2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Identification and control of a two-level open quantum system

Zhengui Xue, Hai Lin, and Tong Heng Lee

***Abstract*— To make full use of the amazing characteristics of quantum states, control of open quantum systems is of great significance in practical applications. This paper studies a more practical control problem where the decoherence parameter is not known exactly. The problem is considered in two phases. First, the unknown decoherence strength is estimated based on continuous measurements and an appropriately designed control. Then, Markovian feedback control is designed to deal with the decoherence effect based on the accurate parameter estimation. It is shown that state transfer with a high probability can be achieved by designing the feedback gains and the measurement. Moreover, good state transfer performance can be guaranteed even there exists an estimation error on the decoherence strength. Simulation studies are included to demonstrate the effectiveness of the proposed approach.**

***Index Terms*— Uncertain open quantum system, identification, measurement, feedback control**

I. INTRODUCTION

Active manipulation of quantum systems has transcended traditional disciplinary boundaries. During the past decades, it has attracted tremendous interest in the communities of control, computer science, physics and chemistry. One can refer to the articles [1]-[4] and the references therein for a comprehensive review. As the first principle of quantum mechanics, the state coherent superposition reflects the unique characteristic of quantum states, which declares that a quantum state could exist in all of its eigenstates simultaneously [5]. It is the state coherent superposition principle that makes quantum systems exhibit amazing properties, and thus quantum systems show an extremely powerful application prospect, e.g., quantum computing [6]. However, if a quantum system is not perfectly isolated from its environment, the coherence will be spoilt. Therefore, control of open quantum system is a significant problem.

After the earlier development stage, in which the research topics range from the fundamental controllability problem to the control of exactly known quantum systems, control of uncertain quantum systems appears to be particularly important due to its practical application importance. For a real system in the laboratory, the coupling between the system and the environment usually is not known well due to many internal and external factors. Instead of studying the control of exactly modeled open quantum systems, this paper considers a more practical control problem that the quantum system is in the presence of an unknown decoherence parameter.

Some celebrated results have been achieved on the estimation of unknown dynamical parameters of open quantum systems, see [7], [8]. Inspired by classical control, closed-loop control design is a natural trend to deal with uncertainties in quantum systems. However, compared to the classical closed-loop control, its main difficulty comes from the measurement effects on quantum states. Once a measurement is performed, it disturbs the system stochastically. On the other hand, one can make use of the unique property introduced by measurements, i.e., the non-unitary evolution, to achieve some special control objectives, e.g., control of decoherence [9], [10]. Continuous measurement based feedback control, which overcomes the limitations of sample preparation in learning control [11], has attracted great attention of researchers. The three general types of feedback strategies include Markovian feedback [12], [13], Bayesian feedback [14], [15] and coherent feedback [16]. In addition, robust control of quantum systems has also been considered in [17]-[21].

In this paper, a two-level system under spontaneous emission is considered. The spontaneous emission is added to the model as decoherence in a phenomenological manner. Since the decoherence depends on both the internal structure of the quantum system and the external filed, the decoherence strength may not be known exactly for a real system. Due to the decoherence effect, the system degenerates to the ground state $|0\rangle$. However, as far as it is known, preparation of the eigenstates $|0\rangle$ and $|1\rangle$ with high fidelities is crucial in quantum computation [6]. Hence, this paper aims to steer the state to the excited state $|1\rangle$ with a high probability in the presence of the unknown decoherence strength. The problem will be studied in two phases. In the first step, the decoherence strength is estimated based on continuous measurements. It is shown that the decoherence strength can be estimated under an appropriately designed control. In the second step, the feedback control to steer the system state to the excited state is studied firstly based on the accurate parameter estimation. The Markovian feedback control to achieve high probability state transfer is designed analytically. It is shown that the state can be driven to the exited state with a high probability by designing the feedback gains and the measurement. The control performance is further analyzed under the case that there exists an estimation error on the decoherence strength. It is shown that it is possible to achieve satisfactory control performance even with the estimation error.

The rest of the paper is organized as follows: In Section II,

Z. Xue is with the NUS Graduate School for Integrative Sciences and Engineering, National University of Singapore, 117456, Singapore zhenguixue@gmail.com

H. Lin is with the Department of Electrical and Computer Engineering, National University of Singapore, 117576, Singapore elelh@nus.edu.sg

T.H. Lee is with the Department of Electrical and Computer Engineering & the NUS Graduate School for Integrative Sciences and Engineering, National University of Singapore, Singapore eleleeth@nus.edu.sg

the problems considered in this paper are formulated. Section III presents the parameter estimation strategy. Section IV further studies the control of the state transfer. Section V concludes the paper.

