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Optimal Synthesis for Finite-time Consensus Under Fixed Graphs

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***Abstract*— Synthesis of optimal controllers achieving finite-time consensus for a network of multiple agents described by a fixed connectivity graph is considered. The solution procedure involves posing a partially nested decentralized control problem and converting it to a constrained convex optimization problem invoking quadratic invariance. The dynamic feedback controller thus synthesized optimizes a transient performance measure and guarantees consensus within a minimal number of steps.**

I. Introduction

Distributed coordination and consensus problems for multi-agent systems have attracted a lot of attention from the scientific community in recent years, mainly due to their applications in several fields such as vehicle formation control, study of flocking theory, and sensor networks [1]. Indeed, various conditions for asymptotic consensus under fixed and time-varying networks have been developed [2]–[6]. However, it was shown in [7] that, under noiseless situations, a network with a fixed connectivity graph can in fact achieve consensus to any function of the initial states of the agents in a finite number of steps, and that this number can be taken to be no larger than the number of agents in the network and no smaller than a connectivity graph–specific lower bound. In this paper, we focus on time-invariant networks, and develop an optimal synthesis procedure for linear dynamic feedback controllers that guarantee consensus within the same number of steps as the least achievable bound presented in [7].

Controller synthesis problems for multi-agent networks under a fixed communication topology has been considered in, e.g., [8]–[10] as well as [7]. In [8], sufficient conditions for the convex but conservative synthesis of time-invariant state feedback controllers guaranteeing asymptotic consensus and minimizing the $\mathcal{H}_2$-norm of the closed-loop system were formulated in the continuous-time domain. In [9], a linear iteration scheme was proposed to achieve asymptotic consensus under noiseless situations; it was also shown that the rate of convergence can be maximized by solving a semidefinite program minimizing the magnitude of the second largest eigenvalue of the state matrix. In [10], finite-time consensus was analyzed based on a continuous-time protocol. While the results in [8], [9] are about asymptotic consensus, those in [7], [10] deal with finite-time consensus. None of these results, however, optimizes a transient performance measure subject to time-optimal finite-time consensus.

In this paper, we first pose a finite-horizon stochastic control problem for a network of agents described by a fixed connectivity graph under noiseless situations. While we assume no disturbance inputs and no measurement noise, this problem is stochastic because the initial states of the agents are considered random. Then we present a convex solution procedure to obtain a decentralized dynamic feedback controller that guarantees finite-time consensus. The significance of the presented synthesis procedure is twofold:

- The procedure guarantees finite-time, time-optimal consensus to any linear function of the agents' initial states;
- The procedure optimizes a transient performance measure given by the expected square sum of the deviation of the agents' states from the desired function of the agents' initial states on the way to consensus.

Due to the decentralization of information within the network, synthesis of optimal controllers is potentially intractable. It was shown in [11] that, for a very simple decentralized control problem under the linear-quadratic framework, a nonlinear controller achieves better performance than any linear controller and no tractable solution is known in general. To guarantee the existence of a linear optimal controller and convexity (and hence tractability) of the synthesis problem, we endow a multi-agent network with a *partially nested* information structure [12]. It is shown in [13] that, as far as finite-horizon problems are concerned, partial nestedness is equivalent to *quadratic invariance* [14], which enables a change of variables that simplifies the problem and yet preserves convexity and the underlying constraint on information flow. Our results are obtained by exploiting the link between these team-theoretic requirements and graph-theoretic considerations.

Notation. The set of real numbers is denoted by $\mathbb{R}$ and the set of nonnegative integers by $\mathbb{N}_0$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is $\|x\| = \sqrt{x^T x}$. The Frobenius norm of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is given by $\|\mathbf{A}\|_F = \sqrt{\mathrm{trace}(\mathbf{A}\mathbf{A}^{\mathrm{T}})}$, where $\mathrm{trace}(\cdot)$ denotes the sum of the entries on the diagonal. If $\mathbf{A} = [a_1 \ \cdots \ a_n]$, where $a_i \in \mathbb{R}^m$ for each i, then the vectorization of $\mathbf{A}$ is given by $\mathrm{vec}(\mathbf{A}) = [a_1^{\mathrm{T}} \ \cdots \ a_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mn}$. For $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$, we write $\mathbf{\Sigma} > \mathbf{0}$ to mean that $\mathbf{\Sigma}$ is symmetric and positive definite. For matrices $\mathbf{A}$ and $\mathbf{B}$ of appropriate dimensions, their Kronecker product and Hadamard product (i.e., entry-wise multiplication) are denoted by $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A} \circ \mathbf{B}$, respectively. If $\mathbf{A}_1, \ldots, \mathbf{A}_k$ are arbitrary matrices, then denoted by $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_k)$ is the block diagonal matrix whose block (i, i) is $\mathbf{A}_i$ for $i = 1, \ldots, k$. Denoted by $\mathbf{1}_m \in \mathbb{R}^m$ is the vector with all its m entries equal to 1. The n-by-n identity matrix is denoted by $\mathbf{I}_n$, or simply by $\mathbf{I}$ if its dimension is understood; similarly, m-by-n zero matrix is denoted by $\mathbf{0}_{m\times n}$ or simply by $\mathbf{0}$.

