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Second-Order Uniform Exact Sliding Mode Control with Uniform Sliding Surface

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Abstract— **In this paper we propose an uniform sliding mode controller for a second order uncertain system providing convergence to an arbitrary small vicinity of origin in finite time, which can be bounded by some constant *independent from initial conditions and uncertainties*. With this aim a nonlinear sliding surface is proposed ensuring during the sliding motions an uniform convergence of the trajectories to any arbitrary small vicinity of origin in finite time bounded by some constant independent from initial conditions on the surface.**

I. INTRODUCTION

The problem of robust prescribed time stabilization is one of the actual tasks in modern control theory. For example, controlling hybrid systems with strictly positive dwell time, it is preferably for control task to provide the robust exact system stabilization before the next of switching or impulse takes place. A reasonable class of controllers providing both: finite time convergence and insensitivity with respect to matched uncertainties/disturbances are sliding mode controllers (see, for example [15]).

Traditional sliding mode control design consists of two steps, [15]: (a) design of the sliding surface ensuring desired behavior of system without uncertainties; (b) design of discontinuous controllers enforcing the sliding motions and compensation of matched uncertainties. The main disadvantage of such methodology is the so called chattering phenomenon restricting the possibilities of usage of first order sliding mode controllers to the hardware, where the switching is a natural mode of work.

A super-twisting controller allows to adjust the chattering problem in the system with Lipschitz continuous uncertainties/disturbances [10]. It opened the door for usage of the properties of sliding modes to practically all continuous controllers. It is necessary to remark that for both classical and super-twisting control design methodologies the convergence time grows together with initial conditions, i.e., (a) the time of convergence to the sliding surface grows together with initial conditions; (b) even if the trajectory is starting on the sliding surface the convergence time to the given vicinity of origin on sliding surface is also growing together with initial conditions on the surface. The uniform super-twisting based differentiator and observer for mechanical systems was designed in [3], [4], ensuring the convergence of differentiator/observer in finite time bounded by some constant independent from initial conditions. For the case of a sliding mode controller, the methodology of these papers can ensure the uniform convergence to the sliding surface only but does not ensure the uniform convergence to origin for the system trajectories on the sliding surface.

On the other hand, the use of nonlinear sliding surfaces instead of linear surfaces of a classical sliding mode control design have proved to enhance the desired performance in closed-loop of the system with sliding mode control algorithms, which can not be always achieved only with linear switching surfaces, (see [2], [1], [14] and references there).

In this paper we propose an exact sliding mode controller for a second order system providing the convergence of the system trajectories to any arbitrary small vicinity of origin in finite time upper bounded by some constant *independent from initial conditions and some class of uncertainties/disturbances*. To achieve this aim: (a) a nonlinear sliding surface is suggested ensuring during sliding motions uniform convergence of the trajectories to any arbitrary small vicinity of origin in finite time upper bounded by some constant independent from initial conditions on the surface; (b) an absolutely continuous super-twisting based controller is suggested providing convergence of the trajectories to the sliding surface in a finite time upper bounded by another constant independent from initial conditions and for a special class of uncertainties/disturbances.

The methodology used in this paper is based on Lyapunov functions proposed in [12], [3], [5], [13], [4]. In the following section we introduce some basic concepts which are useful for the understanding of the paper.

A. Basic Definitions

Consider the following non autonomous dynamic system

$$\dot{x} = f(x) + w(x,t), \tag{1}$$

where $x \in R^n$ are the system states, the functions $f : R^n \to R^n$, $w : R \times R^n \to R^n$, $w(x,t) \in \mathscr{W}_{nv}$ is an uncertainty/disturbance and the class $\mathscr{W}_{nv}$ functions represents a family of non vanishing perturbations at the origin. The functions $f(\cdot)$ and $w(\cdot,\cdot)$ ensure the existence of solutions to the system (1) in the sense of Filippov (1988). Denote a solution trajectory of (1) with the initial condition $x(t_0) = x_0$ and $t_0 \in [0,\infty)$ by $x_s(t,x_0,t_0)$.

Let $B_\mu = \{x : ||x|| < \mu\}$ be a ball centered at the origin with radius $\mu > 0$ and let $T(x_0,\mu)$ be the convergence time

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of some trajectory of (1) from any initial conditions (x_0,t_0) to B_μ, which means that $\forall t > T(x_0,\mu)$ the solution belongs to the ball B_μ,i.e. $x_s(t,x_0,t_0) \in B_\mu$.

Definition 1: Suppose that $w(t,x) = 0$. The origin of (1) is *asymptotically uniformly stable w.r.t. initial conditions (AUS)* if for each μ there exists $T_\mu > T(x_0,\mu) > 0$ such that for any initial condition $x_0 \in B^n$, the trajectory $x_s(t,x_0) \in B_\mu$, $\forall t \geq T_\mu$.

Definition 2: The origin of (1) is

- *uniformly exactly stable (UES) w.r.t. initial conditions (t_0,x_0)* if for any μ and any disturbance $w(x,t) \in \mathscr{W}_{nv}$ there exists $T_\mu > T(x_0,\mu) > 0$ such that for any initial conditions $x_0 \in R^n$ and $t_0 \in [0,\infty)$, $x_s(t,x_0,t_0) \in B_\mu$, $\forall t \geq t_0 + T_\mu$.
- *uniformly finite-time exactly stable (UFTES) w.r.t. initial conditions (t_0,x_0)* if for any $x_0 \in R^n$, any $t_0 \in [0,\infty)$ and any disturbance $w(x,t) \in \mathscr{W}_{nv}$ there exists a constant $T > 0$ *independent from* (x_0,t_0), such that $x_s(t,x_0,t_0) \equiv 0$ holds, $\forall t \geq t_0 + T$.