The following notations will be adopted in the sequel. The eigenstates are denoted as $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The operators $F_{x,y,z}$ have the forms

$$F_x = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F_y = -\frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, F_z = -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The superoperators $\mathcal{D}$ and $\mathcal{H}$ are defined as

$$\begin{aligned} \mathcal{D}[A]\rho &= A\rho A^\dagger - \frac{1}{2}(A^\dagger A\rho + \rho A^\dagger A) \\ \mathcal{H}[A]\rho &= A\rho + \rho A^\dagger - Tr(A\rho + \rho A^\dagger)\rho. \end{aligned}$$

II. Problem formulation

Quantum electrodynamics enables one to explain spontaneous emission by adding it to the model of the system as decoherence phenomenologically. Consider the following system under spontaneous emission [22], [23]

$$\frac{d\rho}{dt} = -is[F_z, \rho] - iu(t)[F_y, \rho] + \gamma\mathcal{D}[\sigma]\rho, \quad (1)$$

where the first term with constant s on the right hand side represents the evolution under internal Hamiltonian. The control $u(t)$ is added to rotate the atom in the Bloch space around the y axis. $\gamma > 0$ represents the decoherence strength. The atomic decay operator σ has the form

$$\sigma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Here, the spontaneous emission is treated by introducing another unobserved field $\tilde{E}$. It is modeled by coupling the atom directly to the external field $\tilde{E}$ through a dipole operator d. Hence, the decoherence strength γ does not only depend on the external field but also on the structure of the atom. As a result, the decoherence strength usually may be not known exactly for a real system controlled in the laboratory.

It is noted that, without control, the system would degenerate to the ground state $|0\rangle$ due to the decoherence. However, preparation of the states $|0\rangle$ and $|1\rangle$ with high fidelities is crucial in quantum computation. The objective of this paper is to steer the state to $|1\rangle$ with a high probability in the presence of the unknown decoherence strength. To achieve the control objective, we will naturally consider the following three problems.

- Decoherence strength estimation: How to obtain the estimate $\hat{\gamma}$ of the decoherence strength based on measurements?
- Real-time feedback control: With the obtained accurate estimate $\hat{\gamma}$, how to achieve the state transfer with a high probability by real-time feedback control?
- Control under estimation error: For the case that there exists an estimation error on $\hat{\gamma}$, can the closed-loop performance be guaranteed?

III. Parameter estimation

To estimate the decoherence strength γ, we continuously measure the system, and then reconstruct γ based on the measurement results.

If the observable F_z is measured, the conditional evolution is described by [22], [24]

$$\begin{aligned} d\rho_t &= -is[F_z, \rho_t]dt - iu(t)[F_y, \rho_t]dt + M\mathcal{D}[F_z]\rho_t dt \\ &\quad + \gamma\mathcal{D}[\sigma]\rho_t dt + \sqrt{M}\mathcal{H}[F_z]\rho_t dW_t, \quad (2) \end{aligned}$$

where M represents the measurement strength. Correspondingly, the observer obtains the measurement result

$$dY_t = 2\sqrt{M}\ Tr(F_z\rho_t)dt + dW_t, \quad (3)$$

where the Wiener process W_t satisfies

$$E(dW(t)) = 0,\ [dW(t)]^2 = dt. \quad (4)$$

Since the density matrix of a two-level system can be expressed as

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} \quad (5)$$

with x, y and z being real numbers, (2) can be equivalently described by

$$\begin{aligned} dx_t &= -\frac{\gamma+M}{2}x_t dt + sy_t dt - u(t)z_t dt + \sqrt{M}x_t z_t dW_t \\ dy_t &= -\frac{\gamma+M}{2}y_t dt - sx_t dt + \sqrt{M}y_t z_t dW_t \\ dz_t &= \gamma(1-z_t)dt + u(t)x_t dt - \sqrt{M}(1-z_t^2)dW_t. \quad (6) \end{aligned}$$

Here, we adopt the master equation to describe the quantum dynamics. In practical implementation, this corresponds to simultaneously observe a sequence of identically prepared systems. From (6), the master equation can be obtained

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\gamma+M}{2}x + sy - u(t)z \\ \frac{dy}{dt} &= -\frac{\gamma+M}{2}y - sx \\ \frac{dz}{dt} &= \gamma(1-z) + u(t)x. \quad (7) \end{aligned}$$

Correspondingly, the measurement output is

$$\begin{aligned} \frac{dY}{dt} &= 2\sqrt{M}\ Tr(F_z\rho) \\ &= -\sqrt{M}\ z. \quad (8) \end{aligned}$$

Subsequently, the decoherence strength γ will be estimated from the measurement outcome (8) by designing the control. If the control signal is chosen as a constant control $u(t) = U$, we have