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II. Problem Formulation

A. Information Structure

A network of agents with its communication topology represented by a directed graph is a decentralized system with a constraint on the information flow within the network. In this subsection, we first present some ideas related to directed graphs and their connectedness. Then we discuss the information structural considerations that form the basis of this work within the framework of sequential teams.

A directed graph G is defined as a pair (V, E), where $V = \{1, \dots, n\}$ denotes the set of nodes in G and $E \subset V \times V$ denotes the set of directed edges in G, so that $(i, j) \in E$ if and only if there is a directed edge from node i to node j in G. Whenever $(i, j) \in E$, node i is called the parent of node j and node j a child of node i. For simplicity, we assume $(i, i) \in E$ for all $i \in V$. In the context of multi-agent network, a node $i \in V$ denotes agent i in the network, and a directed edge $(i, j) \in E$ indicates that agent i conveys its decision and/or information to agent j in a unit time step. The set of parents of a node j in G is called the set of *neighbors* of agent j and is denoted by $\mathcal{N}_j$. By assumption, we have $j \in \mathcal{N}_j$ for all $j \in V$.

A node (or agent) i is said to be connected to node (or agent) j in G if there is a directed path $(i_0, \dots, i_K) \in V^{K+1}$ such that $i_0 = i$, $i_K = j$, and $(i_k, i_{k+1}) \in E$ for every $k \in \{0, \dots, K-1\}$. The directed graph G is said to be *connected from node* $i \in V$ if node i is connected to all other nodes in G or, equivalently, there is a subset of edges $\tilde{E} \subset E$ such that $(V, \tilde{E})$ forms a spanning tree with its root at node i. The graph G is said to be *connected* if it is connected from some node in G. If there is a unique $i^* \in V$ such that G is connected from i^*, then agent i^* is called the *leader* and all the other agents the *followers* in the context of leader-follower networks [2], [4], [6]. On the other hand, if every agent has at least one neighbor aside from itself (i.e., $\mathcal{N}_j \setminus \{j\} \neq \varnothing$ for each $j \in V$), then the network is said to be *leaderless*.

To guarantee our synthesis problem is convex and hence potentially tractable, we endow a multi-agent network represented by a directed graph with a *partially nested* information structure [12]. This implies that a parent must convey all the information it has to its children at each time step. For finite-horizon problems, partial nestedness is equivalent to the *quadratic invariance* condition [13], [14], which imposes a constraint on the interconnections between the sensors and actuators that the agents are equipped with. As we will see later, quadratic invariance enables a change of variables which simplifies the problem and yet preserves its convexity and the underlying constraint on information flow.

B. State-Space Model

In this subsection, we give a state-space model of a multi-agent network whose communication topology is defined by a fixed directed graph $G = (V, E)$. For $t \in \mathbb{N}_0$ and $(i, j) \in E$, denoted by $u_{ji}(t) \in \mathbb{R}$ is the decision of agent i at time t that affects the state of agent j at time $t+1$. Then the state $x_j(t) \in \mathbb{R}$ of agent j at time t evolves to the next state $x_j(t+1)$ at time $t+1$ according to

$$x_j(t+1) = \sum_{i \in \mathcal{N}_j} u_{ji}(t), \quad t \in \mathbb{N}_0. \tag{1}$$

Note again that $j \in \mathcal{N}_j$ for all $j \in V$. Let $I^i = \{i_1, \dots, i_{m_i}\}$ denote the set of children of node i; that is, I^i is the set of agents whose states are directly affected by the decision of agent i. Note that $i \in I^i$ for all $i \in V$ by (1), and that m_i is the number of the children of node i. Assuming $i_1 < i_2 < \cdots < i_{m_i}$ for $i \in V$ and denoting $m = \sum_{i=1}^{n} m_i$, define

$$u(t) = [u_{1_1 1}(t) \quad \cdots \quad u_{1_{m_1} 1}(t) \quad \cdots \quad u_{n_1 n}(t) \quad \cdots \quad u_{n_{m_n} n}(t)]^{\mathrm{T}} \in \mathbb{R}^m.$$

