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Robustness in the Face of Polytopic Initial Conditions Uncertainty and Polytopic System Matrices Uncertainty in Finite-horizon Linear H_∞-Analysis

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***Abstract*— This paper addresses a linear finite-horizon robust optimal H_∞ analysis problem, where the system matrices and the system initial conditions (ICs, x_0) are concurrently uncertain, both in a polytopic manner. Current finite-horizon H_∞ analysis practice assumes $x_0 \in \mathcal{R}^n$; that is, allows infinite ICs uncertainty. This assumption is unrealistically conservative, and incompatible with the prevalent robust H_∞ analysis practice of attributing finite uncertainty to the systems's parameters/matrices. Here, the ICs uncertainty model is analogous to the (convex) uncertainty model of the system matrices. The development applies H_∞ 'first principles', and exploits convexity over the matrices uncertainty polytope, over the ICs uncertainty polytope, and over time ('time-convexity'). A conjecture regarding polytopic final-state convexity in this setup is given, and applied, to overcome non-convexity of the state-transition matrix with respect to the system matrices. A detailed numerical example shows a dramatic advantage of the methods which do not constrain the final Lyapunov function.**

I. INTRODUCTION

H_∞ control with initial state uncertainty has been investigated by several researchers (e.g. [16][27]), however a bound on the initial state uncertainty was not introduced. The more recent work [21] deals with H_∞ attenuation of both disturbances and initial-state uncertainty for LTI systems in the infinite-horizon case. Tadmor [26] was probably the first to give a solution via state-space methods to the H_∞ control problem for linear time-varying systems for the finite-horizon case; the results are in terms of two coupled indefinite Riccati equations. A game-theoretic solution for the same problem was given in [18]. State-space solutions for time-invariant systems in finite horizon, while (for the first time) taking initial conditions into account, were given in terms of one differential Riccati equation (state-feedback) or two coupled ones (output-feedback) in [16]. These works, and many later ones which pose their results in differential linear matrix inequalities (LMI) form, e.g. [9], do not provide a method for actually solving the relevant LMIs for the (symmetric) matrix time-function $P(t)$.

A quite known approach for finding a general time-varying $P(t)$ is the difference-LMIs (DLMIs) method [12]. However, this method is extremely time-consuming because the computation time-step must be minute and an inversion of $P(t)$ is performed in each time-step. In contrast, the time-convexity method proposed in [7] is computationally efficient, simple, and flexibly allows (in its piecewise form) elaborate variations in $P(t)$ with very little numerical overhead. The core idea is to search for a time-linear $P(t)$ and to exploit convexity in (normalized) time over the scenario duration in order to reduce the (originally differential) problem to 'static' LMIs at the scenario end-points.

Piecewise-quadratic/linear Lyapunov functions have been extensively used for analyzing linear time-varying and uncertain systems, mainly piecewise-linear/affine systems (cell/region-dependent) and switched/hybrid systems (case-dependent). For example, [28] aimed at reducing the conservatism of the quadratic stability test for uncertain time-varying systems by using two-term piecewise quadratic Lyapunov functions; the results involve the convex combination of two matrices. In [14] piecewise-linear systems are analyzed by piecewise-quadratic, region-dependent Lyapunov functions. [2] addresses uncertain linear systems affected (within a given polytope) by time-varying uncertain parameters. Note that [8] is the first work applying a linear time-variation of $P(t)$ and fully exploiting time-convexity.

ICs-uncertainty seems to be no less important, in practical situations, than parameter-uncertainty. In this paper, finite uncertainty affects the system's ICs. Addressing finite ICs-uncertainty, analogously to the standard treatment of parameter-uncertainty, removes severe conservatism inherent in assuming $x_0 \in \mathcal{R}^n$. It is shown that the resulting LMIs depend upon the ICs in a convex manner; this profound fact admits conditions regarding a (joint) performance bound over both the parameters and the ICs uncertainty regions. Convexity over the parameters uncertainty region does not formally hold when $P(t_f)$ is unconstrained, and a suitable 'covering' conjecture is proposed and applied.

In [8], a tracking performance analysis paper, the ICs uncertainty was specified as an interval for each component of x_0, thus the ICs uncertainty region was a hyper-rectangle in $\mathcal{R}^n$; the system itself was not uncertain; and the development hinged on a theorem from [12]. Here, in contrast, we deal with a general analysis problem; both the system matrices and the system's ICs are uncertain, both in a general convex polytopic manner; and no use is made of the latter theorem: the development applies 'first principles'.

Notation: The signal norm addressed is the standard $\mathcal{L}_2$-norm, $||w||^2_{2,\,[t_0 \quad t_f]} = \int_{t_0}^{t_f} w^T(\tau)w(\tau)d\tau$, where w^T is the transpose of w. The space of continuous-time functions in $\mathcal{R}^p$ that are square integrable over $[t_0 \quad t_f]$ is $\mathcal{L}_2^p[t_0 \quad t_f]$. $\mathcal{C}o\{G^{(j)}\}$ denotes the convex hull of $G^{(j)}$, $j=1,\ldots,N$.