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\gamma+M}{2}x + sy - Uz \\ \frac{dy}{dt} &= -\frac{\gamma+M}{2}y - sx \\ \frac{dz}{dt} &= \gamma(1-z) + Ux. \quad (9) \end{aligned}$$

The equilibrium point of (9) is

$$
\begin{aligned}
x_0 &= -\frac{2\gamma U(\gamma+M)}{2(\gamma+M)U^2+M^2\gamma+\gamma^3+4s^2\gamma+2\gamma^2 M} \\
y_0 &= \frac{4s\gamma U}{2(\gamma+M)U^2+M^2\gamma+\gamma^3+4s^2\gamma+2\gamma^2 M} \\
z_0 &= \frac{\gamma(4s^2+2M\gamma+M^2+\gamma^2)}{2(\gamma+M)U^2+M^2\gamma+\gamma^3+4s^2\gamma+2\gamma^2 M}. \quad (10)
\end{aligned}
$$

To analyze the stability of the equilibrium point, define

$$x_1 = x - x_0,\ y_1 = y - y_0,\ z_1 = z - z_0. \quad (11)$$

Under the coordinate transformation, (9) is equivalent to

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dy_1}{dt} \\ \frac{dz_1}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{\gamma+M}{2} & s & -U \\ -s & -\frac{\gamma+M}{2} & 0 \\ U & 0 & -\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}. \quad (12)$$

By Routh-Hurwitz stability criterion, it can be concluded that the equilibrium point (x_0, y_0, z_0) is stable for an arbitrary control U.

Furthermore, based on the measurement result of z_0, three estimate values of γ can be deduced by solving (10), only one of which is the desired one, and the other two are not in conformity with the expectation. Specifically, the following estimate values can be obtained

$$
\begin{aligned}
\hat{\gamma}_1 &= \gamma \\
\hat{\gamma}_2 &= \frac{-\gamma^2-3M\gamma-2M^2+\sqrt{(M+\gamma)(\gamma^3+\gamma^2 M-16Ms^2)}}{2(\gamma+M)} \\
\hat{\gamma}_3 &= \frac{-\gamma^2-3M\gamma-2M^2-\sqrt{(M+\gamma)(\gamma^3+\gamma^2 M-16Ms^2)}}{2(\gamma+M)}
\end{aligned} \quad (13)
$$

It is obvious that $Re\{\hat{\gamma}_2\} < 0$, and $Re\{\hat{\gamma}_3\} < 0$, which are contradict with the requirement that $\gamma > 0$. Therefore, the accurate estimate of the decoherence strength can be obtained theoretically. In the following, a numerical example is given to further illustrate the idea.

Consider the system with parameters $\gamma = 1$ and $s = 1$. Design the control as $U = 1$, and the measurement strength as $M = 1$. In a real experimental implementation, a single system evolves conditionally according to the stochastic equation (6). Specially, Fig. 1 shows the evolution trajectories for three identically prepared systems. The proposed estimation is based on the average evolution of a sequence of systems. Ideally, the average evolution can be described by the master equation (7). For (7), the state evolution is shown in Fig. 2. As illustrated, the state z finally converge to $z_0 = 0.6667$ under the measurements and the constant control. By simple calculation, the following estimates can be obtained: $\hat{\gamma}_1 = 1.000$, $\hat{\gamma}_2 = -1.500 + 1.3228i$ and $\hat{\gamma}_3 = -1.500 - 1.3228i$. It is obvious that only $\hat{\gamma}_1$ meets the requirement.

IV. Feedback control

In this section, we will study the problem of the state transfer to $|1\rangle$, i.e., $x = 0,\ y = 0,\ z = -1$ in the Bloch space. The control is investigated by designing a continuous measurement and a feedback.

