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Nonlinear Simultaneous Input and State Estimation with Application to Flow Field Estimation

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***Abstract*— This paper studies the problem of simultaneous input and state estimation for a nonlinear dynamical system. A Bayesian paradigm is proposed to provide statistical derivation of joint input and state estimators. Using the Bayesian paradigm, a *Maximum a Posteriori* (MAP) based estimation scheme is developed as a joint estimator. The scheme involves nonlinear MAP optimization, which is addressed by a classical Gauss-Newton method. The effectiveness of the proposed scheme is illustrated via a simulation based study on ocean flow field estimation using submersible drogues that can measure position and acceleration intermittently.**

I. Introduction

For linear systems, the Kalman filter (KF) is debatably the most popular technique for state estimation [1]. A large variety of systems involve nonlinearities, and the extensions of the KF to nonlinear systems still form a major family of solutions. Among them the extended Kalman filter (EKF) is used widely for its computational efficiency and easiness to implement [1]. Nevertheless, it may have poor performance, because the introduced linearization is only valid around the current operating condition of the state estimate. To reduce the linearization error, the iterated extended Kalman filter (IEKF) refines the procedure by using the most recent state estimate iteratively [1], [2]. The IEKF update was found in [3] to be equivalent to applying the Gauss-Newton method to approximately calculating a maximum likelihood estimate.

The EKF and IEKF uses both input and output observations to estimate the state of a nonlinear system. In this paper, we are interested in extending the notion of state estimation to the situation where the input u_k is also unknown, and investigate the problem of *nonlinear simultaneous input and state estimation (N-SISE)*. Consider the nonlinear system of the form

$$\Sigma : \begin{cases} x_{k+1} = f(u_k, x_k) & \text{(1a)} \\ y_k = h(u_k, x_k) & \text{(1b)} \end{cases}$$

where $k \in \mathbb{N}^+$ is the time index, $u \in \mathbb{R}^m$ is the input vector, $x \in \mathbb{R}^n$ is the state vector, $y \in R^p$ is measurement vector. The mappings $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ define the state transition and measurement functions, respectively; both f and h have continuous partial derivatives of the first order. The N-SISE problem is: *For the system Σ in (1), given the measurements $\{y_0, y_1, \cdots, y_k\}$, how to obtain the estimates of u_k and x_k?* Our motivation comes from the problem of velocity estimation for underwater ocean flow based on a submersible drogue that can only measure position and acceleration intermittently [4].

State estimation with unknown input has been studied for systems where the input is unmeasurable, see [5], [6], [7], [8] and references therein. More useful for understanding the system behavior, simultaneous input and state estimation (SISE) has been given much emphasis in recent years. Most of the works consider linear discrete-time systems, with solutions derived from existing estimation techniques. To mention them, the KF is used in [9], moving horizon estimation (MHE) in [10], H_∞ filtering in [11], sliding mode observer in [12], and minimum variance unbiased estimation (MVUE) in [13], [14]. For nonlinear systems, N-SISE is discussed in [15], [16], but the attention is restricted to only a special class of systems consisting of a nominally linear part and a nonlinear part.

We reformulate the N-SISE problem using statistical analysis and derive a Bayesian framework for estimation. From a statistical point of view, the state variables and measurements form two stochastic processes, the propagation of which depends on the state space equations. The Bayesian approach based on analysis of probability density distributions has been effectively used for nonlinear state estimation [1]. Different from its standard counterpart in the literature, the Bayesian paradigm proposed in this work incorporates not only the state but also the unknown input. This provides a sound theoretical basis for the ensuing design of the N-SISE scheme. With the derived Bayesian paradigm and under Gaussian distribution assumptions, a *Maximum a Posteriori* (MAP) cost function is established, which leads to the MAP estimates of the input and state. As the nonlinearity hinders analytical calculation of the estimates, a Gauss-Newton method will be used for approximate calculation [3], [17].

The remainder of this paper is organized as follows. A Bayesian framework for N-SISE is developed in Section II. Section III proceeds to derive a scheme for input and state estimation under the proposed Bayesian framework. The scheme yields nonlinear MAP estimates of the input and state, employing the Gauss-Newton method for computation. An application study on the drogue based flow field estimation is described in Section IV. Finally, Section V concludes the paper.