II. PROBLEMS FORMULATION

Consider the uncertain (stable or unstable) time-invariant linear system (some time-variance will later be introduced)

$$\begin{array}{rcl}\dot{x} &=& A\,x + B\,w, \quad x(t_0) = x_0 \\ z &=& C\,x + D\,w\end{array} \qquad \text{(1a,b)}$$

where $x \in \mathcal{R}^n$ is the system state, $w \in \mathcal{R}^p$ is a deterministic energy-bounded disturbance in $\mathcal{L}_2^p[t_0 \quad t_f]$, and $z \in \mathcal{R}^m$ is the signal to be regulated. The matrices $\{A, B, C, D\}$, collectively denoted

$$\Omega \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad (2)$$

belong to the convex hull $\mathcal{C}o\{\Omega^{(j)}, j = 1, \ldots, N\}$ of the given N vertices (vertex-systems)

$$\Omega^{(j)} \triangleq \begin{bmatrix} A^{(j)} & B^{(j)} \\ C^{(j)} & D^{(j)} \end{bmatrix}, \quad j = 1, \ldots, N\,. \qquad (3)$$

That is,

$$\Omega = \textstyle\sum_{j=1}^{N} f_j\, \Omega^{(j)}, \quad 0 \le f_j \le 1, \quad \sum_{j=1}^{N} f_j = 1\,. \qquad (4)$$

The variables f_j are called the convex coordinates (CC) of Ω in $\mathcal{C}o\{\Omega^{(j)}\}$.

The initial conditions (ICs) vector x_0 is also uncertain in our setup. This is not a new idea, of course. However, the common finite-horizon H_∞ analysis practice assumes $x_0 \in \mathcal{R}^n$; that is, allows infinite x_0 uncertainty. This assumption is most unrealistic since there must be 'limits' to x_0, and control designers usually can make plausible practical statements about them; thus, infinite x_0 uncertainty is a prohibitively conservative assumption. Moreover, this assumption is incompatible with the prevalent robust H_∞ analysis practice of attributing finite/limited uncertainty to the systems's parameters. For these reasons, in our setup we do not allow x_0 to reside anywhere in $\mathcal{R}^n$; its uncertainty is finite. In fact, the x_0 uncertainty model undertaken here is analogous to the uncertainty of $\{A, B, C, D\}$: there are M given vertices $x_0^{(i)} \in \mathcal{R}^n$ that define a convex ICs polytope $\mathcal{C}o\{x_0^{(i)}, i = 1, \ldots, M\}$, (only) in which x_0 may reside:

$$x_0 = \textstyle\sum_{i=1}^{M} g_i\, x_0^{(i)}, \quad 0 \le g_i \le 1, \quad \sum_{i=1}^{M} g_i = 1, \qquad (5)$$

where g_i are the CC of x_0 in $\mathcal{C}o\{x_0^{(i)}\}$.

Denote by $J(\Omega, x_0)$ the standard finite-horizon H_∞ cost (see [13]) corresponding to some Ω, that is a single system in $\mathcal{C}o\{\Omega^{(j)}\}$, and to a single $x_0 \in \mathcal{C}o\{x_0^{(i)}\}$:

$$J(\Omega, x_0) = \|z(\Omega, x_0)\|^2_{2,\,[t_0 \ \ t_f]} - \gamma^2 \|w\|^2_{2,\,[t_0 \ \ t_f]} + {x_f}^T \Delta\, x_f\,. \qquad (6)$$

$\|w\|_{2,\,[t_0 \ \ t_f]}$ is the finite-horizon $\mathcal{L}_2$-norm of w. $z(\Omega, x_0)$ is the z which emanates, in response to w, from a system having some Ω and some x_0; and $\|z(\Omega, x_0)\|_{2,\,[t_0 \ \ t_f]}$ is its $\mathcal{L}_2$-norm. Both $\mathcal{L}_2$-norms are computed with $t \in [t_0 \ \ t_f]$, where t_0 and t_f are given. The scalar $\gamma > 0$, also given, is under infinite-horizon H_∞ scenario (where x_0 is zero and $J < 0$ is assured) the bound on the disturbance attenuation level (or induced $\mathcal{L}_2$-norm) $\sup_{w \in \mathcal{L}_2\,[0 \ \ \infty),\, w \neq 0} \|z\|_2 / \|w\|_2$. Δ is a given nonnegative definite 'x_f-weight' matrix, where $x_f = x(t_f)$. An analysis problem can now be posed:

Problem: Find the minimal $J(\Omega, x_0)$ that can be jointly assured over given $\mathcal{C}o\{\Omega^{(j)}\}$ and $\mathcal{C}o\{x_0^{(i)}\}$, for all $w \in \mathcal{L}_{2\,[t_0 \ \ t_f]}$, $w \neq 0$, with given $\gamma > 0$ and $\Delta \ge 0$.

III. Solution

A. The prevailing LMIs

In the sequel, the very simple, conservative 'quadratically stabilizing' method will be used because it facilitates a clear exposition of our main idea. The latter, however, is by no means limited to a single Lyapunov-function.