Note that when $w(t,x) = 0$, (1) is an autonomous system.

B. Standard Sliding Mode Control Design

Consider a controllable single-input uncertain second-order linear time invariant system in regular form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + u + w(x,t), \end{aligned} \tag{2}$$

where $[x_1,x_2] \in \mathbb{R}^2$ is the state vector, $u \in \mathbb{R}^1$ is the control input, the parameters $a_{11}, a_{12} \neq 0, a_{21}.a_{22}$ are constants, $w(x,t) \in \mathscr{W}_0$ is a uncertainty/disturbance in the system (2), which belongs to the class $\mathscr{W}_0 = \{w(x,t) : |w(x,t)| \leq \rho_0, \rho_0 > 0\}$. To design the sliding mode control in the classical sense, it is necessary to design firstly the sliding surface. The linear sliding surface for the system (2) can be usually designed as

$$s = x_2 + c_1 x_1, \tag{3}$$

by some discontinuous control law (see, [15], for example), sliding mode is ensured in a finite time. During the sliding mode $s(x) = 0$, it implies

$$\dot{x}_1 = (a_{11} - a_{12}c_1)x_1, \tag{4}$$

and on the sliding, the above equation describes system dynamics completely determined by c_1. The constant c_1 could be chosen ensuring the desired eigenvalue of (4). Therefore, the convergence time to a neighborhood of the origin becomes unbounded even if the trajectory is started on the sliding surface when the sliding motion is started each time far away. After the design of the sliding surface, a control input must be obtained ensuring the sliding motion in a finite time from some initial condition. The standard sliding mode control design suggests that the control law which enforces the sliding mode should be designed in such a way the sufficient condition for sliding mode existence $s\dot{s} < -n|s|$ is satisfied [15]. For the system (2) a control law that satisfies this condition is designed as

$$u = -(c_1 a_{11} + a_{21})x_1 - (c_1 a_{12} + a_{22})x_2 - Q\,\mathrm{sign}(s), \tag{5}$$

Q is chosen from certain upper bound of the uncertainty, i.e, $\rho_0 \leq Q$.

Using the main idea of the sliding mode control design and considering a certain class of uncertainties/disturbances $\mathscr{W}_{nv}$, our aim is for the linear system (2) is: (a) design a nonlinear sliding surface, such that, when the motion is restricted to the manifold $s = 0$, the reduced-order model converge uniformly (with respect to the initial conditions) to a neighborhood of the origin (Section *II*); (b) design the control input to ensure sliding motion in prescribed time from any arbitrary initial condition and with chattering alleviation (Section *III*).

II. Uniform Sliding Surface

Let us take an uniform sliding dynamic in the form

$$s = x_2 + c_1 x_1 + c_2|x_1|^q\,\mathrm{sign}(x_1), \tag{6}$$

where $c_1, c_2 > 0$ and $q > 1$ are positive scalars. Let us show that if q is bigger than one, the rate convergence makes so strong that it becomes uniform with respect to the initial conditions. Note that in the case $q = 1$, the usual linear sliding surface is obtained.

Introducing (s,x_1) as new state variables (where s is defined as in (6)) and applying the control input

$$u = u_{eq} + v, \tag{7}$$

$$\begin{aligned} u_{eq} = &-[c_1(a_{11} - a_{12}c_1) + (a_{21} - a_{22}c_1)]x_1 - (a_{12}c_1 + a_{22})s \\ &-qc_2|x_1|^{q-1}[(a_{11} - a_{12}c_1)x_1 - a_{12}c_2|x_1|^q\,\mathrm{sign}(x_1) + a_{12}s] \\ &+(c_1 a_{12} + a_{22})c_2|x_1|^q\,\mathrm{sign}(x_1), \end{aligned} \tag{8}$$

we can rewrite the system (2) in the form

$$\begin{aligned} \dot{x}_1 &= (a_{11} - a_{12}c_1)x_1 - a_{12}c_2|x_1|^q\,\mathrm{sign}(x_1) + a_{12}s \\ \dot{s} &= v + w(x,t), \end{aligned} \tag{9}$$

the term u_{eq} compensates the nominal system dynamic. It is easy to see that the dynamics of the system (2), in the sliding mode (i.e., $s = 0$), is governed by the differential equation

$$\dot{x}_1 = (a_{11} - a_{12}c_1)x_1 - a_{12}c_2|x_1|^q\,\mathrm{sign}(x_1). \tag{10}$$

And $\forall q > 1$ and $\forall x_1$ started far away from the origin, the high degree term $|x_1|^q\,\mathrm{sign}(x_1)$ does the convergence faster and uniform. The uniform convergence of the system (10) in sliding mode is showed in the following theorem.

Theorem 3: If c_1 and c_2 are selected such that $a_{11} - a_{12}c_1 < 0$ and $a_{12}c_2 > 0$, the reduced system (10) is *asymptotic stability uniform w.r.t initial conditions* and every trajectory reaches a neighborhood of the origin of radius $\mu > 0$ before

$$T_\mu = -\frac{1}{(q-1)\beta}\ln\left(\frac{\beta\mu^{q-1}}{\alpha + \beta\mu^{q-1}}\right), \tag{11}$$

where $\alpha = -a_{11} + a_{12}c_1$ and $\beta = a_{12}c_2$.

