(b)-(c)) is evident, as well as the quite good behavior of the IMHE. Fig. 1(c) also shows that the IMHE filter is able to correctly handle the constraint (21).

VI. Conclusions

A method for the design of an improved MHE estimator for nonlinear systems has been presented in the paper. Such an estimator is defined as the sum of a convex MHE and of a nonlinear DVS part. The convex MHE achieves good estimation performance when the system operates close to linear conditions. The DVS part compensates the mismatch between the convex MHE and the nonlinear behavior of the system outside these linear conditions. The improved MHE estimator is stable by construction and is able to account for constraints on the variable to be estimated. It has also been proven that, under mild assumptions, this estimator is optimal (in the sense of worst-case estimation error minimization). The effectiveness of the proposed method has been shown through a simulation example related to state estimation of a nonlinear mass-spring-damper system.

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