It can easily be shown that, by using the single Lyapunov function $x^T P(t)\, x$, $0 < P(t) = P^T(t)$, and 'completing the squares', the following holds for the cost defined in (6), computed for the single system (1) [9]:

$$J(\Omega, x_0) = \textstyle\int_{t_0}^{t_f} x^T(\tau) \Psi(P(\tau)) x(\tau) d\tau + x_0^T P(t_0) x_0$$
$$-\gamma^2 \textstyle\int_{t_0}^{t_f} (w - w^*)^T (w - w^*)\, d\tau + x^T(t_f)(\Delta - P(t_f)) x(t_f),$$
$$R = \gamma^2 I - D^T D, \quad w^* = \gamma^{-2} B^T P x,$$
$$\Psi(P(t)) = P(A + BR^{-1}D^T C) + (A + BR^{-1}D^T C)^T P$$
$$+PBR^{-1}B^T P + C^T(I + DR^{-1}D^T)C + \dot{P}. \qquad \text{(7a-d)}$$

An upper bound on $J(\Omega, x_0)$ is identified as follows. The term $-\gamma^2 \int_{t_0}^{t_f} (w - w^*)^T (w - w^*)\, d\tau$ can obviously only reduce J (i.e. $J(w) < J(w^*)$), hence can be eliminated from the bound's expression; this is tantamount to choosing $w = w^*$. If a $P(t) > 0$ can be found such that $\Psi(P(t)) < 0$ is assured, as in the celebrated bounded-real lemma (BRL) [13], then the (negative) term $\int_{t_0}^{t_f} x^T(\tau) \Psi(P(\tau)) x(\tau) d\tau$ can also be eliminated. (Theoretical details regarding the negative-definiteness of $\Psi(P(t))$ can be found in [19].) Finally, constrain $P(t)$ by choosing

$$P(t_f) = \Delta. \qquad \text{(8a-c)}$$

This constraint will later be alleviated, enabling better performance at the price of complication of the analysis and some loss of convexity. The following is obtained:

$$J(\Omega, x_0) < x_0^T P(t_0) x_0. \qquad (9)$$

That is, $x_0^T P(t_0) x_0$ (unsurprisingly) constitutes the sought upper bound on $J(\Omega, x_0)$ under the worst-case disturbance w^*, $P(t) > 0$, $P(t_f) = \Delta$ (this requires Δ to be positive-definite, rather than nonnegative definite), and $\Psi(P(t)) < 0$, where $t \in [t_0 \ \ t_f]$.

The matrix inequality $\Psi(P(t)) < 0$ is equivalent, by the Schur complements formula, to the LMI

$$\tilde{\Psi}(P(t)) \triangleq \begin{bmatrix} \dot{P} + A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \qquad (10)$$

That is, the requirement $\Psi(P(t)) < 0$ can be fulfilled by assuring $\tilde{\Psi}(P(t)) < 0$ for some $P(t) > 0$ during $[t_0 \ \ t_f]$.

Note that if the system is unstable the steady-state version of (10), where $\dot{P} \equiv 0$ and $\tilde{\Psi} = \tilde{\Psi}(P)$, cannot be applied since $\tilde{\Psi}(P) < 0$ cannot be assured [13]. If the system is asymptotically stable a constant P may be tried, but it is usually too conservative in a finite-horizon setting. The latter setting can be better handled by a time-varying $P(t)$ since the term $\dot{P}$ and the changing $P(t)$ may make $\tilde{\Psi}(P(t))$ negative-definite; the system may then be unstable, and even time-varying.

Define $Q(t) \triangleq P^{-1}(t) > 0$ and $S \triangleq diag\{Q(t), I, I\}$. Note that (10) can be transformed into an equivalent LMI in $Q(t)$ by multiplying $\tilde{\Psi}(P(t))$ by $S = S^T > 0$ on both sides, since the negative-definiteness of $\tilde{\Psi}(P(t))$, now denoted $\bar{\Psi}(Q(t))$, is not altered by this operation. The result is

$$\bar{\Psi}(Q(t)) \triangleq \begin{bmatrix} -\dot{Q}+QA^T+AQ & B & QC^T \\ B^T & -\gamma I & D^T \\ CQ & D & -\gamma I \end{bmatrix} < 0, \quad (11)$$

where $\dot{Q} = (\dot{P^{-1}}) = -P^{-1}\dot{P}P^{-1} = -Q\dot{P}Q$ has been used. With $Q(t)$, (9) can be rephrased as

$$J(\Omega, x_0) < x_0^T Q^{-1}(t_0)x_0. \quad (12)$$

The (positive) bound $x_0^T Q^{-1}(t_0)x_0$ can be minimized, using again the Schur complements formula, by minimizing the scalar $\rho > 0$ while maintaining

$$\begin{bmatrix} \rho & x_0^T \\ x_0 & Q(t_0) \end{bmatrix} > 0\,. \quad (13)$$

Thus, for a single known (possibly time-varying) Ω and a single x_0, the LMIs (11) and (13) need simply be solved together for $\rho > 0$ and $Q(t) > 0$ while minimizing ρ. Note that the resulting ρ is, in fact, the value of the (lowest) bound $x_0^T Q^{-1}(t_0)x_0$ on $J(\Omega, x_0)$. Note also that if γ is free, rather than given, a lower bound on J is found, usually along with a very high γ (which makes the result almost an H_2 result); however, it turns out that the decrease in ρ is often small. Minimizing $\rho + \alpha\gamma$, where $\alpha > 0$ is an arbitrary weight, is a reasonable option. The minimal γ is usually associated with Q of extreme eigenvalues, thus it is common practice to 'pull back' from the minimal γ.