Proof: When $a_{11} - a_{12}c_1 < 0$ and $a_{12}c_2 > 0$ the equilibrium point is asymptotically stable. It can be easily demonstrated using the Lyapunov function $V_s = |x_1|$. The time derivative along the trajectories of (10) is given by $\dot{V}_s = -\alpha V_s - \beta V_s^q$, $V_s(x_1(0)) = v_0 \geq 0$. Then, $\dot{V}_s$ is negative definite. For $q>1$, the solution of the differential equation is given by

$$V_s(t)^{1-q} = \exp(-(1-q)\beta t)v_0^{1-q} - \frac{\alpha}{\beta}\exp(-(1-q)\beta t)\cdot [\exp((1-q)\beta t)-1]. \tag{12}$$

The time T_1 taken of every trajectory from any initial state $x_1(0)$ to a level set $V_s=\mu$, where $0<\mu<V_s(x_1(0))$, is determined by

$$T_1(v_0,\mu)=\frac{1}{(q-1)\beta}\left[\ln\left(\frac{\beta v_0^{q-1}}{\alpha+\beta v_0^{q-1}}\right)-\ln\left(\frac{\beta\mu^{q-1}}{\alpha+\beta\mu^{q-1}}\right)\right].$$

The reaching time $T_1(v_0,\mu)$ depends on the parameters c_1,c_2,μ and is upper bounded by a constant which is independent on the initial condition $x_1(0)\in V_s(x_0)$,that is, the convergence time $T_1(v_0,\mu)$ of every trajectory can be upper bounded by (11), i.e., $T_1(v_0,\mu)\le T_\mu$. ■

If $\mu\to 0$, the convergence time to a neighborhood of the origin is infinite, which correspond to the asymptotic (exponential) convergence. By there exists some constant $T_\mu>0$ independent of the initial conditions we can conclude that the sliding motion is uniformly convergent. Hence, $\forall x_0\in R^2$ and $\forall t\ge T_\mu$, the system trajectories of (10) are in the interior of a ball $B_\mu=\{x_1:|x_1|<\mu\}$ centered at the origin. Once the surface is specified with some performance requirements in the system, the new control input v must be designed in such a way it guarantees the sliding mode. For instance, there exists some control laws based on the classical design that enforces the sliding motion (see, for example [8]). These control laws are obtained by the so-called reaching law approach in which the switching function dynamics are specified a priori and basically based on a first-order sliding mode (FOSM).

III. Uniform Exact Super-Twisting Based Sliding Mode Enforcement

The use of the relay (or unit controllers) sliding mode in the control law to enforce the sliding motion produces the chattering effect. To eliminate the high frequency component of this controllers in the first sliding mode a low-pass filter is frequently used. Another approach to deal with this problem is using high order sliding modes. Up to now only few 2-sliding controllers have been proposed [11]. The so-called *Super-Twisting Algorithm* (STA) has been widely used as an absolutely continuous controller because it ensures all the main properties of first order sliding mode control for the system with Lipschitz continuous matched bounded uncertainties/disturbances, i.e., with bounded gradients. Taking into account this advantages we will introduce a controller based on the STA to alleviate the chattering effect. The second-order uniformly exact controller (SOUEC) is given now by

$$v=-k_1\phi_1(s)-k_2\int_0^t\phi_2(s)\,dt, \tag{13}$$

where k_1,k_2 are the gains to be designed,

$$\begin{aligned}\phi_1(s)&=\mu_1|s|^{\frac{1}{2}}\operatorname{sign}(s)+\mu_2 s+\mu_3|s|^{\frac{3}{2}}\operatorname{sign}(s),\\ \phi_2(s)&=\frac{1}{2}\mu_1^2\operatorname{sign}(s)+\frac{3}{2}\mu_1\mu_2|s|^{\frac{1}{2}}\operatorname{sign}(s)+(\mu_2^2+2\mu_1\mu_3)s\\ &\quad+\frac{5}{2}\mu_2\mu_3|s|^{\frac{3}{2}}\operatorname{sign}(s)+\frac{3}{2}\mu_3^2|s|^2\operatorname{sign}(s),\end{aligned}$$

being μ_1,μ_2 and μ_3 positive constants. Actually, every trajectory will have achieved the sliding surface before some time, which is uniformly bounded by a constant. The SOUEC is based on standard Super-Twisting Controller (STC) and it is clear to see that the standard STA based controller can be recovered, when $\mu_2,\mu_3=0$, and the gains $k_1,k_2>0$. Also, the control law v inherits the robustness properties of a STC, such that, it is *absolutely continuous* control law and the class of uncertainties/disturbances supported by this controller does not include bounded discontinuous functions, such that, the control law v can not compensate these class of functions. To determinate the class of uncertainties/disturbances supported by the new control law v, let us assume that $w(x,t)$ can always be written as

$$w(x_1,x_2,t)=g_1(x_1,s,t)+g_2(x_1,x_2,t), \tag{14}$$

and we assume that $w\in\mathscr{W}_2$, where the class of uncertainties supported by v is the set $\mathscr{W}_2=\{w:|g_1(x_1,s,t)|\le\rho_1|\phi_1(s)|\;,\left|\frac{d}{dt}g_2(x_1,t)\right|\le\rho_2|\phi_2(s)|\}$, for some known constants $\rho_1,\rho_2\ge 0$. Also, the system (9) with the controller (13) and the disturbance $w\in\mathscr{W}_2$, can be written as

$$\begin{aligned}\dot{x}_1&=(a_{11}-a_{12}c_1)x_1-a_{12}c_2|x_1|^q\operatorname{sign}(x_1))+a_{12}s\\ \dot{s}&=-k_1\phi_1(s)+\varsigma+g_1(x_1,x_2,t),\\ \dot{\varsigma}&=-k_2\phi_2(s)+\tfrac{d}{dt}g_2(x_1,x_2,t).\end{aligned} \tag{15}$$

The existence of control law is discussed in the following theorem.