Then, with

$$x(t) = \begin{bmatrix} x_1(t) & \cdots & x_n(t) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n,$$

we may write the state equation (1) as

$$x(t+1) = \mathbf{B} u(t), \tag{2}$$

where $\mathbf{B} = [b_{ji}] \in \{0, 1\}^{n \times m}$ is an appropriate binary matrix. The initial state $x(0) \in \mathbb{R}^n$ is considered to be a random vector with $E[x(0)] = 0$ and $E\big[x(0)x(0)^{\mathrm{T}}\big] > \mathbf{0}$. The zero mean assumption simplifies the proof of the main result and its relaxation is presented as a corollary to the main result.

We consider dynamic feedback controllers of the form

$$\begin{aligned} x_K(t+1) &= \mathbf{A}_K(t) x_K(t) + \mathbf{B}_K(t) x(t), \\ u(t) &= \mathbf{C}_K(t) x_K(t) + \mathbf{D}_K(t) x(t), \end{aligned} \tag{3}$$

where the matrices $\mathbf{A}_K \in \mathbb{R}^{n_K \times n_K}$, $\mathbf{B}_K \in \mathbb{R}^{n_K \times n}$, $\mathbf{C}_K \in \mathbb{R}^{m \times n_K}$, and $\mathbf{D}_K \in \mathbb{R}^{m \times n}$ must respect the constraint on information flow as dictated by the directed graph G. However, to impose partially nested information structure, we do not restrict the controller order n_K and allow any information about the overall state $x(t)$ to be passed on from agent i to agent j whenever the decision of agent i directly affects the state of agent j (i.e., whenever the appropriate entry in the jth row of $\mathbf{B}$ equals 1).

C. Problem Statement

Our notion of finite-time consensus is defined as follows: The multi-agent network (2) with a random initial state $x(0)$ is said to achieve *finite-time consensus* at time $N \in \mathbb{N}_0$ if there exists a function $x_f \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$x(t) = x_f\big(x(0)\big)\mathbf{1}_n, \quad t = N,\ N+1,\ \dots,$$

with probability one. The transient performance measure that will be minimized subject to finite-time consensus over all controllers K of the form (3) is given by

$$J(K) = \sum_{t=0}^{T-1} E\,\|z(t)\|^2, \tag{4a}$$

where T is the control horizon, $E(\cdot)$ denotes the expectation with respect to the probabilistic distribution of the initial state $x(0)$, and the error output $z(t)$ takes the form of

$$z(t) = \mathbf{B}u(t) + \mathbf{F}x(0) = x(t+1) + \mathbf{F}x(0) \tag{4b}$$

for some matrix $\mathbf{F} \in \mathbb{R}^{n\times n}$.

For the purpose of achieving consensus, we restrict our attention to matrices $\mathbf{F}$ having same rows; that is, we have $\mathbf{F} = [-\alpha_1 \mathbf{1}_n \ \cdots \ -\alpha_n \mathbf{1}_n]$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Depending on whether the network is of leaderless or leader-follower type, an additional condition needs to be imposed on $\mathbf{F}$. That is, if $i \in V$ is not the root of any spanning tree in G, then we must have $\alpha_i = 0$. This is because information about $x_i(0)$ cannot be conveyed to every agent in the network and consensus to a function of $x_i(0)$ is impossible in this case. In particular, if agent i^* is the leader in a leader-follower network, then $\alpha_i \neq 0$ if and only if $i = i^*$. If finite-time consensus is achieved at time N, then we have

$$x(t) = -\mathbf{F}x(0) = \left(\sum_{i=1}^{n} \alpha_i x_i(0)\right)\mathbf{1}_n$$

for all $t \geq N$, which means that $z(t) = 0$ for all $t \geq N-1$ (with probability one). Also, the directed graph G is necessarily connected for this to hold for any $\mathbf{F}$ satisfying the above condition.

Problem Statement. For a decentralized multi-agent network (2) whose communication topology is described by a connected directed graph, our objective is to synthesize a controller K of the form (3), which minimizes the transient performance measure (4) subject to finite-time consensus at some time $N \leq T$.

III. Main Result

In this section we present the main result with the sketch of its proof postponed to the next section. Let $G = (V, E)$ be a directed graph, where $V = \{1, \ldots, n\}$. Define the indicator matrix $\mathcal{M}(G) = [m(G)_{ij}] \in \mathbb{R}^{n\times n}$ of G by

$$m(G)_{ij} = \begin{cases} 1 & \text{if } (j,i) \in E; \\ 0 & \text{if } (j,i) \notin E. \end{cases}$$

Since we assume each node in G has a self-loop, we have that all the entries on the diagonal of $\mathcal{M}(G)$ equal one, and that $\mathcal{M}(G)$ is the adjacency matrix of G.