The problem posed at the end of Section II requires finding the minimal $J(\Omega, x_0)$ over the whole of $\mathcal{C}o\{\Omega^{(j)}\}$ and of $\mathcal{C}o\{x_0^{(i)}\}$, jointly. This can now, obviously, be readily achieved by applying standard convexity practices since (11) is convex in Ω and (13) is convex in x_0. Thus, the solution to the problem consists in simultaneously solving the two set of LMIs (nicknamed 'BRL LMIs' and 'ICs LMIs')

$$\begin{bmatrix} -\dot{Q(t)}+Q(t){A^{(j)}}^T+A^{(j)}Q(t) & B^{(j)} & Q(t){C^{(j)}}^T \\ {B^{(j)}}^T & -\gamma I & {D^{(j)}}^T \\ C^{(j)}Q(t) & D^{(j)} & -\gamma I \end{bmatrix} < 0,$$

$$j=1,\dots,N, \quad (14)$$

$$\begin{bmatrix} \rho & {x_0^{(i)}}^T \\ x_0^{(i)} & Q(t_0) \end{bmatrix} > 0, \quad i=1,\dots,M \quad (15)$$

for the matrix function $Q(t)$, which needs to satisfy

$$Q(t) > 0 \quad \forall t \in [t_0 \;\; t_f], \quad Q(t_f) = \Delta^{-1} > 0, \quad (16a,b)$$

and for the scalar $\rho > 0$, while minimizing ρ. (A procedure for finding $Q(t)$ will be outlined in the next subsection.)

Note that $Q(t)$ has, as customary in robust control, to accommodate the Ω uncertainty; but, and this is new, it has (simultaneously) to accommodate only a finite, given x_0 uncertainty region. This makes the solution inherently 'tighter' than methods which address $x_0 \in \mathcal{R}^n$. Note also that $\{A, B, C, D\}^{(j)}$ may, in principle, be time-varying; time-linear $\{B, D\}^{(j)}$ will be addressed later. If one wishes to solve 'freely' for $Q(t)$, without the constraint $Q(t_f) = \Delta^{-1}$, the term $x^T(t_f)(\Delta - P(t_f))x(t_f)$ remains in (7a) and requires further treatment, that will be shown later.

B. Solutions Utilizing Time-convexity

A well-known approach for finding a general time-varying $P(t)$ (or $Q(t)$) is the difference-LMIs (DLMIs) method [12], which is inherently suitable for time-varying systems. However, this method is extremely time-consuming because the computation time-step must be minute and an inversion of $P(t)$ is performed in each time-step. In contrast, the time-convexity method proposed at length in [7] is computationally efficient, simple, and flexibly allows (in its piecewise form) elaborate variations in $P(t)$ with very little numerical overhead.

In its generic form, the time-convexity approach allows $Q(t)$ to vary linearly with time over $[t_0 \;\; t_f]$, and (11) is assured for all $t \in [t_0 \;\; t_f]$ by convexity. To this end, define

$$\begin{aligned} &Q(t) = g_1(t)Q(t_0) + g_2(t)Q(t_f)\,, \\ &0 \le g_1(t) \le 1\,, \;\; 0 \le g_2(t) \le 1\,, \\ &g_1(t) + g_2(t) = 1\,, \;\; g_2(t) \triangleq \tfrac{t-t_0}{t_f-t_0}\,, \end{aligned} \quad (17a\text{-}e)$$

where in our setup $Q(t_0)$ is an unknown constant matrix, $Q(t_f)$ is the given Δ^{-1} (see (16b)), and $g_1(t)$, $g_2(t)$ are the convex coordinates of $Q(t)$ in $\mathcal{C}o\{Q(t_0), Q(t_f)\}$. In fact, g_2 is merely the normalized time, ranging from 0 at t_0 to 1 at t_f. Note that $Q(t_0) > 0$ and $Q(t_f) > 0$ assure, by the convex description (17a-d), that $Q(t) > 0$ for $t_0 \le t \le t_f$, as required by (16a). Obviously, $\dot{Q} = \frac{Q(t_f)-Q(t_0)}{t_f-t_0} = \text{Constant}$.

This convex characterization of $Q(t)$ over $[t_0 \;\; t_f]$ is inherited to $\bar{\Psi}(Q(t))$ because the latter is affine in $Q(t)$ and otherwise contains constants. Thus,

$$\bar{\Psi}(Q(t)) = g_1(t)\bar{\Psi}(Q(t_0)) + g_2(t)\bar{\Psi}(Q(t_f)) \quad (18)$$

i.e. $\bar{\Psi}(Q(t)) \in \mathcal{C}o\{\bar{\Psi}(Q(t_0)), \bar{\Psi}(Q(t_f))\}$. Thus, it suffices to find a $Q(t_0) > 0$ for which $\bar{\Psi}(Q(t_0)) < 0$ and to verify that $\bar{\Psi}(Q(t_f)) < 0$ for the given $Q(t_f) = \Delta^{-1} > 0$, since then $\bar{\Psi}(Q(t)) < 0 \;\forall t \in [t_0 \;\; t_f]$ by (18).