II. Bayesian Paradigm

Bayesian statistics has historically provided a significant framework for developing the estimation schemes such as

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the classical KF [18] and particle filter [1]. A Bayesian estimator estimates the probability density function (pdf) of unknown variables conditioned on available measurements. Denote $Y_k = \{y_0, y_1, \cdots, y_k\}$. It is obvious that Y_k contains all measurement information available until time instant k. In our case, both the input u_k and state x_k are to be estimated from Y_k. In fact, they can be estimated in appropriate ways, once the joint conditional pdf $p(u_k, x_k|Y_k)$ is obtained for each time instant k. The goal here is to sequentially compute $p(u_k, x_k|Y_k)$ from $p(u_{k-1}, x_{k-1}|Y_{k-1})$. This can be accomplished in a two-step procedure of *prediction* and *update*.

Prediction is to determine $p(x_k|Y_{k-1})$ – the *a priori* conditional pdf of x_k given measurements up until time instant $k-1$. By the Chapman-Kolmogorov equation [19], we have

$$p(x_k|Y_{k-1}) = \int\int p(x_k|u_{k-1}, x_{k-1}, Y_{k-1}) \cdot p(u_{k-1}, x_{k-1}|Y_{k-1}) du_{k-1} dx_{k-1}.$$

It is noted that

$$p(x_k|u_{k-1}, x_{k-1}, Y_{k-1}) = p(x_k|u_{k-1}, x_{k-1})$$

since x_k entirely depends on u_{k-1} and x_{k-1}, as the state equation (1a) is Markovian with order one. Hence it follows that

$$p(x_k|Y_{k-1}) = \int\int p(x_k|u_{k-1}, x_{k-1}) \cdot p(u_{k-1}, x_{k-1}|Y_{k-1}) du_{k-1} dx_{k-1} \tag{2}$$

where the pdf $p(x_k|u_{k-1}, x_{k-1})$ can be determined from (1a).

At time instant k, the measurement y_k is usable to update $p(x_k|Y_{k-1})$ to obtain the *a posteriori* conditional pdf $p(u_k, x_k|Y_k)$. To proceed further, we make the following assumption:

(A1) $\{u_k\}$ is a *white* process, independent of x_0, $\{w_k\}$ and $\{v_k\}$

where 'white' means that u_k and u_l are independent random variables for $k \neq l$. Unknown to us, u_k may assume all possible values. Hence a natural way to address the estimation of u_k is to treat it as a random variable. We adopt (A1) to disentangle u_k from the state and measurement processes. From (A1), it can be easily seen that u_k is independent of x_k and Y_{k-1}.

Using the Bayes' rule repeatedly, we obtain

$$\begin{aligned} p(u_k, x_k|Y_k) &= \frac{p(u_k, x_k, Y_k)}{p(Y_k)} \\ &= \frac{p(u_k, x_k, y_k, Y_{k-1})}{p(y_k, Y_{k-1})} \\ &= \frac{p(y_k|u_k, x_k, Y_{k-1})p(u_k, x_k|Y_{k-1})p(Y_{k-1})}{p(y_k|Y_{k-1})p(Y_{k-1})} \\ &= \frac{p(y_k|u_k, x_k, Y_{k-1})p(u_k, x_k|Y_{k-1})}{p(y_k|Y_{k-1})}. \end{aligned}$$

Note that

$$p(y_k|u_k, x_k, Y_{k-1}) = p(y_k|u_k, x_k) \tag{3}$$

$$p(u_k, x_k|Y_{k-1}) = p(x_k|Y_{k-1})p(u_k). \tag{4}$$

Here, (3) is due to the fact that y_k entirely depends on u_k and x_k, and (4) is due to (A1), which indicates u_k's independence from x_k and Y_{k-1}. Consequently,

$$\begin{aligned} p(u_k, x_k|Y_k) &= \frac{p(y_k|u_k, x_k)p(x_k|Y_{k-1})p(u_k)}{p(y_k|Y_{k-1})} \\ &\propto p(y_k|u_k, x_k)p(x_k|Y_{k-1}). \end{aligned} \tag{5}$$

To clarify, $p(y_k|u_k, x_k)$ can be determined from (1b), $p(x_k|Y_{k-1})$ can be obtained in the prediction procedure. Since $p(u_k)$ is unknown to us, it can be neglected in the estimation procedures and treated like $p(y_k|Y_{k-1})$ as a proportionality coefficient.