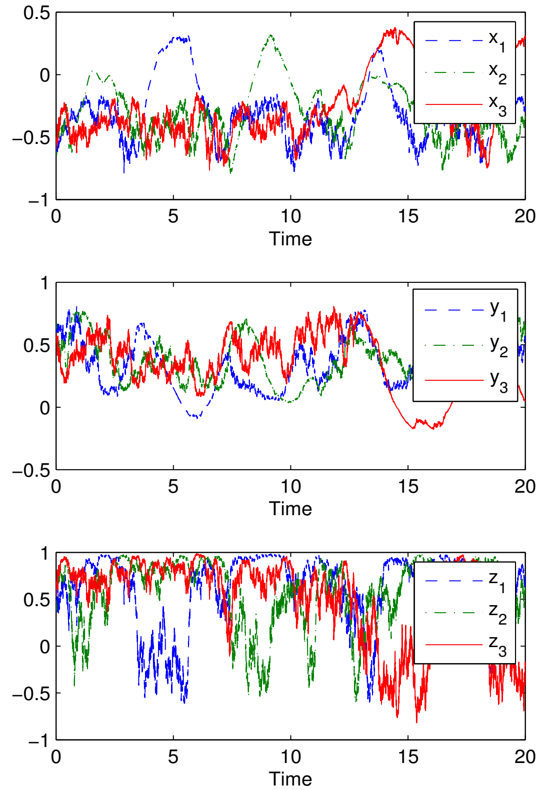

Fig. 1. Conditional evolutions of three systems

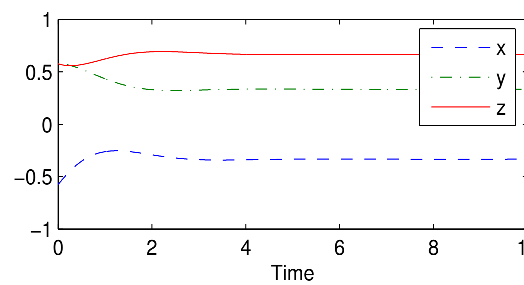

Fig. 2. Average evolution

A. Markovian feedback control design

Instead of measuring the observable F_z, the one related to F_x is measured in this section. The quantum trajectory and the corresponding measurement result are given as follows

$$
\begin{aligned}
d\rho_t &= -is[F_z, \rho_t]dt + M\mathcal{D}[\sigma_m]\rho_t dt + \gamma\mathcal{D}[\sigma]\rho_t dt \\
&\quad + \sqrt{M}\mathcal{H}[\sigma_m]\rho_t dW_t, \quad (14) \\
dY_t &= \sqrt{M}\ Tr\left((\sigma_m + \sigma_m^\dagger)\rho_t\right) dt + dW_t, \quad (15)
\end{aligned}
$$

where the operator $\sigma_m = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and W_t is a Wiener process. Furthermore, define the measurement output current as

$$I(t) = \frac{dY_t}{dt}. \quad (16)$$

Design the feedback of the output current $I(t)$ to the system along x and y axes with feedback gains λ_x and λ_y. The evolution of the system can be described by [12]

$$\begin{aligned} d\rho_t &= -is[F_z,\rho_t]dt + M\mathcal{D}[\sigma_m]\rho_t dt + \gamma\mathcal{D}[\sigma]\rho_t dt \\ &\quad -i\lambda_x\sqrt{M}[F_x,\sigma_m\rho_t+\rho_t\sigma_m^\dagger]dt + \lambda_x^2\mathcal{D}[F_x]\rho_t dt \\ &\quad -i\lambda_y\sqrt{M}[F_y,\sigma_m\rho_t+\rho_t\sigma_m^\dagger]dt + \lambda_y^2\mathcal{D}[F_y]\rho_t dt \\ &\quad +\mathcal{H}[\sqrt{M}\sigma_m - i\lambda_x F_x - i\lambda_y F_y]\rho_t dW_t. \end{aligned} \tag{17}$$

The following theorem indicates how the state can be driven to $|1\rangle$ with a high probability by designing the feedback gains λ_x and λ_y, and the measurement strength M .

Theorem 1: Consider the closed-loop system (17) under the continuous measurement and the Markovian feedback control. Suppose there is a constraint on the measurement strength M, $M \le M_{max}$. Design the feedback gains as

$$\lambda_x = 0 \tag{18}$$

$$\lambda_y = -\frac{2}{\sqrt{M}}\gamma. \tag{19}$$

Choose the measurement strength as

$$M = M_{max}. \tag{20}$$

Then, the state $z(t)$ will converge to the minimal value

$$z_{T_{min}} = -1 + G_{min}, \tag{21}$$

where

$$G_{min} = \frac{2(\gamma/M_{max})^2 + 2\gamma/M_{max}}{2(\gamma/M_{max})^2 + 3\gamma/M_{max} + 1}.$$

Correspondingly, the quantum state is driven to $|1\rangle$ with maximal probability $1-\frac{G_{min}}{2}$.