Thus, under basic time-convexity and with $Q_0 = Q(t_0)$, $Q_f = Q(t_f)$, the general equations (14)-(16) governing the solution to the problem specify to

$$\begin{bmatrix} -\frac{Q_f-Q_0}{t_f-t_0}+Q_0{A^{(j)}}^T+A^{(j)}Q_0 & B^{(j)} & Q_0{C^{(j)}}^T \\ {B^{(j)}}^T & -\gamma I & {D^{(j)}}^T \\ C^{(j)}Q_0 & D^{(j)} & -\gamma I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\frac{Q_f-Q_0}{t_f-t_0}+Q_f{A^{(j)}}^T+A^{(j)}Q_f & B^{(j)} & Q_f{C^{(j)}}^T \\ {B^{(j)}}^T & -\gamma I & {D^{(j)}}^T \\ C^{(j)}Q_f & D^{(j)} & -\gamma I \end{bmatrix} < 0,$$

$$j=1,\dots,N, \quad (19a,b)$$

$$\begin{bmatrix} \rho & {x_0^{(i)}}^T \\ x_0^{(i)} & Q_0 \end{bmatrix} > 0, \quad i=1,\dots,M \quad (20)$$

$$Q_0 > 0, \quad Q_f = \Delta^{-1} > 0. \quad (21a\text{-}c)$$

The solution is obtained by solving the above for the unknowns Q_0 and ρ, while minimizing ρ (Q_f is given by (21c)). The resulting $Q(t)$, $t \in [t_0 \;\; t_f]$, is given by (17). Note that the choice of Δ is restricted beyond just being positive-definite, since $Q_f = \Delta^{-1}$ appears in both (19b) and (19a).

As noted earlier, if γ is free rather than given, a lower ρ is attained. One can find the minimal ρ for progressively smaller values of γ (better induced $\mathcal{L}_2$-norms), or minimize $\rho + \alpha\gamma$ where $\alpha > 0$ is a suitable arbitrary 'weight'.

If B and D are linearly time-dependent over $\mathcal{C}o\{\Omega^{(j)}\}$,

$$\begin{aligned} B(t) &= g_1(t)B(t_0) + g_2(t)B(t_f)\,, \\ D(t) &= g_1(t)D(t_0) + g_2(t)D(t_f)\,, \end{aligned} \qquad (22)$$

the latter LMIs provide the solution to such a time-varying system, but with $B^{(j)}, D^{(j)}$ replaced by $B_0^{(j)} = B^{(j)}(t_0)$, $D_0^{(j)} = D^{(j)}(t_0)$ in (19a), and by $B_f^{(j)} = B^{(j)}(t_f)$, $D_f^{(j)} = D^{(j)}(t_f)$ in (19b).

Several advanced versions of the time-convexity approach are given in [7], which may be directly and easily applied to our problem in order to obtain better results than those offered by the generic time-convexity approach, described above. The general time-convexity method addresses a polytopically uncertain time-varying system and applies a piecewise-linear $Q(t)$ with Q-jumps at the intermediate time-instants; it even allows piecewise-constant vertex-dependent $\gamma^{(j)}(t)$. Details regarding the (straightforward) application of the latter method to our problem will not be given here.

C. Time-convexity With $Q_f \neq \Delta^{-1}$

The choice $P(t_f) = \Delta$, or $Q_f = \Delta^{-1}$, leads to the simple and elegant (9), (12), (13), (15), and (20). However, this choice constrains $Q(t)$, thus leading to more conservative results than those that may be obtained with a 'free' $Q_f > 0$. This choice also constrains (by (19)) Δ, the 'x_f-weight' matrix in (6), which in principle should be chosen freely by the control designer.

If $P(t_f)$ is not required to equal Δ, the term $x^T(t_f)(\Delta - P(t_f))\,x(t_f)$, which is of an indeterminate sign(!), does not vanish from $J(\Omega, x_0)$ and (9) becomes

$$J(\Omega, x_0) < x_0^T P(t_0) x_0 + x^T(t_f)(\Delta - P(t_f))x(t_f). \qquad (23)$$

The RHS of (23) will now be reduced to a quadratic expression in x_0 only (as in (9)), which enables transforming (23) into an LMI like (13), by utilizing the state-transition matrix which relates $x(t_f)$ to x_0 and by applying (7c) to express w^* in terms of x. Δ may again be nonnegative definite.

Since $w = w^*$ maximizes the RHS of (7a), it should be applied to find the bound on $J(\Omega, x_0)$. So, substitute $w = \gamma^{-2}B^T P x$ into (1a):

$$\dot{x} = A\,x + B\,(\gamma^{-2}B^T P x) = (A + \gamma^{-2}BB^T P)\,x\,. \qquad (24)$$

Define $\hat{A} = A + \gamma^{-2}BB^T P$ and note that here $\hat{A} = \hat{A}(t)$, even though the system (1) is time-invariant, since here $P = P(t)$. Constrain $P(t)$ to be linear in time (compare (17)):

$$P(t) = P(t_0) + (t - t_0)\dot{P}\,, \quad \dot{P} = \frac{P(t_f) - P(t_0)}{t_f - t_0}\,. \qquad (25a,b)$$

This results in linear dependence of $\hat{A}(t)$ on time:

$$\begin{aligned} \hat{A}(t) &= \hat{A}_0 + (t - t_0)\dot{\hat{A}}\,, \\ \hat{A}_0 &= A + \gamma^{-2}BB^T P_0\,, \\ \dot{\hat{A}} &= \gamma^{-2}BB^T \dot{P}\,, \end{aligned} \qquad (26a\text{-}c)$$

where $\hat{A}_0 = \hat{A}(t_0)$, $P_0 = P(t_0)$.