Sequentially updating (2) and (5) yields the Bayesian solution to estimating the input and state simultaneously. The proposed Bayesian paradigm describes a statistics based framework for nonlinear input and state estimation, within which different methods can be derived potentially. Guided by the paradigm, we would establish an MAP based scheme for input and state estimation.

III. ITERATED SIMULTANEOUS INPUT AND STATE ESTIMATION

Let $\hat{u}_k$ be the estimation of u_k, $\hat{x}_k^-$ be the *a priori* estimate of x_k given Y_{k-1}, and $\hat{x}_k^+$ be the *a posteriori* estimate of x_k given Y_k. We assume multivariate Gaussian distributions for the following conditional pdf's, i.e.,

(A2) $$p(u_k, x_k|Y_k) \sim \mathcal{N}\left(\begin{bmatrix} \hat{u}_k \\ \hat{x}_k^+ \end{bmatrix}, \begin{bmatrix} P_k^u & P_k^{ux} \\ (P_k^{ux})^{\mathrm{T}} & P_k^{x+} \end{bmatrix}\right) \tag{6}$$

(A3) $$p(y_k) \sim \mathcal{N}\left(h(u_k, x_k), R_k)\right) \tag{7}$$

(A4) $$p(x_k|Y_{k-1}) \sim \mathcal{N}\left(\hat{x}_k^-, P_k^{x-}\right) \tag{8}$$

(A5) u_k is a Gaussian random variable with no *a priori* knowledge available,

where the covariance matrices, P_k^u, P_k^{x+}, $\begin{bmatrix} P_k^u & P_k^{ux} \\ (P_k^{ux})^{\mathrm{T}} & P_k^{x+} \end{bmatrix}$, R_k and P_k^{x-}, are symmetric positive definite (SPD) for each time instant k.

Under the assumptions (A2)-(A5), an N-SISE scheme is to be developed by MAP estimation within the Bayesian framework in Section II. In other words, the scheme is Bayesian for multivariate Gaussian distributions.

A. Prediction

To develop the state prediction procedure, let us begin with (2). Define

$$\hat{x}_k^- = \arg\max_{x_k} p(x_k|Y_{k-1}). \tag{9}$$

The above maximization requires the knowledge of $p(x_k|u_{k-1}, x_{k-1})$ and $p(u_{k-1}, x_{k-1}|Y_{k-1})$. We see that the assumption (A1) indicates

$$p(u_{k-1}, x_{k-1}|Y_{k-1}) \sim \mathcal{N}\left(\begin{bmatrix} \hat{u}_{k-1} \\ \hat{x}_{k-1}^+ \end{bmatrix}, \begin{bmatrix} P_{k-1}^u & P_{k-1}^{ux} \\ (P_{k-1}^{ux})^{\mathrm{T}} & P_{k-1}^{x+} \end{bmatrix}\right). \tag{10}$$

Furthermore, we approximately have

$$p(x_k|u_{k-1},x_{k-1}) \sim \mathcal{N}\left(f(\hat{u}_{k-1},\hat{x}^+_{k-1}),W_{k-1}\right) \quad (11)$$

where W_{k-1} is given in (12). The result in (11) follows from the assumption (A1) and the first-order Taylor series expansion of $f(u_{k-1},x_{k-1})$ about $(\hat{u}_{k-1},\hat{x}^+_{k-1})$ in (13).

Substituting the pdf's of Gaussian distributions in (10) and (11) into (2), we obtain

$$p(x_k|Y_{k-1}) \sim \mathcal{N}\left(f(\hat{u}_{k-1},\hat{x}^+_{k-1}),W_{k-1}\right)$$

which is maximized by

$$\hat{x}^-_k = f(\hat{u}_{k-1},\hat{x}^+_{k-1}) \quad (14)$$

associated with the prediction error covariance W_{k-1}. However, W_k may suffer singularity in computation, so we consider using the following prediction covariance instead

$$P^{x-}_k = W_{k-1} + Q_{k-1} \quad (15)$$

where Q_k is designer-specified and adjustable as R_k.