Proof: The corresponding master equation of (17) can be equivalently expressed as

$$\begin{aligned} \frac{dx}{dt} &= -\frac{M+\gamma}{2}x + \lambda_y\sqrt{M}x - \frac{1}{2}\lambda_y^2 x + sy \\ \frac{dy}{dt} &= -\frac{M+\gamma}{2}y - \lambda_x\sqrt{M}x - \frac{1}{2}\lambda_x^2 y - sx \\ \frac{dz}{dt} &= -M(1+z) + \gamma(1-z) - \frac{1}{2}\lambda_y^2 z - \frac{1}{2}\lambda_x^2 z + (1+z)\lambda_y\sqrt{M}. \end{aligned} \tag{22}$$

The objective is to steer the state to $x=0,\ y=0,\ z=-1$. First, we analyze the derivative of z.

$$\begin{aligned} \frac{dz}{dt} &= -(M+\gamma+\frac{1}{2}\lambda_y^2 - \lambda_y\sqrt{M} + \frac{1}{2}\lambda_x^2)\times \\ &\quad \left(1+z-\frac{2\gamma+\frac{1}{2}\lambda_y^2+\frac{1}{2}\lambda_x^2}{M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M}+\frac{1}{2}\lambda_x^2}\right) \end{aligned} \tag{23}$$

It is obvious that $M+\gamma+\frac{1}{2}\lambda_y^2+\frac{1}{2}\lambda_x^2-\lambda_y\sqrt{M}>0$. Therefore, under the continuous measurement and the feedback, z will converge to

$$z_T = -1 + \frac{2\gamma+\frac{1}{2}\lambda_y^2+\frac{1}{2}\lambda_x^2}{M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M}+\frac{1}{2}\lambda_x^2}. \tag{24}$$

Denote

$$G := \frac{2\gamma+\frac{1}{2}\lambda_y^2+\frac{1}{2}\lambda_x^2}{M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M}+\frac{1}{2}\lambda_x^2}. \tag{25}$$

Next, it will be shown that G can be minimized by designing the feedback gains λ_x and λ_y, and the measurement strength M.

Since $2\gamma+\frac{1}{2}\lambda_y^2>0$, $M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M}>0$, and we expect $M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M} > 2\gamma+\frac{1}{2}\lambda_y^2$, we should design $\lambda_x=0$ to minimize G. Then, we calculate the following partial derivatives

$$\frac{\partial G}{\partial \lambda_y} = \frac{-\frac{1}{2}\lambda_y^2\sqrt{M}+\lambda_y(M-\gamma)+2\sqrt{M}\gamma}{(M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M})^2} \tag{26}$$

$$\frac{\partial G}{\partial M} = \frac{(2\gamma+\frac{1}{2}\lambda_y^2)(1-\frac{1}{2}\lambda_y/\sqrt{M})}{(M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M})^2}. \tag{27}$$

Let $\frac{\partial G}{\partial \lambda_y}=0$, we can obtain $\lambda_y = 2\sqrt{M}$ *or* $\lambda_y = -\frac{2}{\sqrt{M}}\gamma$. Let $\frac{\partial G}{\partial M}=0$, we have $\lambda_y = 2\sqrt{M}$. For $\lambda_y=2\sqrt{M}$, G reaches the maximum value $G=2$. Therefore, we design $\lambda_y = -\frac{2}{\sqrt{M}}\gamma$. Then, we have

$$G = \frac{2(\gamma/M)^2+2\gamma/M}{2(\gamma/M)^2+3\gamma/M+1}. \tag{28}$$

Correspondingly,

$$z_T = -1 + \frac{2(\gamma/M)^2+2\gamma/M}{2(\gamma/M)^2+3\gamma/M+1}. \tag{29}$$

It can be checked that G monotonously decreases with respect to $\frac{\gamma}{M}$. Therefore, we design $M=M_{max}$ to obtain the minimal value of G, G_{min}.

Furthermore, for $\lambda_y = -\frac{2}{\sqrt{M}}\gamma$, we have

$$\begin{aligned} \frac{dx}{dt} &= -\frac{M+\gamma}{2}x - 2\gamma x - \frac{2}{M}\gamma^2 x + sy \\ \frac{dy}{dt} &= -\frac{M+\gamma}{2}y - sx. \end{aligned} \tag{30}$$

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}x^2 + \frac{1}{2}y^2. \tag{31}$$

The derivative of V is

$$\frac{dV}{dt} = -\frac{M+\gamma}{2}x^2 - 2\gamma x^2 - \frac{2}{M}\gamma^2 x^2 - \frac{M+\gamma}{2}y^2$$

Hence, x, y converge to zero. The obtained density matrix ρ has the following form

$$\rho = \begin{bmatrix} \frac{G_{min}}{2} & 0 \\ 0 & 1-\frac{Gmin}{2} \end{bmatrix}, \tag{32}$$

which means that the state can be steered to $|1\rangle$ with maximal probability $1-\frac{G_{min}}{2}$. ∎

Remark 1: If choose the feedback gains as $\lambda_x=0$, $\lambda_y=0$ and the measurement strength as $M=\infty$, we have $G=0$ and $z_T=-1$. This is consistent with the quantum Zeno effect [9], which means that one can defeat the decoherent effect by very frequent instant observations.