Denote by Φ the state-transition matrix associated with $\hat{A}(t)$, which satisfies

$$\frac{d}{dt}\Phi(t, t_0) = \hat{A}(t)\,\Phi(t, t_0), \quad \Phi(t_0, t_0) = I. \qquad (27a\text{-}b)$$

Since $\hat{A}(t)$ is linear in time, (27) has an explicit solution:

$$\Phi(t, t_0) = e^{(t-t_0)\hat{A}_0 + \frac{1}{2}(t-t_0)^2\dot{\hat{A}}}. \qquad (28)$$

Thus, for $w = w^*$ the state equation (1a) becomes the homogeneous equation (24), and with the time-linear $P(t)$ prescribed in (25) it has the explicit solution

$$x(t) = \Phi(t, t_0)\,x(t_0) = e^{(t-t_0)\hat{A}_0 + \frac{1}{2}(t-t_0)^2\dot{\hat{A}}}\,x_0\,. \qquad (29)$$

For $t = t_f$ we have

$$x_f = x(t_f) = e^{(t_f-t_0)\hat{A}_0 + \frac{1}{2}(t_f-t_0)^2\dot{\hat{A}}}\,x_0\,. \qquad (30)$$

Note that (24), (26) and (28)-(30) are (also) functions of the system matrices A and B, that is are Ω-dependent. The above expressions have the following versions at the vertices $\Omega^{(j)}$ of the system uncertainty region (with $P_f = P(t_f)$):

$$\begin{aligned} \hat{A}^{(j)}(t) &= \hat{A}_0^{(j)} + (t - t_0)\dot{\hat{A}}^{(j)} \\ \hat{A}_0^{(j)} &= A^{(j)} + \gamma^{-2}B^{(j)}{B^{(j)}}^T P_0 \\ \dot{\hat{A}}^{(j)} &= \gamma^{-2}B^{(j)}{B^{(j)}}^T\dot{P} = \gamma^{-2}B^{(j)}{B^{(j)}}^T\tfrac{P_f - P_0}{t_f - t_0} \end{aligned} \qquad (31a\text{-}c)$$

$$\begin{aligned} \Phi^{(j)}(t, t_0) &= e^{(t-t_0)\hat{A}_0^{(j)} + \frac{1}{2}(t-t_0)^2\dot{\hat{A}}^{(j)}} \\ \Phi_f^{(j)} &= \Phi^{(j)}(t_f, t_0) = e^{(t_f-t_0)\hat{A}_0^{(j)} + \frac{1}{2}(t_f-t_0)^2\dot{\hat{A}}^{(j)}} \end{aligned} \qquad (32a,b)$$

It is important to realize that $\Phi^{(j)}$ entails no convexity with respect to Ω, because of the exponentiation of the system matrices and because of the product $B^{(j)}{B^{(j)}}^T$. As for x_f, we denote the x_f corresponding to the j^{th} vertex-Ω and to the i^{th} vertex-x_0 as

$$x_f^{(i,j)} = \Phi^{(j)}(t_f, t_0)\,x_0^{(i)}, \quad i = 1, \ldots, M, \; j = 1, \ldots, N. \qquad (33)$$

The solution in this section again consists of a set of BRL LMIs like (19), now in terms of P rather that Q, and a set of ICs LMIs broader than (20), which emanate from (23). Because of (33), the ICs LMIs now involve also $\Omega^{(j)}$, but no convexity over $\mathcal{C}o\{\Omega^{(j)}\}$ can be guaranteed.

The governing BRL LMIs are:

$$\begin{bmatrix} \frac{P_f - P_0}{t_f - t_0} + {A^{(j)}}^T P_0 + P_0 A^{(j)} & P_0 B^{(j)} & {C^{(j)}}^T \\ {B^{(j)}}^T P_0 & -\gamma I & {D^{(j)}}^T \\ C^{(j)} & D^{(j)} & -\gamma I \end{bmatrix} < 0\,,$$

$$\begin{bmatrix} \frac{P_f - P_0}{t_f - t_0} + {A^{(j)}}^T P_f + P_f A^{(j)} & P_f B^{(j)} & {C^{(j)}}^T \\ {B^{(j)}}^T P_f & -\gamma I & {D^{(j)}}^T \\ C^{(j)} & D^{(j)} & -\gamma I \end{bmatrix} < 0\,,$$

$$j = 1, \ldots, N\,. \qquad (34a,b)$$

The RHS of (23) is the (possibly negative) bound on the cost. It can be minimized by minimizing the (possibly negative) scalar ρ in

$$\begin{aligned} \rho > x_0^T P(t_0) x_0 + x^T(t_f)(\Delta - P(t_f))x(t_f) \quad &\Rightarrow \\ \rho - x_0^T P(t_0) x_0 > -x^T(t_f)(P(t_f) - \Delta)x(t_f). & \end{aligned}$$