Similarly, for any matrix $\mathbf{A} = [a_{ij}]$, the indicator matrix $\mathcal{M}(\mathbf{A}) = [m(\mathbf{A})_{ij}]$ is defined by

$$m(\mathbf{A})_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0; \\ 0 & \text{if } a_{ij} = 0. \end{cases}$$

Throughout the paper, we will use the abuse of notation $m(G)^t_{ij}$ to denote entry (i,j) of $\mathcal{M}(G)^t$.

For binary matrices $\mathbf{A} = [a_{ij}] \in \{0,1\}^{n\times n}$, define $\mathcal{G}(\mathbf{A}) \in \{0,1\}^{n^2\times n}$ as

$$\mathcal{G}(\mathbf{A}) = \begin{bmatrix} \mathbf{G}_1 & \cdots & \mathbf{G}_n \end{bmatrix}^{\mathrm{T}},$$

where each block $\mathbf{G}_i = [g^i_{jk}] \in \{0,1\}^{n\times n}$ is diagonal with $g^i_{jj} = a_{ij}$. Also, for binary matrices $\mathbf{A} = [a_{ij}] \in \{0,1\}^{n\times n}$, we define the matrix $\mathcal{H}(\mathbf{A}) \in \{0,1\}^{n^2\times n^2}$ as

$$\mathcal{H}(\mathbf{A}) = \begin{bmatrix} \mathbf{H}_{11} & \cdots & \mathbf{H}_{1n} \\ \vdots & & \vdots \\ \mathbf{H}_{n1} & \cdots & \mathbf{H}_{nn} \end{bmatrix},$$

where each block $\mathbf{H}_{ij} \in \{0,1\}^{n\times n}$ is given by

$$\mathbf{H}_{ij} = \begin{cases} \mathbf{I}_n & \text{if } a_{ij} = 1; \\ \mathbf{0} & \text{if } a_{ij} = 0. \end{cases}$$

Theorem 1: Consider the state-space model (2) of an n-agent network, whose communication topology is defined by a connected directed graph $G = (V, E)$ with $V = \{1, \ldots, n\}$. Suppose the initial state $x(0) \in \mathbb{R}^n$ is a random vector with $E[x(0)] = 0$ and $E\big[x(0)x(0)^{\mathrm{T}}\big] = \mathbf{\Sigma} > \mathbf{0}$. Suppose the matrix $\mathbf{F}$ in (4b) satisfies the following:

1) For each $j \in V$, the jth column of $\mathbf{F}$ is equal to $-\alpha_j \mathbf{1}_n$ for some $\alpha_j \in \mathbb{R}$;
2) If $j \in V$ is not the root of any spanning tree in G, then $\alpha_j = 0$.

Let matrices $\mathbf{K}_{t+1} \in \mathbb{R}^{m\times n}$ minimize

$$\big\|(\mathbf{F} + \mathbf{B}\mathbf{K}_{t+1})\mathbf{\Sigma}^{1/2}\big\|_F^2 \tag{5}$$

subject to the convex sparsity constraints

$$\begin{aligned} &\mathcal{M}\big(\mathbf{K}_{t+1}\big) \\ &\quad = \mathcal{M}\big(\mathbf{K}_{t+1}\big) \circ \mathcal{M}\big(\operatorname{diag}(\mathbf{1}_{m_1}, \ldots, \mathbf{1}_{m_n})\mathcal{M}(G)^t\big) \end{aligned} \tag{6}$$

for $t = 0, \ldots, T-1$ separately. For each t, partition $\mathbf{K}_{t+1}$ into $\mathbf{K}^i_{t+1} \in \mathbb{R}^{m_i\times n}$, $i \in V$, so that

$$\mathbf{K}_{t+1} = \big[(\mathbf{K}^1_{t+1})^{\mathrm{T}} \quad \cdots \quad (\mathbf{K}^n_{t+1})^{\mathrm{T}}\big]^{\mathrm{T}}.$$