Theorem 4: The control law (7), with u_{eq} as in (8) and v as in (13), enforces every trajectory of (9) to move from any initial condition to the sliding surface and thereafter to remain on it. Moreover, $\forall w\in\mathscr{W}_2$, the sliding manifold $s=0$ is *UFTES w.r.t initial conditions* (t_0,x_0) and every trajectory reaches it before

$$T_{stc}=\frac{6}{\kappa_3}\frac{1}{\mu_{ss}^{\frac{1}{6}}}+\frac{2}{\kappa_1}\ln\left(\frac{\kappa_2}{\kappa_1}\mu_{ss}^{\frac{1}{2}}+1\right) \tag{16}$$

where

$$\begin{aligned}&\kappa_1=\mu_1^2\varepsilon/2(\lambda_{\max}\{P\}+C_2)^{\frac{1}{2}},\;\kappa_2=\mu_2\varepsilon/(\lambda_{\max}\{P\}+C_2),\\ &\kappa_3=(1/2)^{\frac{1}{14}}v_{\min}C_3/(\lambda_{\max}\{P\}+C_2)^{\frac{7}{6}},\end{aligned} \tag{17}$$

and μ_{ss} is the positive real root of

$$\mu_{ss}^{\frac{1}{2}}+(\kappa_1/\kappa_2)=(\kappa_3/\kappa_2)\mu_{ss}^{\frac{2}{3}}. \tag{18}$$

The SOUEC enforces the sliding mode for every trajectory of the system (2), taking into account their respective uncertainties/disturbances. Moreover, all the system trajectories achieve an arbitrary ball B_μ ($\mu>0$) in prescribed time. First, the sliding motion is enforced in a time T_{stc}, which is a scalar constant. Once in the sliding motion, the trajectories converge in a time T_μ to an arbitrary neighborhood of the origin of radius $\mu > 0$. All the trajectories converge to B_μ before a time which has a constant value, if they start out the ball B_μ, i.e., the system driven by the SOUEC is *UES w.r.t. initial conditions*.

A. A Lyapunov based approach

We propose a Lyapunov method to show the uniform exact convergence of the system trajectories from any initial condition to the nonlinear sliding surface. Consider the continuous function

$$W(s,\varsigma) = V_1(\zeta) + V_2(s,\varsigma), \quad (19)$$

like a candidate Lyapunov function. The function $V_1(\zeta) = \zeta^T P \zeta$ is quadratic in the vector $\zeta = \begin{bmatrix}\phi_1(s) & \varsigma\end{bmatrix}^T$ and $P = P^T > 0$ is a definite positive symmetric matrix, which is solution of a Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^TP + PA + R + \varepsilon I & PB \\ B^TP & -\Theta \end{bmatrix} < 0, \quad (20)$$

where $R = (\theta_1\rho_1^2 + \theta_2\rho_2^2)C^TC$, $C = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and

$$A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}, \quad (21)$$

$\forall \theta_1, \theta_2 \geq 0$ and being $\varepsilon > 0$. The function $V_2(s,\varsigma) = \delta k_2|\phi_1(s)|^2 - |\phi_1(s)|^{\frac{2}{3}}\operatorname{sign}(s)|\varsigma|^{\frac{4}{3}}\operatorname{sign}(\varsigma) + \delta|\varsigma|^2$ and δ is a positive constant which satisfies

$$\begin{aligned} &\delta > (4/27k_2)^{\frac{1}{3}}, \\ &7\delta k_2(k_1-\rho_1) > 4(3)^{\frac{3}{2}}\rho_2^{\frac{7}{4}}\delta^{\frac{7}{4}} + \tilde{k}_1^{\frac{7}{3}} + 4\tilde{k}_2^{\frac{7}{6}}, \end{aligned} \quad (22)$$

where $\tilde{k}_1 = k_1 + \rho_1$ and $\tilde{k}_2 = k_2 + \rho_2$. The following result shows that the subsystem (s,ζ) of (15) is robust against uncertainties/disturbances $w \in \mathscr{W}_2$, when the gains (k_1,k_2) and δ are selected properly.

Proposition 5: The Lyapunov function (19) is a *strong,robust Lyapunov function* for the subsystem (s,ς) of (15). Moreover, the derivative $\dot{W}$ of the Lyapunov function taken along the trajectories of the subsystem satisfies the differential inequality

$$\dot{W}(s,\varsigma) \leq -\kappa_1 W^{\frac{1}{2}}(s,\varsigma) - \kappa_2 W(s,\varsigma) - \kappa_3 W^{\frac{7}{6}}(s,\varsigma), \quad (23)$$

where $\kappa_3, \kappa_2, \kappa_1$ are as in (17).

Proof: To show positive definiteness of $V_2(s,\varsigma)$. Since $|\phi_1(s)|^{\frac{2}{3}}\operatorname{sign}(s)|\varsigma|^{\frac{4}{3}}\operatorname{sign}(\varsigma) \leq |\phi_1(s)|^{\frac{2}{3}}|\varsigma|^{\frac{4}{3}}$, from Lemma 6 (see Appendix), let γ_{00} and γ_{01} be constants which satisfy

$$|\phi_1(s)|^{\frac{2}{3}}|\varsigma|^{\frac{4}{3}} \leq \gamma_{0i}^3|\phi_1(s)|^2/3 + 2\gamma_{0i}^{-\frac{3}{2}}|\varsigma|^2/3,$$