The governing ICs LMIs are now obtained by applying Schur complements on both sides of the latter inequality, and using (33):

$$\begin{bmatrix} \rho & {x_0^{(i)}}^T \\ x_0^{(i)} & {P_0}^{-1} \end{bmatrix} > \begin{bmatrix} 0 & {x_0^{(i)}}^T {\Phi_f^{(j)}}^T \\ \Phi_f^{(j)} x_0^{(i)} & (P_f - \Delta)^{-1} \end{bmatrix}, \tag{35}$$

$$i=1,\ldots,M, \quad j=1,\ldots,N.$$

The solution is obtained, in principle, by solving (34)-(35), where $\Phi_f^{(j)}$ is defined in (31)-(32), for the unknown matrices $P_0 > 0$ and $P_f > 0$ and the scalar ρ, while minimizing ρ. The resulting minimal ρ is the minimal $J(\Omega, x_0)$ that can be assured over the whole ICs uncertainty region and (simultaneously) over the finite set of vertices of the system-matrices uncertainty region. One may claim that addressing the whole $\mathcal{C}o\{\Omega^{(j)}\}$ in (34), by convexity, and all $\Omega^{(j)}$ in (35), provides 'some' assurance over the whole $\mathcal{C}o\{\Omega^{(j)}\}$. One may even add points within $\mathcal{C}o\{\Omega^{(j)}\}$, e.g. its 'center' defined by $f_j = 1/N \ \forall j$ (see [5],[6]), to the set solved in (35) in order to 'improve the coverage' of $\mathcal{C}o\{\Omega^{(j)}\}$. The resulting $P(t)$, $t\in[t_0 \ \ t_f]$, is given by (25).

Obviously, (34) and (35) cannot be solved together by LMI solvers because P_0 and P_f appear nonlinearly in (35) and in (32b) (see (31b-c)). A plausible procedure to obtain a sub-optimal solution consists of the following three steps:

1) Solve (34) for $P_0 > 0$ and $P_f > 0$;
2) Use these P_0 and P_f to compute all $\Phi_f^{(j)}$ according to (32b) and (31b-c);
3) Plug these $\Phi_f^{(j)}$, P_0 and P_f in (35), and minimize ρ.

An alternative approach, which attempts to find the minimal cost over $\mathcal{C}o\{\Omega^{(j)}\}$ by 'full' convexity, hinges upon the following intuitive x_f-convexification conjecture:

Conjecture 1: Find among the MN points $x_f^{(i,j)}$ in $\mathcal{R}^n$ (see (33)) a subset of L vertices ($L\leq MN$) such that all these MN points belong to $\mathcal{C}o\{x_f^{(k)},\ k=1,\ldots,L\}$. Then, for any Ω in $\mathcal{C}o\{\Omega^{(j)}\}$ and any x_0 in $\mathcal{C}o\{x_0^{(i)}\}$, the corresponding x_f (see (30)) is in $\mathcal{C}o\{x_f^{(k)},\ k=1,\ldots,L\}$.

The conjecture is applied as follows. First, re-write (35) in terms of $x_f^{(k)}$:

$$\begin{bmatrix} \rho & {x_0^{(i)}}^T \\ x_0^{(i)} & {P_0}^{-1} \end{bmatrix} > \begin{bmatrix} 0 & {x_f^{(k)}}^T \\ x_f^{(k)} & (P_f - \Delta)^{-1} \end{bmatrix}, \tag{36}$$

$$i=1,\ldots,M, \quad k=1,\ldots,L.$$

Then, obtain a sub-optimal solution by the following five steps (steps 1 and 2 are unchanged):

1) Solve for $P_0 > 0$ and $P_f > 0$;
2) Use these P_0 and P_f to compute all $\Phi_f^{(j)}$;
3) Use these $\Phi_f^{(j)}$ and all the given $x_0^{(i)}$ to compute the MN points $x_f^{(i,j)}$ (see (33)); the final-states can also be found by direct integration of the N vertex-systems (with the P_0 and P_f of step 1) from the M vertex-ICs;
4) Find among these MN points a subset of L vertices ($L\leq MN$) such that all MN points $x_f^{(i,j)}$ belong to $\mathcal{C}o\{x_f^{(k)},\ k=1,\ldots,L\}$;
5) Plug these L points $x_f^{(k)}$, together with the P_0 and P_f found in step 1, in (36), and minimize ρ.

IV. Numerical Example

Consider these $\Omega^{(1:3)}$, $x_0^{(1:3)}$, and problem parameters:

$$A^{(1)} = \begin{bmatrix} 0 & 1 \\ -1.8 & -0.5 \end{bmatrix}, A^{(2)} = \begin{bmatrix} -0.9 & 0.2 \\ 0.6 & -0.4 \end{bmatrix}, A^{(3)} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$B^{(1)} = B^{(3)} = \begin{bmatrix} -1.4 \\ 1 \end{bmatrix},\ B^{(2)} = \begin{bmatrix} -0.6 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \ t_0 = 0,\ t_f = 2,\ \ \Delta = \begin{bmatrix} 2 & 0.2 \\ 0.2 & 1.5 \end{bmatrix},$$

$$x_0^{(1)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix},\ x_0^{(2)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ x_0^{(3)} = \begin{bmatrix} 0 \\ -2 \end{bmatrix},\ \gamma = 20\,.$$

Applying (19)-(21), the minimal $J(\Omega, x_0)$ that can be jointly assured over the ICs and parameters uncertainty regions is $\rho = 3996$, and $Q_0 = \begin{bmatrix} 0.0425 & 0.0463 \\ 0.0463 & 0.0550 \end{bmatrix}$. If $x_0^{(1:3)}$ is enlarged tenfold, ρ (unsurprisingly) grows by 10^2 and Q_0 remains the same. If γ is free and $\rho + 10\gamma$ is minimized, the solution is $\rho = 70.68$, $\gamma = 25.4$, and $Q_0 = \begin{bmatrix} 0.1301 & 0.0208 \\ 0.0208 & 0.1381 \end{bmatrix}$. If $x_0^{(1:3)}$ is enlarged tenfold, and $\rho+10\gamma$ is minimized, we obtain $\rho = 2547.3$, $\gamma = 62.7$, and $Q_0 = \begin{bmatrix} 0.2938 & -0.0000 \\ -0.0000 & 0.3374 \end{bmatrix}$.