Then, whenever the control horizon $T \geq n$, a decentralized dynamic output feedback controller K that minimizes the transient performance measure (4a) subject to finite-time consensus at some time $N \leq n-1$ is given by (3) with $x_K(0) = 0$, and

$$\mathbf{A}_K(t) = \mathcal{H}(\mathcal{M}(G)), \tag{7a}$$

$$\mathbf{B}_K(t) = \begin{cases} \mathcal{G}(\mathcal{M}(G)), & t = 0; \\ \mathbf{0}, & t > 0, \end{cases} \tag{7b}$$

$$\mathbf{C}_K(t) = \operatorname{diag}\big(\widetilde{\mathbf{K}}^1_{t+1}, \ldots, \widetilde{\mathbf{K}}^n_{t+1}\big), \tag{7c}$$

$$\mathbf{D}_K(t) = \begin{cases} \mathbf{K}_1, & t = 0; \\ \mathbf{0}, & t > 0, \end{cases} \tag{7d}$$

where the jth column of $\widetilde{\mathbf{K}}^i_{t+1}$, denoted by $(\widetilde{\mathbf{K}}^i_{t+1})_j$, is related to the jth column of $\mathbf{K}^i_{t+1}$, denoted by $(\mathbf{K}^i_{t+1})_j$, as follows:

$$(\widetilde{\mathbf{K}}^i_{t+1})_j = \begin{cases} \dfrac{1}{m(G)^t_{ij}}(\mathbf{K}^i_{t+1})_j, & m(G)^t_{ij} \neq 0; \\ \mathbf{0}, & m(G)^t_{ij} = 0 \end{cases}$$

for $i, j \in V$ and for $t = 0, \ldots, T-1$.

The sparsity constraint (6) guarantees that the communication topology defined by the graph G is respected by the controller. The order of the controller in Theorem 1 is equal to the square of the number of agents. This means that the controller state $x_K(t)$ is partitioned into n subvectors of length n for all t, and that the jth agent has direct access to the jth subvector of $x_K(t)$ for each t and j. On the other hand, the first agent generates the first m_1 components of $u(t)$ for each t, the second agent the next m_2 components of $u(t)$ for each t, and so on.

In (7), the structure of matrix $\mathbf{B}_K(0)$ at time $t = 0$ indicates that, whenever the decision of agent j directly affects the state of agent i (i.e., $m(G)_{ij} = 1$), agent j also conveys its own initial state to agent i. Agent i then updates its own controller state to store the information thus received. Similarly, the structure of $\mathbf{A}_K(t)$ dictates that, whenever an agent conveys its decision to another, it also conveys its controller state (i.e., all the information it has received so far). This way, the agents fully exploit the partially nested information structure. Yet, matrices $\mathbf{B}_K(t)$ and $\mathbf{D}_K(t)$ being zero for $t > 0$ indicates that, at optimum, the agents do not convey unnecessary information. Finally, the block diagonal structure of $\mathbf{C}_K(t)$ indicates that the decision of an agent at time $t > 0$ depends solely on the state of the controller that the agent has direct access to.

The convex problem of minimizing (5) over all $\mathbf{K}_{t+1}$ subject to (6) can be readily solved using, e.g., the vectorization approach described in [14, Theorem 29]. In the next section, we prove Theorem 1 and show that the time to consensus equals N^* under the optimal controller.

Corollary to Main Result

Let the matrices $\mathbf{A}_K(\cdot)$, $\mathbf{B}_K(\cdot)$, $\mathbf{C}_K(\cdot)$, and $\mathbf{D}_K(\cdot)$ be as in (7). If $\mu = E[x(0)] \neq \mathbf{0}$, then it is readily seen that an optimal controller has the form (3) with the second equation replaced by

$$u(t) = \mathbf{C}_K(t)x_K(t) + \mathbf{D}_K(t)x(t) - \mathbf{B}^{\mathrm{T}}(\mathbf{B}\mathbf{B}^{\mathrm{T}})^{-1}\mathbf{F}\mu$$

and the initial controller state given by $x_K(0) = -\mathcal{G}(\mathbf{I}_n)\mu$.

IV. Proof of Main Result

In this section, we sketch a proof of Theorem 1. In the first subsection, we invoke quadratic invariance to employ a change variables and show the optimality part of the theorem. Then a guarantee for time-optimal, finite-time consensus is established in the second subsection. The proofs of the lemmas presented in this section are omitted due to space constraints.

A. Optimality

We first present an augmented version of the state-space model, so that our finite-horizon stochastic control problem is converted to an equivalent static optimization problem. Define the augmented vectors $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Z}$ as

$$\mathcal{X} = \begin{bmatrix} x(0) \\ \vdots \\ x(T-1) \end{bmatrix}, \ \mathcal{U} = \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}, \ \mathcal{Z} = \begin{bmatrix} z(0) \\ \vdots \\ z(T-1) \end{bmatrix}.$$