Remark 2: The measurement operator plays a significant role in the control design. Suppose we still measure the observable F_z as in Section III. Under Markovian feedback, the following master equations can be obtained

$$\begin{aligned}\frac{dx}{dt} &= -\frac{M+\gamma+\lambda_y^2}{2}x+sy+\lambda_y\sqrt{M}\\ \frac{dy}{dt} &= -\frac{M+\gamma+\lambda_x^2}{2}y-sx-\lambda_x\sqrt{M}\\ \frac{dz}{dt} &= \gamma(1-z)-\frac{1}{2}\lambda_x^2 z-\frac{1}{2}\lambda_y^2 z. \end{aligned}\tag{33}$$

Analyzing the derivative of z

$$\frac{dz}{dt}=\gamma-(\gamma+\frac{1}{2}\lambda_x^2+\frac{1}{2}\lambda_y^2)z, \tag{34}$$

the following equilibrium point can be obtained

$$z_T=\frac{\gamma}{\gamma+\frac{1}{2}\lambda_x^2+\frac{1}{2}\lambda_y^2} \tag{35}$$

which is far away from the desired state $z=-1$.

B. Control analysis in the presence of an estimation error

In this section, the performance of the closed-loop system will be further investigated under a parameter estimation error in $\hat{\gamma}$. We have the following theorem.

Theorem 2: Suppose there exists a small estimation error in $\hat{\gamma}$, which satisfies

$$|\hat{\gamma}-\gamma|<\epsilon. \tag{36}$$

Design the feedback gains as

$$\begin{aligned}\lambda_x &= 0\\ \lambda_y &= -\frac{2}{\sqrt{M}}\hat{\gamma}.\end{aligned}\tag{37}$$

Choose the measurement strength as

$$M=M_{max}. \tag{38}$$

Then, the state $z(t)$ converges to a small neighborhood around the ideal one $z_{T_{min}}$ in the form of (21). Specifically, we have

$$z_{\gamma T_{min}}-z_{T_{min}}<K\epsilon^2, \tag{39}$$

where

$$K=\frac{2}{M_{max}\left(1+\frac{2\hat{\gamma}+2\hat{\gamma}^2/M_{max}}{M_{max}+\gamma}\right)\left(1+\frac{2\gamma+2\gamma^2/M_{max}}{M_{max}+\gamma}\right)},$$

and $z_{\gamma T_{min}}$ represents the final value of $z(t)$ obtained in the presence of the estimation error.

Proof: With the control (37), we have

$$\begin{aligned}G_\gamma &= \frac{2\gamma+\frac{1}{2}\lambda_y^2+\frac{1}{2}\lambda_x^2}{M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M}+\frac{1}{2}\lambda_x^2}\\ &= \frac{2\gamma+2\hat{\gamma}^2/M}{M+2\hat{\gamma}+\gamma+2\hat{\gamma}^2/M}.\end{aligned}\tag{40}$$

The derivative of z is

$$\frac{dz}{dt}=-(M+\gamma+\frac{1}{2}\lambda_y^2-\lambda_y\sqrt{M})\left(1+z-G_\gamma\right). \tag{41}$$

Hence, z will converge to

$$z_{\gamma T}=-1+G_\gamma. \tag{42}$$

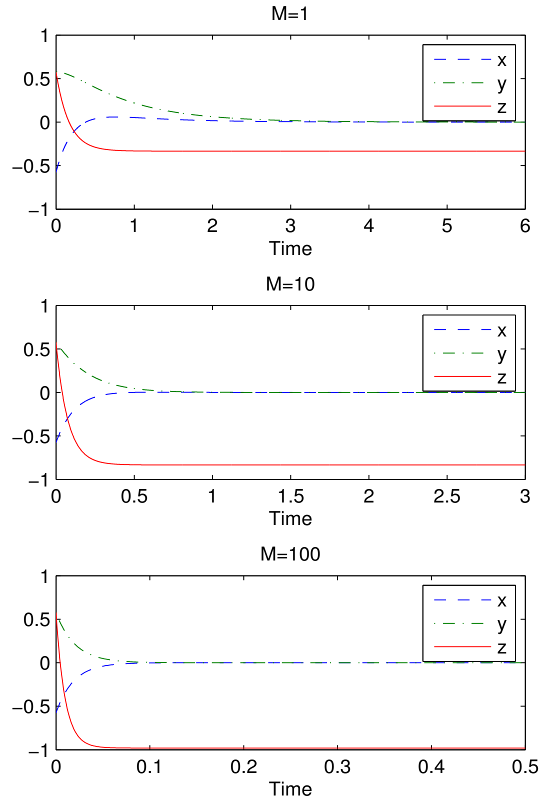

Fig. 3. Control performance with different measurement strengths

The state error between the obtained state $z_{\gamma T}$ and the ideal one z_T is