The equations (14) and (15) constitute the prediction formula together, computing the predicted values of of the state and error covariance, respectively. The computation at time instant k utilizes only $\hat{u}_{k-1}$ and $\hat{x}^+_{k-1}$, thus cutting down on storage of the past data to reduce computational complexity.

B. Update

The next step is to update $\hat{x}^-_k$ to $\hat{x}^+_k$ with the newly arriving measurement y_k. Define

$$\begin{bmatrix} \hat{u}_k \\ \hat{x}^+_k \end{bmatrix} = \arg\max_{u_k,x_k} p(u_k,x_k|Y_k). \quad (16)$$

The approach used in [3], [17] can be extended to the above maximization problem to obtain $\hat{u}_k$ and $\hat{x}^+_k$, by applying the Gauss-Newton method for approximating MAP estimates.

Define $p(u_k,x_k|Y_k)$ as the MAP cost function $L(u_k,x_k)$, i.e.,

$$L(u_k,x_k) := p(u_k,x_k|Y_k)$$

which can be rewritten as in (17). Note that λ in (17) combines all the constants.

It is easier to deal with the logarithmic cost function

$$\ell(u_k,x_k) = \ln L(u_k,x_k) = \delta + r^{\mathrm{T}}(u_k,x_k)r(u_k,x_k) \quad (18)$$

where $\delta = \ln\lambda$ and

$$r(x_k,u_k) = \begin{bmatrix} R_k^{-\frac{1}{2}}\left(y_k - h(x_k,u_k)\right) \\ (P^{x-}_k)^{-\frac{1}{2}}(x_k - \hat{x}^-_k) \end{bmatrix}.$$

Then by (16), the input estimate $\hat{u}_k$ and state estimate $\hat{x}^+_k$ can be equivalently defined as

$$\begin{bmatrix} \hat{u}_k \\ \hat{x}^+_k \end{bmatrix} = \arg\max_{u_k,x_k} \ell(u_k,x_k). \quad (19)$$

The MAP estimation problem in (19) requires nonlinear optimization, which makes the derivation of an analytical solution difficult. However, it can be numerically solved by the Gauss-Newton method if treated as a nonlinear least squares problem.

Given $\hat{x}^-_k$ at time instant k, the Gauss-Newton method gives the input estimate $\hat{u}_k$ and updated state estimate $\hat{x}^+_k$, by defining the sequences of approximations $\hat{u}^{(i)}_k$ and $\hat{x}^{+(i)}_k$ as the updated estimate of $\hat{u}_k$ and $\hat{x}^+_k$, where (i) denotes the iteration step.

The iterated computation is shown in (20), where the initial guesses $\hat{u}^{(0)}_k = 0$ and $\hat{x}^{+(0)}_k = \hat{x}^-_k$. The iteration process continues until the iteration step (i) reaches the preselected maximum $i_{\max}$ or the difference between two consecutive iterations is less than than a preselected value, with $\hat{u}^{(i)}_k$ and $\hat{x}^{+(i)}_k$ obtained in the final iteration assigned to $\hat{u}_k$ and $\hat{x}^+_k$, respectively.

The iteration process in (20) refines the input and state estimates continually by re-evaluating the joint estimator around the latest estimated input and state operating point. Though it demands more computational power, the iteration based refinement enhances not only the estimation performance but also the robustness to nonlinearities.

The error covariance associated with (20) is equal to the inverse of the Fisher information matrix, as common in MAP estimators under Gaussian distributions [20]. The Fisher information matrix $\mathcal{I}$ is defined as

$$\mathcal{I}(u_k,x_k) = \begin{bmatrix} \mathcal{I}^u & \mathcal{I}^{ux} \\ (\mathcal{I}^{ux})^{\mathrm{T}} & \mathcal{I}^x \end{bmatrix} = \mathrm{E}\left(\begin{bmatrix} \nabla_u^{\mathrm{T}}\ell \\ \nabla_x^{\mathrm{T}}\ell \end{bmatrix}\begin{bmatrix} \nabla_u\ell & \nabla_x\ell \end{bmatrix}\right)$$

followed by

$$\begin{bmatrix} P^u_k & P^{ux}_k \\ (P^{ux}_k)^{\mathrm{T}} & P^{x+}_k \end{bmatrix} = \mathcal{I}^{-1}(\hat{u}_k,\hat{x}^+_k). \quad (21)$$