Then the augmented system has the state-space description

$$\mathcal{Z} = \mathbf{P}_{11}x(0) + \mathbf{P}_{12}\mathcal{U},$$
$$\mathcal{X} = \mathbf{P}_{21}x(0) + \mathbf{P}_{22}\mathcal{U},$$

where

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F} \\ \vdots \\ \mathbf{F} \end{bmatrix}, \quad \mathbf{P}_{12} = \begin{bmatrix} \mathbf{B} & & & \\ & \mathbf{B} & & \\ & & \ddots & \\ & & & \mathbf{B} \end{bmatrix},$$

$$\mathbf{P}_{21} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_{22} = \begin{bmatrix} \mathbf{0} & & & \\ \mathbf{B} & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & \mathbf{B} & \mathbf{0} \end{bmatrix}.$$

With $E[(x(0))] = \mathbf{0}$, the initial controller state $x_K(0)$ can be taken to be equal to zero and a linear feedback control law takes the form

$$\mathcal{U} = \mathbf{K}\mathcal{X}$$

with $\mathbf{K} \in \mathbb{R}^{mT\times nT}$. If $\mathbf{I} - \mathbf{P}_{22}\mathbf{K}$ is invertible, we have

$$\mathcal{Z} = \left(\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\right)x(0).$$

The cost function for the augmented system is then given as

$$\sum_{t=0}^{T-1} E\left\|z(t)\right\|^2 = E\left\|\mathcal{Z}\right\|^2.$$

With $E\left[x(0)x(0)^{\mathrm{T}}\right] = \mathbf{\Sigma} > \mathbf{0}$, this is equal to

$$\left\|(\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21})\mathbf{\Sigma}^{1/2}\right\|_F^2. \tag{8}$$

For a binary matrix $\mathbf{A} = [a_{ij}] \in \{0,1\}^{m\times n}$, we define the subspace $S(\mathbf{A})$ of $\mathbb{R}^{m\times n}$ as

$$S(\mathbf{A}) = \{\mathbf{S} = [s_{ij}] \colon a_{ij} = 0 \text{ implies } s_{ij} = 0\}$$
$$= \{\mathbf{S} \in \mathbb{R}^{m\times n} \colon \mathcal{M}(\mathbf{S}) = \mathcal{M}(\mathbf{S}) \circ \mathcal{M}(\mathbf{A})\}.$$

Since the information structure within the network is assumed to be partially nested, the information of a parent is allowed to be fully transmitted to its children at each time step. This guarantees the existence of a linear optimal controller minimizing the cost (8) subject to an appropriate constraint on the sparsity pattern of $\mathbf{K}$ [12]. We partition $\mathbf{K}$ into T^2 blocks $\mathbf{K}_{ij} \in \mathbb{R}^{m\times n}$, so that $\mathbf{K}_{ij}$ denotes the linear mapping that maps $x(j-1)$ to $u(i-1)$. By causality, it immediately follows that $\mathbf{K}_{ij} = \mathbf{0}$ for all $i < j$. The block $\mathbf{K}_{ii}$ represents the mapping from $x(i-1)$ to $u(i-1)$. Since the information from a parent takes a unit time step to reach its children, it is evident that only $x_j(t)$ can directly affect $u_{ij}(t)$. Define

$$\mathbf{N}_{11}(t) = \mathrm{diag}(\mathbf{1}_{m_1}, \ldots, \mathbf{1}_{m_n}) \in \{0,1\}^{m\times n}$$

and

$$\mathbf{N}_{ij} = \mathcal{M}(\mathbf{N}_{11}\mathcal{M}(G)^{i-j}) \in \{0,1\}^{m\times n}$$

for i, $j \in V$ with $i \geq j$. Since the decision of an agent depends only on the information available to its parents, we have that $\mathbf{K}_{ii} \in S(\mathbf{N}_{11})$ and $\mathbf{K}_{ij} \in S(\mathbf{N}_{ij})$. Let $\mathbf{N} \in$

$\mathbb{R}^{mT\times nT}$ be the block lower triangular matrix whose block (i,j) is given by $\mathbf{N}_{ij}$ whenever i, $j\in V$ with $i\geq j$. Then our partially nested decentralized control problem reduces to the static optimization problem to minimize (8) subject to $\mathbf{K}\in S(\mathbf{N})$.

Now we invoke the notion of quadratic invariance [14], which enables us to perform a change of variables and convert this problem into an equivalent convex problem.

Lemma 2: The following hold true:

1) The subspace $S(\mathbf{N})$ is quadratically invariant under $\mathbf{P}_{22}$; i.e., we have $\mathbf{K}\mathbf{P}_{22}\mathbf{K}\in S(\mathbf{N})$ for all $\mathbf{K}\in S(\mathbf{N})$.
2) We have $\mathbf{K}\in S(\mathbf{N})$ if and only if $\mathbf{Q}=\mathbf{K}(\mathbf{I}-\mathbf{P}_{22}\mathbf{K})^{-1}\in S(\mathbf{N})$.