$$z_{\gamma T}-z_T=G_\gamma-G. \tag{43}$$

From (25) and (40), we have

$$\begin{aligned}G_\gamma - G &= \frac{2(\hat{\gamma}-\gamma)^2(M+\gamma)}{M(M+2\hat{\gamma}+\gamma+2\hat{\gamma}^2/M)(M+3\gamma+2\gamma^2/M)}\\ &< \frac{2}{M\left(1+\frac{2\hat{\gamma}+2\hat{\gamma}^2/M}{M+\gamma}\right)\left(1+\frac{2\gamma+2\gamma^2/M}{M+\gamma}\right)}\epsilon^2.\end{aligned}\tag{44}$$

Denote

$$K:=\frac{2}{M\left(1+\frac{2\hat{\gamma}+2\hat{\gamma}^2/M}{M+\gamma}\right)\left(1+\frac{2\gamma+2\gamma^2/M}{M+\gamma}\right)}. \tag{45}$$

Then, $G_\gamma-G<K\epsilon^2$ with bounded K. For an arbitrary small ϵ, $G_\gamma-G$ is small. Therefore, $z_{\gamma T}-z_T$ is small. From (38), we can further obtain (39). ■

C. Simulation studies

To verify the proposed control approach, the average evolution is simulated firstly based on the master equation. Consider the system with parameters $\gamma=1$, $s=1$ in the simulation. The initial state is $[x(0),y(0),z(0)]=\frac{1}{3}[-\sqrt{3},\sqrt{3},\sqrt{3}\ ]$. Control results with different values of the measurement strength M are shown in Fig. 3. As shown, both $x(t)$ and $y(t)$ converge to zero. The state is finally transferred to $|1\rangle$ with a higher probability with the increase of the measurement strength. In addition, the control

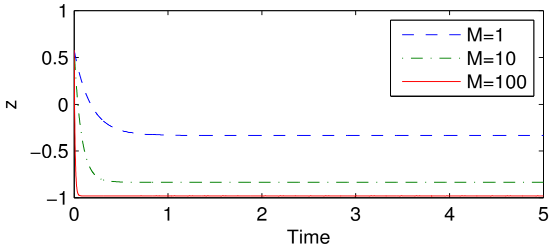

Fig. 4. Control performance in the presence of an estimation error

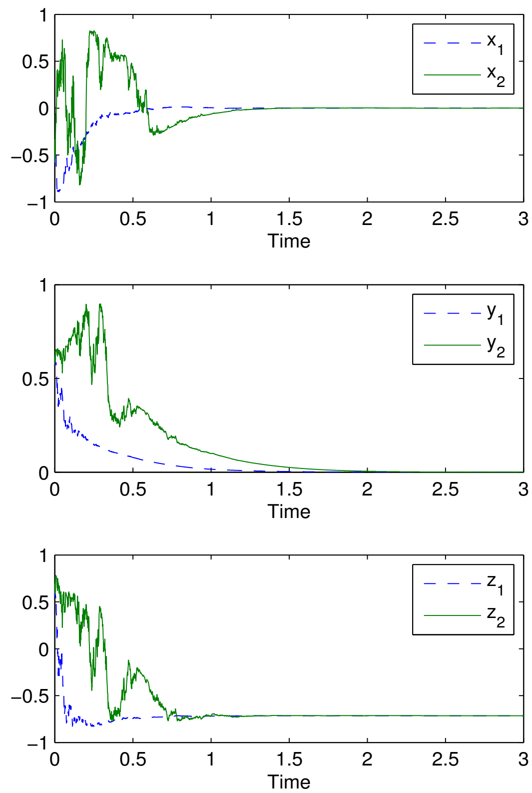

Fig. 5. Closed-loop evolution for two systems

performance with an estimation error in $\hat{\gamma}$ is simulated. Here, we assume $\hat{\gamma} = 0.95$. As seen in Fig. 4, the control results have no much difference compared to the results in Fig. 3.

As to the real experimental feedback for a single quantum system, the system evolves under conditional evolution. To illustrate the closed-loop performance of a single system, the evolutions of two identically prepared systems are shown in Fig. 5. The measurement strength is chosen as $M = 5$. For a single system, it is shown that good state transfer performance can be guaranteed with the designed control.