Using results in matrix analysis, we give the explicit formulae for the gradients:

$$\begin{aligned} \nabla_u r &= \begin{bmatrix} -R_k^{-\frac{1}{2}}\nabla_u h(u_k,x_k) \\ 0 \end{bmatrix} \\ \nabla_x r &= \begin{bmatrix} -R_k^{-\frac{1}{2}}\nabla_x h(u_k,x_k) \\ \left(P^{x-}_k\right)^{-\frac{1}{2}} \end{bmatrix} \\ \nabla_u \ell &= r^{\mathrm{T}}\nabla_u r \\ &= \left(y_k - h(u_k,x_k)\right)^{\mathrm{T}} R_k^{-1}\nabla_u h(u_k,x_k) \\ \nabla_x \ell &= r^{\mathrm{T}}\nabla_x r \\ &= \left(y_k - h(u_k,x_k)\right)^{\mathrm{T}} R_k^{-1}\nabla_x h(u_k,x_k) \\ &\quad + \left(x_k - \hat{x}^-_k\right)^{\mathrm{T}}\left(P^{x-}_k\right)^{-1}. \end{aligned}$$

Hence $\mathcal{I}(u_k,x_k)$ is given by (22).

Within the proposed Bayesian framework, we have thus far developed a MAP based N-SISE scheme for the nonlinear system Σ in (1). It includes the prediction procedure (14)-(15) and the update procedure (20)-(22), while the latter is iteratively implemented. The proposed algorithm is summarized below.

IV. Application to Flow Field Estimation

A. The Drogue Based Flow Field Estimation

The buoyancy-controlled drogue [4] (Fig. 1(a)) allows submergence up to a depth of 150 feet. We consider an arbitrary 2D flow field $v_d(z,t)$ (Fig. 1(b)) for the sake of simplicity, since the results to be obtained are easily

$$W_{k-1} \approx \left[\, \nabla_u f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) \quad \nabla_x f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) \,\right] \begin{bmatrix} P^u_{k-1} & P^{ux}_{k-1} \\ (P^{ux}_{k-1})^{\mathrm{T}} & P^{x+}_{k-1} \end{bmatrix} \begin{bmatrix} \nabla_u^{\mathrm{T}} f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) \\ \nabla_x^{\mathrm{T}} f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) \end{bmatrix} \tag{12}$$

$$f(u_{k-1}, x_{k-1}) = f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) + \left[\, \nabla_u f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) \quad \nabla_x f(\hat{u}_{k-1}, \hat{x}^+_{k-1}) \,\right] \begin{bmatrix} u_{k-1} - \hat{u}_{k-1} \\ x_{k-1} - \hat{x}^+_{k-1} \end{bmatrix} + O(\cdot) \tag{13}$$

$$L(u_k, x_k) = \lambda \cdot \exp\left[\left(y_k - h(u_k, x_k)\right)^{\mathrm{T}} R_k^{-1} \left(y_k - h(u_k, x_k)\right) + \left(x_k - \hat{x}_k^-\right)^{\mathrm{T}} (P_k^{x-})^{-1} \left(x_k - \hat{x}_k^-\right) \right] \tag{17}$$

$$\begin{bmatrix} \hat{u}_k^{(i+1)} \\ \hat{x}_k^{+(i+1)} \end{bmatrix} = \begin{bmatrix} \hat{u}_k^{(i)} \\ \hat{x}_k^{+(i)} \end{bmatrix} - \left[\begin{bmatrix} \nabla_u^{\mathrm{T}} r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \\ \nabla_x^{\mathrm{T}} r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \end{bmatrix} \left[\, \nabla_u r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \quad \nabla_x r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \,\right] \right]^{-1} \cdot \begin{bmatrix} \nabla_u^{\mathrm{T}} r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \\ \nabla_x^{\mathrm{T}} r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \end{bmatrix} r(\hat{u}_k^{(i)}, \hat{x}_k^{+(i)}) \tag{20}$$

$$\mathcal{I}(u_k, x_k) = \begin{bmatrix} \nabla_u^{\mathrm{T}} h(u_k, x_k) R_k^{-1} \nabla_u h(u_k, x_k) & \nabla_u^{\mathrm{T}} h(u_k, x_k) R_k^{-1} \nabla_x h(u_k, x_k) \\ \nabla_x^{\mathrm{T}} h(u_k, x_k) R_k^{-1} \nabla_u h(u_k, x_k) & \nabla_x^{\mathrm{T}} h(u_k, x_k) R_k^{-1} \nabla_x h(u_k, x_k) + \left(P_k^{x-}\right)^{-1} \end{bmatrix} \tag{22}$$