Lemma 3: The problem of minimizing (8) over all $\mathbf{K}\in S(\mathbf{N})\subset\mathbb{R}^{mT\times nT}$ is equivalent to minimizing (5) over all $\mathbf{K}_{t+1}\in\mathbb{R}^{m\times n}$ subject to (6) for each $t=0,1,\ldots,T-1$ separately. The minimizers $\mathbf{K}$ and $\mathbf{K}_{t+1}$, $t=0,\ldots,T-1$ are related by

$$\mathbf{K}=\begin{bmatrix}\mathbf{K}_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & & \vdots\\ \mathbf{K}_T & \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}$$

with each block $\mathbf{K}_t=\mathbf{Q}_{t1}\in S(\mathbf{N}_{t1})$ for $t=1,\ldots,T$.

Lemma 3 is a consequence of Lemma 2. The matrix $\mathbf{K}$ that represents an optimal feedback controller is then obtained based on Lemma 3. The fact that the state-space description in (7) leads to this $\mathbf{K}$ can be shown by verifying that the product $\mathbf{C}_K(t)\mathbf{A}_K(t-1)\cdots\mathbf{A}_K(1)\mathbf{B}_K(0)$ equals $\mathbf{K}_{t+1}$. This equality can be established exploiting the diagonal structure of the blocks of $\mathbf{B}_K(0)$ and $\mathbf{A}_K(t)$.

B. Guarantee for Consensus

It remains to show that the optimal controller guarantees finite-time consensus in an optimal number of steps. The following lemma says that the number of "effective" communication channels is nondecreasing in time t, and that the number of such channels is maximum at time $t=n-1$. That is, if there is no information flow from agent i to agent j at some time $t\geq n-1$, then no information flow is allowed from i to j at the next time step $t+1$.

Lemma 4: The following hold true:

1) $S(\mathcal{M}(G)^k)\subset S(\mathcal{M}(G)^{k+1})$ for $k\in\mathbb{N}_0$.
2) $\mathcal{M}(\mathcal{M}(G)^k)=\mathcal{M}(\mathcal{M}(G)^{k+1})$ for $k=n-1, n,\ldots$.

The following lemma gives a relation between the information flow within the network and the spanning trees within the network's connectivity graph.

Lemma 5: If node $j\in V$ is the root of some spanning tree in G, then $m(G)^{n-1}_{ij}\neq 0$ for any $i\in V$.

Whenever G is connected from a node $i\in V$, define a sequence of disjoint subsets $V_i(0), V_i(1),\ldots$ of V as $V_i(0)=\{i\}$ and

$$V_i(t+1)=\left\{j\in V\setminus\bigcup_{\tau=0}^{t}V_i(\tau)\colon m_{jk}(G)=1, k\in V_i(t)\right\}$$

whenever $t\in\mathbb{N}_0$. Then, under the partially nested information structure, $V_i(t)$ is the set of agents that will receive

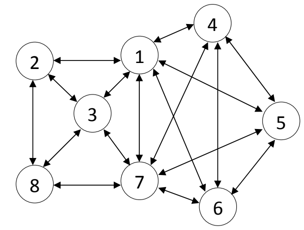

Fig. 1. A leaderless network for Example 1.

the information about agent i's initial state in exactly t time steps. Let

$$N_i=\min\{t\in\mathbb{N}_0\colon V_i(0)\cup\cdots\cup V_i(t)=V\}$$

if G is connected from i; otherwise, put $N_i=0$. If $\alpha_i=0$ for some $i\in V$, then the initial state of agent i needs not be conveyed among the agents. Thus, the quantity

$$N^*=\max\{N_i\colon \alpha_i\neq 0, i\in V\}$$

is the minimum number of time steps required for a network of n agents to reach consensus under any distributed protocol [7, Corollary 1]. Clearly, $N^*\leq n-1$ in general.

Lemmas 4 and 5 establish the following proposition, which says that the controller of the form (3) with its coefficients given in (7) makes the network reach consensus within N^* steps. We will denote this controller by K^*.

Proposition 6: Suppose that the matrix $\mathbf{F}$ in (4b) satisfies conditions 1) and 2) in Theorem 1, and that the control horizon $T\geq n$. Then controller K^* achieves finite-time consensus at time $t=N^*$, so that $x_i(t)=\sum_{j=1}^{n}\alpha_j x_j(0)$ for all $i\in V$ and for all $t\geq N^*$ (with probability one).