$\forall \gamma_{0i} > 0 \ \ , i = 0,1$. Therefore, $\forall (s,\varsigma)$, the function V_2 can be bounded as $\alpha_1(\gamma_{00})|\phi_1(s)|^2 + \alpha_2(\gamma_{00})|\varsigma|^2 \leq V_2(s,\varsigma) \leq \alpha_2(\gamma_{01})|\phi_1(s)|^2 + \alpha_4(\gamma_{01})|\varsigma|^2$, where $\alpha_1(\gamma_{00}) = \left(\delta k_2 - \gamma_{00}^3/3\right)\ , \alpha_2(\gamma_{00}) = (\delta - 2\gamma_{00}^{-\frac{3}{2}}/3)\ , \alpha_3(\gamma_{01}) = (\delta k_2 + \gamma_{01}^3/3)$ and $\alpha_4(\gamma_{01}) = (\delta + 2\gamma_{01}^{-\frac{3}{2}}/3)$. Also, $V_2(s,\varsigma)$ is positive definite if and only if $\alpha_1(\gamma_{00}) > 0$ and $\alpha_2(\gamma_{00}) > 0$, it is always possible if δ satisfies (22). Moreover, V_2 can be bounded as

$$C_1\|\zeta\|_2^2 \leq V_2(s,\varsigma) \leq C_2\|\zeta\|_2^2, \quad (24)$$

where $C_1 = \delta k_2 - z_m^2/3$ and $C_2 = \delta k_2 + z_M^2/3$ are positive constants. And $z_m = \gamma_{00}^{\frac{2}{3}}$ is the positive real root of $2 + 3\delta(k_2-1)z_m = z_m^3$, and $z_M = \gamma_{01}^{\frac{2}{3}}$ is the positive real root of $2 + 3\delta(1-k_2)z_M = z_M^3$. Since the uncertainties/disturbances term $w \in \mathscr{W}_2$, the time derivative of V_2 is given by

$$\begin{aligned} \dot{V}_2 \ \leq & -|\phi_1(s)|^{\frac{-1}{3}}\phi_1'(s)\{2\delta k_2(k_1-g_1)|\phi_1(s)|^{\frac{7}{3}} \\ & -2g_2\delta|\phi_1(s)|^{\frac{4}{3}}\varsigma - \tfrac{2}{3}(k_1+g_1)\phi_1(s)|\varsigma|^{\frac{4}{3}}\operatorname{sign}(\varsigma) \\ & -\tfrac{4}{3}(k_2+g_2)|\phi_1(s)|^2|\varsigma|^{\frac{1}{3}} + \tfrac{2}{3}|\varsigma|^{\frac{7}{3}}\}. \end{aligned}$$

Using the following inequalities derived from Lemma (6)

$$\begin{aligned} &|\phi_1(s)|^{\frac{4}{3}}|\varsigma| \leq 4\gamma_1^{\frac{7}{4}}|\phi_1(s)|^{\frac{7}{3}}/7 + 3\gamma_1^{-\frac{7}{3}}|\varsigma|^{\frac{7}{3}}/7\ , \forall \gamma_1 > 0\ , \\ &|\phi_1(s)||\varsigma|^{\frac{4}{3}} \leq 3\gamma_2^{\frac{7}{3}}|\phi_1(s)|^{\frac{7}{3}}/7 + 4\gamma_2^{-\frac{7}{4}}|\varsigma|^{\frac{7}{3}}/7\ , \forall \gamma_2 = \tilde{k}_1^{\frac{4}{7}} > 0, \\ &|\phi_1(s)|^2|\varsigma|^{\frac{1}{3}} \leq 6\gamma_3^{\frac{7}{6}}|\phi_1(s)|^{\frac{7}{3}}/7 + \gamma_3^{-7}|\varsigma|^{\frac{7}{3}}/7, \forall \gamma_3 = \tilde{k}_2^{\frac{1}{7}} > 0, \end{aligned}$$

we obtain

$$\dot{V}_2(s,\varsigma) = -|\phi_1(s)|^{\frac{-1}{3}}\phi_1'(s)\left(\psi_1|\phi_1(s)|^{\frac{7}{3}} + \psi_2|\varsigma|^{\frac{7}{3}}\right)\ ,$$

where $\psi_1 = 2(\Upsilon_1 - 4\rho_2\delta\gamma_1^{\frac{7}{4}}/7)$, $\psi_2 = 2(1 - 9\rho_2\delta\gamma_1^{-\frac{7}{3}})/21$, $\Upsilon_1 = \delta k_2(k_1-\rho_1) - \tilde{k}_1^{\frac{7}{3}}/7 - 4\tilde{k}_2^{\frac{7}{6}}/7$ and γ_1 always exits iff it satisties $(7\Upsilon_1/8\rho_2\delta)^{\frac{4}{7}} > \gamma_1 > (9\rho_2\delta)^{\frac{3}{7}}$. For negative definiteness of $\dot{V}_2$ is required that $\psi_1, \psi_2 > 0$, it is always possible if (22) is satisfied. Note that for any $s \in R$, the function

$$v_1(s) = -\frac{\phi_1'(s)}{|\phi_1(s)|^{\frac{1}{3}}} = \frac{\left(\frac{1}{2}\mu_1|s|^{\frac{-1}{2}} + \mu_2 + \frac{3}{2}\mu_3|s|^{\frac{1}{2}}\right)}{(\mu_1|s|^{\frac{1}{2}} + \mu_2|s| + \mu_3|s|^{\frac{3}{2}})^{\frac{1}{3}}}$$

has a minimum. Since $v_1(s) > 0\ , \forall s$

$$\lim_{|s|\to\infty} v(s) = \frac{3}{2}\mu_3^{\frac{2}{3}}\ , \lim_{|s|\to 0} v(s) = \frac{\frac{3}{2}\mu_1^{\frac{2}{3}}}{|s|^{\frac{2}{3}}} = \infty\ ,$$