Algorithm 1: The N-SISE Algorithm

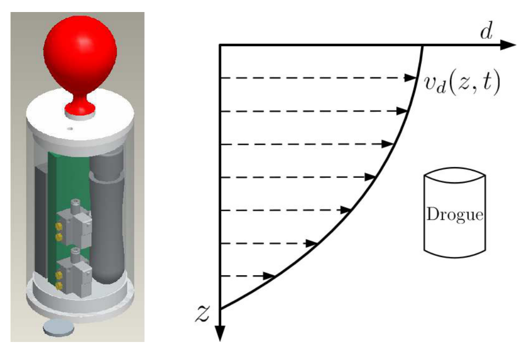
Initialize: $\hat{x}_0^+ = \mathrm{E}(x_0)$, $P_0^{x+} = p_0 I$, where p_0 is a large positive value ;
for $k = 1$ *to* N **do**
  Prediction:
  State prediction via (14);
  Computation of approximate prediction error covariance via (15) ;

  Update:
  Initialize: $i = 0$, $\hat{u}_k^{(0)} = 0$, $\hat{x}_k^{+(0)} = \hat{x}_k^-$;
  while $i < i_{max}$ **do**
    Gauss-Newton based joint input and state estimation via (20) ;
    $i = i + 1$;
  end
  $\hat{u}_k = \hat{u}_k^{(i_{\max})}$, $\hat{x}_k^+ = \hat{x}_k^{+(i_{\max})}$;
  Computation of approximate joint estimation error covariance via (21)-(22);
end

Note: An alternative for the stop condition in the iteration process is that the difference between two consecutive iterations is less than some preselected tolerance level.

generalizable to the 3D case. As shown in Fig. 1(b), $v_d(z,t)$ is along the d-direction and assumed time-stationary and dependent only on the drogue depth z. The dynamics of the drogue within the flow field can be described by the differential equation [21]

$$m\ddot{d}(t) = c \cdot \left(v_d(z,t) - \dot{d}(t)\right) \cdot \left|v_d(z,t) - \dot{d}(t)\right| \tag{23}$$



(a) (b)

Fig. 1. (a) Schematic view of the buoyancy-controlled drogue; (b) flow field estimation using the drogue.

where m is the constant rigid mass, c is the drag coefficient that quantifies the drag or resistance applied on the drogue in the flow field.

For (23), two state variables $x_1(t) := d(t)$ and $x_2(t) := \dot{d}(t)$ can be defined. Further, $v_d(z,t)$ can be viewed as the unknown external input into the drogue dynamics, naturally implying the definition of $u(t) := v_d(z,t)$. Then (23) can be transformed into the state space equations

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{c}{m} \cdot \operatorname{sign}\left(u(t) - x_2(t)\right) \cdot \left(u(t) - x_2(t)\right)^2 \end{cases} \tag{24}$$

Its discrete-time representation, by assuming zero-order hold for the input variable $u(t)$ and using finite difference to approximate the differentiation over half open intervals $[kT, (k+1)T)$, can be written as

$$\begin{cases} x_{1,k+1} = x_{1,k} + T \cdot x_{2,k} \\ x_{2,k+1} = x_{2,k} + T \cdot \frac{c}{m} \cdot (u_k - x_{2,k}) \cdot |u_k - x_{2,k}| \end{cases} \tag{25}$$

where $k \in \mathbb{N}^+$, $u_k := u(kT)$ and $x_{i,k} := x_i(kT)$ for $i = 1, 2$.