Since $N^*\leq n-1$, Proposition 6 completes the proof of Theorem 1.

V. Numerical Examples

In this section, we present a couple of numerical examples. In all of the following examples, we assume that the initial state vector has zero mean and identity covariance. This assumption only affects the optimality of the transient performance; it does not affect the time-optimality of finite-time consensus.

1) Example 1: Consider the network of eight agents whose connectivity graph is as shown in Fig. 1. This network was used previously in, e.g., [9], [15]. A decentralized controller of order 64, where each agent has direct access to an eight-variable subvector of the controller state, has been obtained based on Theorem 1 with $\mathbf{F}=-\frac{1}{8}\begin{bmatrix}\mathbf{1}_8 & \cdots & \mathbf{1}_8\end{bmatrix}$. Then an initial state $x(0)$ is generated randomly and the initial controller state $x_K(0)$ is set to zero. While the protocol in [9] for this network achieved asymptotic consensus to the average value of the initial states, finite-time consensus was achieved within 6 steps in [15]. However, Fig. 2 shows that our controller achieves consensus at $t=2$, which is equal to the minimum number of steps N^* required for this network to achieve consensus.

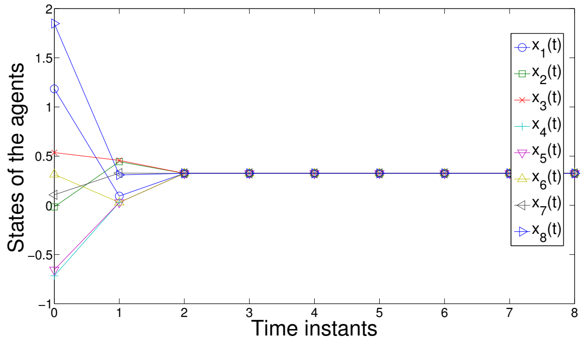

Fig. 2. Finite-time consensus on the graph in Fig. 1.

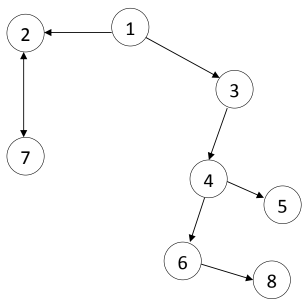

Fig. 3. A leader-follower network for Example 2.

2) Example 2: Consider the eight-agent leader-follower network described by the graph in Fig. 3. In this example, agent 1 is the leader and the objective of each follower is to follow the state of the leader perfectly. An optimal controller is obtained using $\mathbf{F} = -\begin{bmatrix}\mathbf{1}_8 & \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}$. As shown in Fig. 4, the network reaches finite-time consensus at $t = 4$.

VI. Conclusions

We considered a finite-time consensus problem for a multi-agent network whose communication topology is defined by a fixed directed graph, and presented an efficient controller synthesis procedure for achieving optimal transient performance while reaching consensus in a minimal number of time steps. The tractability of this procedure was ensured by invoking information structural concepts such as partial nestedness and quadratic invariance. The initial states of the agents were assumed random and unknown, but disturbance inputs and measurement noise were assumed absent.

As is typical in decentralized stochastic control problems, what is implicit in our problem formulation is that each agent knows *a priori* what the underlying communication topology is. This is a reasonable assumption as far as networks with stationary information structure are concerned. However, such an assumption is no longer valid for time-varying networks. Thus, a potential future research direction is to generalize the concepts of partial nestedness and quadratic invariance to time-varying situations and provide a tractable synthesis procedure for networks with time-varying graphs.

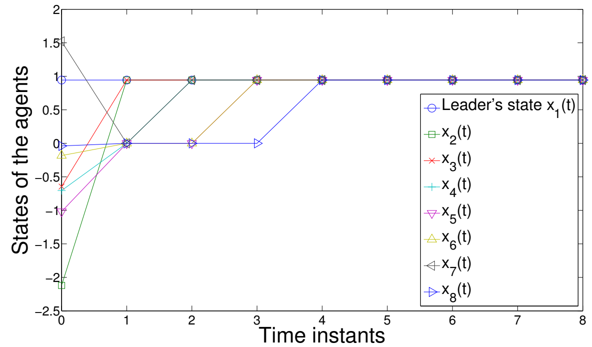

Fig. 4. Finite-time consensus on the graph in Fig. 3.

In the presence of disturbance inputs and measurement noise, on the other hand, the requirement of finite-time consensus is not attainable, and one needs to settle for an asymptotic result. Such a result will have to rely on a concept of asymptotic consensus with added robustness requirements against disturbances and noise, and thus requires further investigation of existing analysis and synthesis conditions for asymptotic consensus.

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