Now we go through the three-step procedure for obtaining a (sub-optimal) solution of (34)-(35) without the conjecture. The first step results in $P_0 = \begin{bmatrix} 2.8245 & 0.9005 \\ 0.9005 & 1.8362 \end{bmatrix}$, $P_f = \begin{bmatrix} 0.2538 & 0.1828 \\ 0.1828 & 0.1536 \end{bmatrix}$. If γ is free, the result is $\gamma = 19.42$, $P_0 = \begin{bmatrix} 2.0013 & 0.5682 \\ 0.5682 & 1.5818 \end{bmatrix}$ and $P_f = \begin{bmatrix} 0.1803 & 0.1573 \\ 0.1573 & 0.1373 \end{bmatrix}$. We proceed with the results applicable to the given γ.

The second step calls for the computation of $\Phi_f^{(1:3)}$, which turn out to be $\Phi_f^{(1)} = \begin{bmatrix} -0.4779 & 0.2235 \\ -0.4035 & -0.5908 \end{bmatrix}$, $\Phi_f^{(2)} = \begin{bmatrix} 0.2246 & 0.1224 \\ 0.3684 & 0.5328 \end{bmatrix}$, and $\Phi_f^{(3)} = \begin{bmatrix} 1.0318 & -0.0019 \\ -0.0091 & 1.2229 \end{bmatrix}$.

To demonstrate the design freedom and better performance afforded by **not** requiring $P(t_f) = \Delta$, we compute the third step (only in which Δ appears) with various Δ's. The original Δ leads to $\rho = 21.61$; a tenfold larger Δ leads to $\rho = 55.75$; a 50% smaller Δ leads to $\rho = 10.39$. Note the drastic drop in ρ relative to 3996 (about 99.5%, with the original Δ). However, the latter small ρ's are not analytically guaranteed over the whole parameters uncertainty region.

Next, we 'trust the conjecture' and go through the five-step procedure for obtaining a (sub-optimal) solution of (34)-(35). Steps 1 and 2 are the same as in the previous procedure. The results of steps 3 and 4 are depicted in Figure 1; the nine final states corresponding to the three vertex-Ω and the three vertex-x_0 are the same as (implicitly addressed) in the previous procedure; the convex hull $\mathcal{C}o\{x_f^{(k)}\}$, based on $L = 5$ final states only, was obtained by Matlab's *convhull* function. Step 5 produces $\rho = 21.61$, the same result as without the conjecture (this intuitively 'corroborates' the conjecture). If the conjecture is true, then under time-convexity over $[t_0 \ \ t_f]$ (without time sub-divisions), this is the minimal $J(\Omega, x_0)$ that can be jointly assured, using the single Lyapunov function $x^T P(t)\, x$, over the given ICs and parameters uncertainty regions, with the given γ and Δ.

V. Conclusions

Tractable solutions, based on time-convexity, have been presented to a practical finite-horizon robust H_∞ analysis problem, where the optimization is with respect to a standard H_∞ cost function with a given/free disturbance attenuation level. The problem is 'practical' because not only the system uncertain matrices lie in finite convex polytope; the uncertain initial conditions lie also in finite convex polytope. This approach removes the severe conservatism incurred by allowing 'any' initial condition.

Full parameters/ICS/time convexity applies when the final-state weight matrix, which appears in the H_∞ cost to be minimized, constrains the final value of matrix time-function used in the quadratic Lyapunov function. Without this constraint, the result is not convex over the parameters uncertainty region because the state-transition matrix is not convex with respect to the system matrices. For this case, a sub-optimal simple solution procedure is suggested. Finally, a final-state convexity conjecture is proposed, under which full convexity may be guaranteed for the solution. The numerical example shows a dramatic advantage of the two procedures which do not require $P(t_f) = \Delta$.

Advanced versions of time-convexity (see [7]), more elaborate Lyapunov functions, and sophisticated BRL LMI-formulations, can be applied within the proposed approach. Extension of the method to state-feedback control design is straightforward. The conjecture might be useful in other robust finite-horizon control problems.

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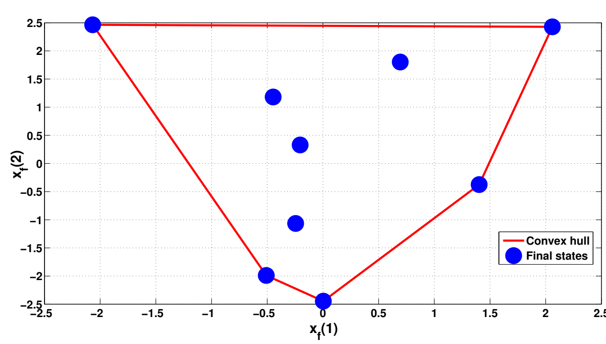

Fig. 1. The nine final vertex-states and their convex hull