The drogue features N-periodical ($N \in \mathbb{N}^+$) submerging-surfacing depth pattern – after submerging, it moves underwater for a duration of $(N-1)T$, and then surfaces for T. No matter whether it is underwater or on the surface, the depth $z_k := z(kT)$ and acceleration $\ddot{d}_k := \ddot{d}(kT)$ are measurable; however, the position $d_k := d(kT)$ can only be measured when it is surfaced. Thus multi-rate measurements arise, with the fast one $y_k := \ddot{d}_k$ and slow one $\eta_k := d_k$ given by, respectively,

$$\begin{cases} y_k = \frac{c}{m} \cdot \mathrm{sign}\left(u_k - x_{2,k}\right) \cdot \left(u_k - x_{2,k}\right)^2 \\ \eta_{kN} = x_{1,kN}. \end{cases} \tag{26}$$

Combining (25) and (26), we obtain the state space model to describe the dynamics of the drogue:

$$\Sigma_m : \begin{cases} x_{k+1} = f(u_k, x_k) \\ y_k = h(u_k, x_k) \\ \eta_{kN} = \varphi(u_{kN}, x_{kN}) \end{cases} \tag{27}$$

where the functions f, h and ϕ can be determined contextually. The model is nonlinear, multi-rate and with unknown input. Multi-rate sampling induces cumbersomeness in usage of the measurement data. It is thus desirable to use the lifting technique to transform the multi-rate system into a single-rate high-order one [22]. The idea of 'lifting' a system was also used in [23] to design Newton observer for nonlinear systems.

Define the lifted input, state and measurement vectors, respectively, as

$$\begin{aligned} \tilde{u}_k &:= \begin{bmatrix} u^{\mathrm{T}}_{(k-1)N+1} & u^{\mathrm{T}}_{(k-1)N+2} & \cdots & u^{\mathrm{T}}_{kN} \end{bmatrix}^{\mathrm{T}} \\ \tilde{x}_k &:= x_{(k-1)N+1} \\ \tilde{y}_k &:= \begin{bmatrix} y^{\mathrm{T}}_{(k-1)N+1} & y^{\mathrm{T}}_{(k-1)N+2} & \cdots & y^{\mathrm{T}}_{kN} & \eta^{\mathrm{T}}_{kN} \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

and define the lifted state transition and measurement functions, respectively, as

$$\begin{aligned} F(\tilde{x}_k, \tilde{u}_k) &:= f^{u_{kN}} \circ f^{u_{kN-1}} \circ \\ &\quad \cdots \circ f^{u_{(k-1)N+1}}(x_{(k-1)N+1}) \\ H(\tilde{x}_k, \tilde{u}_k) &:= \begin{bmatrix} h^{u_{(k-1)N+1}}(x_{(k-1)N+1}) \\ h^{u_{(k-1)N+2}} \circ f^{u_{(k-1)N+1}}(x_{(k-1)N+1}) \\ \vdots \\ h^{u_{kN}} \circ \cdots \circ f^{u_{(k-1)N+1}}(x_{(k-1)N+1}) \\ \varphi^{u_{kN}} \circ \cdots \circ f^{u_{(k-1)N+1}}(x_{(k-1)N+1}) \end{bmatrix}. \end{aligned}$$

Then it is straightforward to obtain the equivalent lifted model as follows:

$$\Sigma_s : \begin{cases} \tilde{x}_{k+1} = F(\tilde{u}_k, \tilde{x}_k) \\ \tilde{y}_k = H(\tilde{u}_k, \tilde{x}_k) \end{cases}. \tag{28}$$

The proposed N-SISE scheme would be implemented to the system Σ_s in (28) to acquire the information estimates of not only the velocities of the flow field (unknown input) but also the motion of the drogue (state variables).

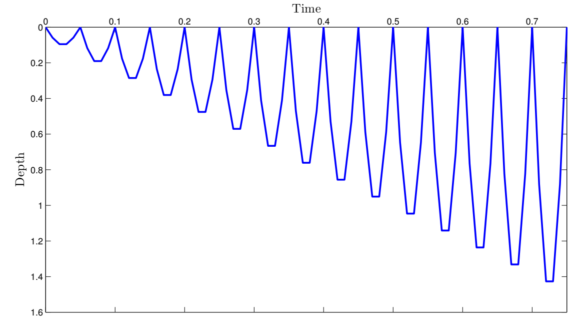


Fig. 2. Depth profile assumed for the drogue.