we immediately establish that the minimum of $v_{\min} = \min_{s\in R} v(s)$ exists and is positive. Then, it leads to

$$\dot{V}_2(s,\varsigma) \leq -v_{\min}(x)C_3\left(|\phi_1(s)|^{\frac{7}{3}} + |\varsigma|^{\frac{7}{3}}\right)$$

where $C_3 = 2(1 - 9\rho_2\delta/y_m)/21$ and $y_m = \gamma_1^{\frac{7}{3}}$ is the positive real root of $3\rho_2\delta + 7(\Upsilon_1 - 1/21)y_m = 4\rho_2\delta y_m^{\frac{7}{4}}$. Using the following standard inequality [7, Thm. 16, Section 2.9]

$$(\alpha|x_1|^s + (1-\alpha)|x_2|^s)^{\frac{1}{s}} \leq (\alpha|x_1|^r + (1-\alpha)|x_2|^r)^{\frac{1}{r}}, \quad (25)$$

$0 \leq \alpha \leq 1 \forall x \in R^2\ \ , s < r$, clearly, we can find that $(1/2)^{\frac{1}{14}}(|\phi_1(s)|^2 + |\varsigma|^2)^{\frac{7}{6}} \leq (|\phi_1(s)|^{\frac{7}{3}} + |\varsigma|^{\frac{7}{3}})$. Finally, from this inequality and using the inequality (24), we find that

$$\begin{aligned} \dot{V}_2 &\leq -(1/2)^{\frac{1}{14}}v_{\min}C_3(|\phi_1(s)|^2 + |\varsigma|^2)^{\frac{7}{6}} \\ &\leq -(1/2)^{\frac{1}{14}}v_{\min}C_3(\|\zeta\|_2^2)^{\frac{7}{6}}. \end{aligned}$$

On other hand, using the vector ζ, the subsystem (15) can be written as $\dot{\zeta} = \phi_1'(s)(A\zeta + B\tilde{\rho})$, with A and B as in (21) and

$$\tilde{\rho}(t,\zeta) = \begin{bmatrix} g_1(x_1,x_2,t) \\ \frac{\frac{d}{dt}g_2(x_1,x_2,t)}{\phi_1'(s)} \end{bmatrix}_{s=\phi^{-1}(\zeta)}$$

To take into account a bigger variety of disturbances, it will be assumed that the components of the (transformed) uncertainties/disturbances term $\tilde{\rho}(t,\zeta)$ satisfy the sector conditions (for $i=1,2$ and $\forall t\geq 0$ and $\forall\zeta\in R^2$)

$$\omega_i(\tilde{\rho}_i,\zeta)=-\tilde{\rho}_i^2(t,\zeta)+\rho_i^2\zeta_1^2=\varpi_i^T\begin{bmatrix}-1 & 0\\ 0 & R_i\end{bmatrix}\varpi_i\geq 0$$

where $\varpi_i^T=\begin{bmatrix}\tilde{\rho}_i & \zeta_i\end{bmatrix}^T$, $R_i=\rho_i^2C^TC$ and C is defined in (21). It means that the class of uncertainty/disturbance $\mathscr{W}_2$ can be expressed as a sector condition (in the ζ coordinates), i.e, in the original variables, $|\tilde{\rho}_i(t,\zeta)|\leq\rho_i|\phi_i(s)|$, is equivalent to $|\tilde{\rho}_i(t,\zeta)|\leq\rho_i|\zeta_i|$, with $\rho_i>0$, in the transformed coordinates. It follows that $\omega(\tilde{\rho},\zeta)=\theta_1\omega_1(\tilde{\rho}_1,\zeta)+\theta_2\omega_2(\tilde{\rho}_2,\zeta)\geq 0,\forall\theta_1,\theta_2\geq 0$, hence

$$\omega(\tilde{\rho},\zeta)=\begin{bmatrix}\tilde{\rho}(t,\zeta)\\ \zeta\end{bmatrix}^T\begin{bmatrix}-\Theta & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}\tilde{\rho}(t,\zeta)\\ \zeta\end{bmatrix},\qquad(26)$$

with Θ and R as in (21). The derivative of the Lyapunov function $V_1(\zeta)$ is

$$\begin{aligned}\dot{V}_1&=\phi_1'\begin{bmatrix}\zeta\\ \tilde{\rho}\end{bmatrix}^T\begin{bmatrix}A^TP+PA & PB\\ B^TP & 0\end{bmatrix}\begin{bmatrix}\zeta\\ \tilde{\rho}\end{bmatrix}\\ &\leq\phi_1'\left\{\begin{bmatrix}\zeta\\ \tilde{\rho}\end{bmatrix}^T\begin{bmatrix}A^TP+PA & PB\\ B^TP & 0\end{bmatrix}\begin{bmatrix}\zeta\\ \tilde{\rho}\end{bmatrix}+\omega(\tilde{\rho},\zeta)\right\}\\ &=\phi_1'\begin{bmatrix}\zeta\\ \tilde{\rho}\end{bmatrix}^T\begin{bmatrix}A^TP+PA+R & PB\\ B^TP & -\Theta\end{bmatrix}\begin{bmatrix}\zeta\\ \tilde{\rho}\end{bmatrix}\leq-\phi_1'\varepsilon\|\zeta\|_2^2.\end{aligned}$$