V. CONCLUSIONS AND FUTURE WORKS

This paper has studied the control of an uncertain open quantum system, which is closely related to practical applications. The proposed estimation method can estimate the decoherence strength based on continuous measurements and an appropriately designed control. The control strategy is capable of steering the system state to the excited state with a high probability by designing the feedback gains and the measurement. It has been shown that satisfactory performance can also been achieved even in the presence of a parameter estimation error.

This paper provides an effective support for the further exploration on more general state transfer problem and more complex uncertain open quantum systems. We wish to extend the work to the observer based estimation and control problems in our future research.

REFERENCES

[1] C. Brif, R. Chakrabarti, and H. Rabitz, Control of quantum phenomena: past, present and future, *New J. Phys.*, vol. 12, 2010, pp 075008.

[2] D. Dong and I.R. Petersen, Quantum control theory and applications: a survey, *IET Control Theor. Appl.*, vol. 12, 2010, pp 2651–2671.

[3] P. Facchi, S. Tasaki, S. Pascazio, H. Nakazato, A. Tokuse, and D.A. Lidar, Control of decoherence: Analysis and comparison of three different strategies, *Phys. Rev. A*, vol. 71, 2005, pp 22302.

[4] H. Mabuchi and N. Khaneja, Principles and applications of control in quantum systems, *Int. J. Robust Nonlinear contr.*, vol. 15, 2005, pp 647–667.

[5] R.P., Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Quantum Mechanics, Vol. 3*, Reading, MA: Addison-Wesley; 1965.

[6] D.P. DiVincenzo, Quantum computation, *Science*, vol. 270, 1995, pp 255.

[7] J. Gambetta and H.M. Wiseman, State and dynamical parameter estimation for open quantum systems, *Phys. Rev. A*, vol. 64, 2001, pp 42105.

[8] H. Mabuchi, Dynamical identification of open quantum systems, *Quantum and Semiclass. Opt.: Journal of the European Optical Society Part B*, vol. 8, 1996, pp 1103.

[9] P. Facchi and S. Pascazio, Quantum Zeno dynamics: mathematical and physical aspects, *J. Phys. A: Math. Theor.*, vol. 41, 2008, pp 493001.

[10] S. Lloyd and L. Viola, Engineering quantum dynamics, *Phys. Rev. A*, vol. 65, 2001, pp 10101.

[11] R.S. Judson and H. Rabitz, Teaching lasers to control molecules, *Phys. Rev. Lett.*, vol. 68, 1992, pp 1500–1503.

[12] H.M. Wiseman, Quantum theory of continuous feedback, *Phys. Rev. A*, vol. 49, 1994, pp 2133–2150.

[13] J. Wang and H.M. Wiseman, Feedback-stabilization of an arbitrary pure state of a two-level atom, *Phys. Rev. A*, vol. 64, 2001, pp 63810.

[14] A.N. Korotkov, Selective quantum evolution of a qubit state due to continuous measurement, *Phys. Rev. B*, vol. 63, 2001, pp. 115403.

[15] R. Van Handel, J.K. Stockton, and H. Mabuchi, Feedback control of quantum state reduction, *IEEE Trans. Automat. Contr.*, vol. 50, 2005, pp 768–780.

[16] S. Lloyd, Coherent quantum feedback, *Phys. Rev. A*, vol. 62, 2000, pp 22108.

[17] C. D'Helon and M.R. James, Stability, gain, and robustness in quantum feedback networks, *Phys. Rev. A*, vol. 73, 2006, pp 53803.

[18] M.R. James, H.I. Nurdin, and I.R. Petersen, H∞ control of linear quantum stochastic systems, *IEEE Trans. Automat. Contr.*, vol. 53, 2008, pp 1787–1803.

[19] D. Dong and I.R. Petersen, Sliding mode control of quantum systems, *New J. Phys.*, vol. 11, 2009, pp 105033.

[20] M. Yanagisawa H. Kimura, Transfer function approach to quantum control-part I: Dynamics of quantum feedback systems, *IEEE Trans. Automat. Contr.*, vol. 48, 2005, pp 2107–2120.

[21] M. Yanagisawa H. Kimura, Transfer function approach to quantum control-part II: Control concepts and applications, *IEEE Trans. Automat. Contr.*, vol. 48, 2005, pp 2121–2132.

[22] R. Handel, J.K. Stockton, and H. Mabuchi, Modelling and feedback control design for quantum state preparation, *J. Opt. B: Quantum Semiclass. Opt.*, vol. 7, 2005, pp S179.

[23] B. Qi and L. Guo, Is measurement-based feedback still better for quantum control systems?, *Syst. Control Lett.*, vol. 59, 2010, pp 333-339.

[24] K. Jacobs D.A. Steck, A straightforward introduction to continuous quantum measurement, *Contemporary Physics*, vol. 47, 2006, pp 279–303.