B. Numerical Simulation

Let the flow field $v_d(z, t)$ be with parabolic velocity distribution, i.e.,

$$v_d(z, t) = -0.2z^2 + 0.8$$

Let $m = 1.5\mathrm{Kg}$, $c = 0.24\mathrm{N/m^2/s^2}$, $T = 0.1\mathrm{s}$ and $N = 5$. The drogue is assumed to follow the depth profile shown in Fig. 2, with the time duration of each underwater travel being $N \cdot T$. The proposed scheme of course also allows for other profiles with equal dive duration.

Substitute the parameters into the state space drogue model in (25)-(26), and then implement the proposed scheme to the simulation data obtained from the model. The simulation results are shown in Figs. 3. Fig. 3 makes comparisons between the actual and estimated values for u, x_1 and x_2, respectively. A quick convergence to the truth is observed for the estimates of all the variables. It is seen that the estimation achieves high accurateness. The simulation shows the power of the proposed scheme to provide reliable estimates for the challenging problem of flow field estimation.

V. Conclusion

Motivated by the needs in drogue based ocean flow field estimation, this paper has investigated the problem of N-SISE – simultaneous input and state estimation for nonlinear systems. The problem is solved via two procedures. First, a Bayesian paradigm for N-SISE is developed. Then with the guidance of the Bayesian paradigm, a scheme based on MAP criterion is proposed to estimate the input and state of the nonlinear system. The scheme is sequential, consisting of prediction and update; furthermore, each update involves iterative searching, as a result of using the Gauss-Newton method for maximizing the nonlinear MAP cost function. The N-SISE scheme is applied to estimate the ocean flow velocity profiles on the basis of multi-rate sampling of position and acceleration of a drogue. Satisfactory performance is observed in simulation results. Regarding N-SISE, another interesting but challenging topic is joint input and state observability, which will be further studied.

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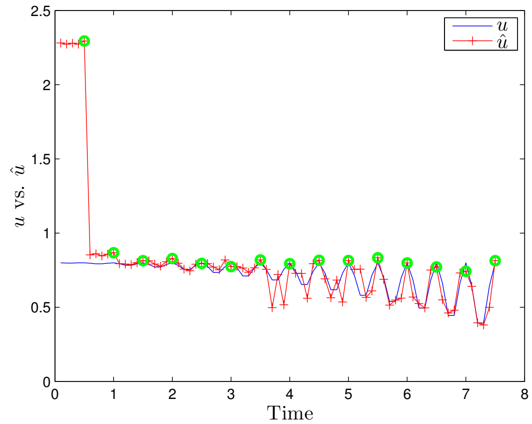
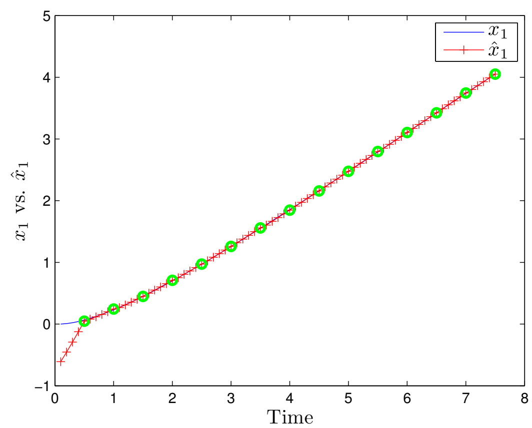
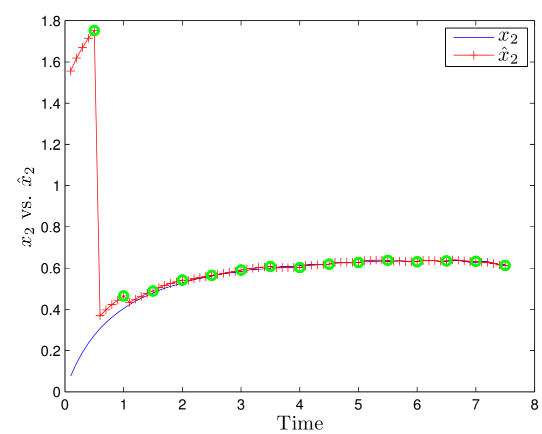


(a)



(b)



(c)

Fig. 3. Actual values and estimated values: (a) u vs. $\hat{u}$; (b) x_1 vs. $\hat{x}_1$; (c) x_2 vs. $\hat{x}_2$. The green circle denotes the time instants when multi-rate sampling occurs.

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