And for negative definiteness of $\dot{V}_1$ is required the feasibility of the LMI (20). Recall the standard inequality from the quadratic forms $\lambda_{\min}\{P\}\|\zeta\|_2^2\leq\zeta^TP\zeta\leq\lambda_{\max}\{P\}\|\zeta\|_2^2$, where $\|\zeta\|_2^2=\phi_1^2(s)+\varsigma^2=\mu_2^2|s|^2+\mu_3^2|s|^3+2\mu_2\mu_3|s|^{\frac{5}{2}}+\mu_1^2|s|+2\mu_1\mu_2|s|^{\frac{3}{2}}+2\mu_1\mu_3|s|^2+\varsigma_2^2$ is the Euclidean norm of ζ. From this, we can see that $(\mu_1|s|^{\frac{1}{2}})^{-1}\leq(\|\zeta\|_2)^{-1}$ is always satisfied. Then, as $\phi_1'=\frac{\mu_1}{2|s|^{\frac{1}{2}}}+\mu_2+\frac{3\mu_3}{2}|s|^{\frac{1}{2}}$, we immediately establish

$$\dot{V}_1\leq-\frac{1}{2}\mu_1^2\varepsilon\|\zeta\|_2-\mu_2\varepsilon\|\zeta\|_2^2-\frac{3}{2}\mu_3\varepsilon|s|^{\frac{1}{2}}\|\zeta\|_2^2.$$

Thanks to the negative definiteness of $\dot{V}_1(\zeta)$ and $\dot{V}_2(s,\varsigma)$,

$$\begin{aligned}\dot{W}(s,\varsigma)&=\dot{V}_1+\dot{V}_2\\ &\leq-\mu_1^2\frac{1}{2}\varepsilon\|\zeta\|_2-\mu_2\varepsilon\|\zeta\|_2^2-(1/2)^{\frac{1}{14}}v_{\min}C_3(\|\zeta\|_2)^{\frac{7}{3}}.\end{aligned}$$

As both Lyapunov functions satisfy $\lambda_{\min}\{P\}\|\zeta\|_2^2\leq V_1(\zeta)\leq\lambda_{\max}\{P\}\|\zeta\|_2^2$ and $C_1\|\zeta\|_2^2\leq V_2(s,\varsigma)\leq C_2\|\zeta\|_2^2$, the function $W(s,\varsigma)$ can be upper bounded as $W(s,\varsigma)\leq(\lambda_{\max}\{P\}+C_2)\|\zeta\|_2^2$, it allows us to obtain (23). ■

It is clear to see that every trajectory converges to zero in finite time. The proof of the uniform exact convergence follows immediately from Proposition 5. Since W satisfies both differential inequalities $\dot{W}(s,\varsigma)\leq-\kappa_1W^{\frac{1}{2}}(s,\varsigma)-\kappa_2W(s,\varsigma)$ and $\dot{W}(s,\varsigma)\leq-\kappa_3W^{\frac{7}{6}}(s,\varsigma)$, the value of $W(t)$ is below the solution of any of both inequalities. Then, for $W_0=W(s(x_0),\varsigma(x_0))$ the solution $W(t)$ satisfies $W(t)\leq\min\{W_1,W_2\}$, where

$$\begin{aligned}W_1&=\exp(-\tfrac{1}{2}\kappa_2t)\left[W_0^{\frac{1}{2}}-\tfrac{\kappa_1}{\kappa_2}\left[\exp(\tfrac{1}{2}\kappa_2t)-1\right]\right]^2,\\ W_2&=\left(W_0^{-\frac{1}{6}}+\left(\tfrac{1}{6}\right)\kappa_3t\right)^{-6}.\end{aligned}$$

This expression allows to estimate the convergence time. First, an upper bound $T_1(x_0,\mu_{ss})$ of the convergence time of a trajectory starting at point x_0 at an energy level W_0 at which it reaches the level set $W(s,\varsigma)=\mu_{ss}$ (for some $0<\mu_{ss}<W_0$), can be calculated from $W_2=\mu_{ss}$ as $T_1(x_0,\mu_{ss})=6\left(\mu_{ss}^{-\frac{1}{6}}-W_0^{-\frac{1}{6}}\right)/\kappa_3$. After, an upper bound $T_2(\mu_{ss})$ at which $s=0$ is reached (starting from this level set $W_0'=\mu_{ss}$), can be calculated from $W_1=0$ as $T_2(\mu_{ss})=2\ln\left(\frac{\kappa_2}{\kappa_1}\mu_{ss}^{\frac{1}{2}}+1\right)/\kappa_1$. Then, every trajectory reaches the sliding surface in a time $T(x_0,\mu_{ss})=T_1(x_0,\mu_{ss})+T_2(\mu_{ss})$. Moreover, the convergence time $T(x_0,\mu_{ss})$ is uniformly upper bounded by a constant, i.e., $T(x_0,\mu_{ss})\leq T_{stc}$, with T_{stc} as in (16). Choose μ_{ss} as in (18) to ensure the best estimation of the prescribed time T_{stc}.

IV. Simulation: Academic Example

Consider the following hybrid linear system modeled by the equations $\dot{x}=A_qx+b(u+w(t))$, where $x\in R^2$ is the continuous state, $q\in[1,2]$ is the discrete state that indexes the subsystems

$$A_1=\begin{bmatrix}0 & 1\\ 2 & -1\end{bmatrix},\ A_2=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix},$$

both A_1 and A_2 are unstable, $b=\begin{bmatrix}0 & 1\end{bmatrix}^T$, $w(t)=0.5\sin(2t)+0.5\cos(5t)$ and the control input u is defined by (7). The control objective is to stabilize the system in $x=0$ with convergence time of every trajectory of the system to the sliding mode upper bounded by a constant. The dynamic system changes every 4 seconds. Without lost of generality we will suppose all state measured but for the simulation the action control will start one second later of each switching takes place. The sliding surfaces for each system are designed as $s_{A_1}=x_2+x_1+0.5|x_1|^{\frac{3}{2}}\operatorname{sign}(x_1)$ and $s_{A_2}=x_2+2x_1+|x_1|^{\frac{3}{2}}\operatorname{sign}(x_1)$ and the initial conditions for the system $x(0)=\begin{bmatrix}1 & 1.5\end{bmatrix}^T$ and the sampling time $\tau=0.001$. We compare the controller (7) with:

1) $v=-M_0\operatorname{sign}(s)$, the first-order sliding controller (FOSC), with $M_0=4$.
2) $v=-k_1\phi_1(s)-k_2\int_0^t\phi_2(s)\,dt$, the super-twisting uniform exact controller (SOUEC), with $\mu_1=2,\mu_2=1,\mu_3=0.5$.

Note that with the SOUEC the disturbance $w\in\mathscr{W}_2=\{w:|g_1(x_1,s,t)|\leq\rho_1|\phi_1(s)|\ ,|\dot{w}|=\left|\frac{d}{dt}g_2(x_1,t)\right|\leq\rho_2|\phi_2(s)|\}$, with $\rho_1=0$ and $\rho_2=1$. For this case, when $\rho_1=0$ (see [13]), the gains k_1 and k_2 are selected in the set $\kappa=\{(k_1,k_2)\in R^2|k_2>\rho_2,k_1^2(k_2-0.25k_1^2)>\rho_2^2\}$. We are choosing $k_1=2$ and $k_2=4$.

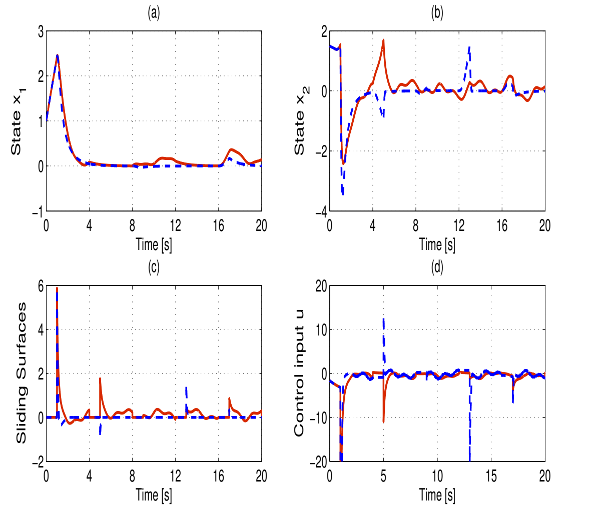

Fig. 1. (a) and (b) shows the states x_1 and x_2 with the: FOSC (solid line) and SOUEC (dotted line); (c) Sliding Surfaces and (d) Control input.

The results are shown in the Fig. 1. Only the SOUEC enforces the sliding mode providing chattering alleviation. Nevertheless, the transient response with the SOUEC is increased enormously, see Fig. 1(d). It is a consequences of the uniform convergence property. Actually, to attract the system trajectories started from far away of the origin to a neighborhood of the origin in prescribed time, the action control has to be so stronger to do this. The Fig. 2(b) shows a linear sliding surface and the uniform surfaces $s_{A_1} = 0$ and $s_{A_2} = 0$, which are clearly nonlinear. Considering that the control action starts since $t = 0$ at the first operation mode during the first 4 seconds, the Fig. 2(a) puts in evidence that the convergence time to the surfaces grows to infinity with the growth of the initial conditions for the FOSC, while the convergence time using the SOUEC is uniformly bounded by a constant.

V. CONCLUSIONS

In this paper three notions of uniform convergence w.r.t initial conditions and uncertainties/disturbances are introduced. A nonlinear sliding surface has been suggested ensuring during the sliding mode asymptotic convergence of the system trajectories uniform w.r.t. initial conditions to any arbitrary small vicinity of origin. An absolutely continuous super-twisting based controller is suggested providing uniform finite-time exact convergence of the trajectories to the sliding surface for a class of uncertainties/disturbances with chattering alleviation. The uniform sliding surface and the controller, guarantee the uniform exact convergence of all the system trajectories.

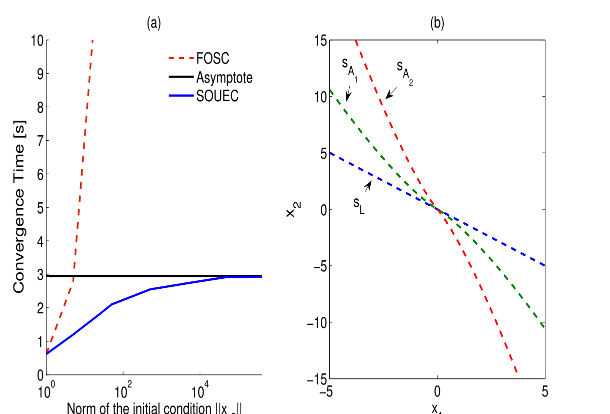

Fig. 2. (a) Convergence time to the sliding surface during the first operation mode (b) a linear sliding surface $s_L = x_2 + x_1$ and the uniform sliding surfaces s_{A_1} and s_{A_2}.

VI. ACKNOWLEDGMENTS

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APPENDIX

Lemma 6: For every real numbers $a > 0$, $b > 0$, $c > 0$, $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$ the following inequality is satisfied

$$ab \leq c^p a^p / p + c^{-q} b^q / q \ .$$

Proof: See [3]. ∎