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Sliding mode observer based control for a class of nonlinear time delay systems with delayed uncertainties*

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Abstract— **In this paper, a class of nonlinear time varying delay systems is considered. A sliding mode observer is proposed by employing the structure of the uncertainty such that the sliding motion associated with the error dynamics is uniformly asymptotically stable. Then, based on the observer, a nonlinear discontinuous control scheme is presented to stabilise the system uniformly asymptotically. The Razuminkin Lyapunov approach is used to deal with the time delay. The accessible part of both the bounds and the nonlinear terms is separated and used in the control design to reduce conservatism. A numerical example is given to illustrate the proposed approach.**

I. INTRODUCTION

Many theoretical studies assume that system states are available for control design, which is not valid for many real systems. In order to implement such developed control schemes, an observer may be constructed to estimate the system states. Unfortunately, the traditional separation principle usually does not hold for nonlinear control systems, which implies that the properties of existing state feedback control law may not be valid when the control law is implemented with the estimate states. Therefore, it is necessary to construct an appropriate control strategy to combine with the dynamical observer if nonlinearity exists in the system.

Recently, observer-based control for time delay systems has received much attention [6], [5], [8], [3], [2]. The backstepping approach is employed in [2]. By choosing an appropriate Lyapunov Krasovskii function, a high gain parameterized linear controller is presented in [3]. In both [3] and [2], it is required that the systems considered have a special structure. An observer-based sliding mode control is proposed in [6] where it is required that the nonlinear term is matched. Luo *et al* study a class of time-delay systems using static and dynamic output feedback strategies in [5] but it is required that the uncertainty is matched. Moreover, all the results mentioned here require that the bounds on the mismatched uncertainties satisfy a linear growth condition. Since uncertainty bounds may have nonlinear forms in reality, it is pertinent to consider the case when the bounds on the uncertainties are nonlinear. More recently, a sliding mode control scheme is proposed for a class of nonlinear system in [8] where the bounds on the uncertainties have been extended to the nonlinear case. However, like most of the existing work, it is required that the input distribution matrix is constant. Compared with [8], the observer used in this paper is sliding mode observer while the control is not sliding mode control.

In this paper, a class of nonlinear time varying delay control systems with uncertainties is considered. The bounds on the uncertainties are nonlinear and time delayed. The accessible parts of the bounds and the nonlinear terms are separated and employed in the control design to reduce the effects of the uncertainty and nonlinearity. By employing the system structure and an appropriate coordinate transformation, a robust sliding mode observer is designed for the system. The error dynamics are uniformly asymptotically stable and completely robust to the uncertainty. Then, based on the designed observer, a discontinuous control law is proposed to stabilise the system uniformly asymptotically even in the presence of uncertainties and time delay. The well known Razuminkhin Lyapunov approach is employed to deal with the time delay in the stability analysis of the closed-loop system Both the nonlinear term and the uncertainty are mismatched. The input distribution matrix is a nonlinear function matrix. The only limitation on the time delay is that it is bounded. There is no limitation on the rate of change (time derivative) of the time varying delay. Simulation results reflect the effectiveness of the approach proposed.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

Definition 1. A continuous function $\alpha : \ [0,a) \mapsto [0,\infty)$ is called a class K function if it is strictly increasing and $\alpha(0) = 0$ (see, [4]). Further, a class K function $\alpha(\cdot)$ is called a strong class K function if $\alpha(r) = \kappa(r)r$ for some continuous function $\kappa(\cdot)$ in $\mathcal{R}^+$.

Definition 2. A function vector/matrix $f(x_1, x_2)$ ($x_i \in \Omega_i \subset \mathcal{R}^{n_i}$ for $i = 1, 2$) is said to satisfy the Lipschitz condition with respect to (w.r.t.) x_2 in Ω_2 if there exists a function $\mathcal{L}_f(\cdot)$ defined in $x_1 \in \Omega_1$ such that for any $x_2, \hat{x}_2 \in \Omega_2$

$$\|f(x_1, x_2) - f(x_1, \hat{x}_2)\| \le \mathcal{L}_f(x_1)\|x_2 - \hat{x}_2\|, \quad x_1 \in \Omega_1$$

where the function $\mathcal{L}_f(\cdot)$ is called the generalised Lipschitz constant (see, [7]).

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Consider a nonlinear system described by

$$\dot{x} = Ax + G(t,y)u + \Phi(t,x,x_d) + \Delta\Phi(t,x,x_d) \quad (1)$$
$$y = Cx, \quad (2)$$

where $x \in \Omega \subset \mathcal{R}^n$, $u, y \in \mathcal{R}^m$ are the states, inputs and outputs respectively; $A \in \mathcal{R}^{n\times n}$ and $C \in \mathcal{R}^{m\times n}$ $(m < n)$ are constant matrices with C being of full rank; $G(\cdot) \in \mathcal{R}^{n\times m}$ is assumed to be known and full rank; $\Phi(\cdot)$ is known and $\Delta\Phi(\cdot)$ includes all the uncertainties. The symbol $x_d := x(t-d)$ is the delayed state where $d := d(t)$ is the time varying delay which is assumed to be known, continuous, nonnegative and bounded in $\mathcal{R}^+ := \{t \mid t \geq 0\}$, that is $\overline{d} := \sup_{t\in\mathcal{R}^+}\{d(t)\} < \infty$. The initial condition is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0] \quad (3)$$

where $\phi(\cdot)$ is continuous in $[-\overline{d}, 0]$. All the nonlinear functions are assumed to be smooth enough, which guarantees that the unforced system has a unique continuous solution.

Assumption 1. The matrix pair (A, C) is observable. The known function $\Phi(\cdot)$ is Lipschitz w.r.t x and x_d.

From Assumption 1, there exists a matrix L such that $A - LC$ is stable, and thus for any $Q > 0$ the following Lyapunov equation has a unique solution $P > 0$

$$(A - LC)^T P + P(A - LC) = -Q \quad (4)$$

Assumption 2. The uncertainty $\Delta\Phi(\cdot)$ has decomposition

$$\Delta\Phi(t,x,x_d) = E\Delta\Psi(t,x,x_d) \quad (5)$$

and the distribution matrix $E \in \mathcal{R}^{n\times p}$ satisfies

$$E^T P = FC \quad (6)$$

where $F \in \mathcal{R}^{p\times m}$, P satisfies (4), and $\Delta\Psi(\cdot)$ satisfies

$$\|\Delta\Psi(t,x,x_d)\| \leq \xi_1(t,y,\|y_d\|)\xi_2(t,x,x_d) \quad (7)$$

where $\xi_1(\cdot,\cdot,r)$ is nondecreasing w.r.t. r in $\mathcal{R}^+$ with $\xi_1(t,0,r) = 0$ and $\xi_2(t,x,x_d)$ is Lipschitz w.r.t. x and x_d.

Assumption 3. There exist continuous $u^a(\cdot) : \mathcal{R}^+ \times \mathcal{R}^n \mapsto \mathcal{R}^m$ which is Lipshitz w.r.t x, C^1 function $V_0(t,x) : \mathcal{R}^+ \times \mathcal{R}^n \mapsto \mathcal{R}^+$, class K functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, and strong class K functions $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ such that

i). $\alpha_1(\|x\|) \leq V_0(t,x) \leq \alpha_2(\|x\|)$;

ii). $\frac{\partial V_0}{\partial t} + \left(\frac{\partial V_0}{\partial x}\right)^T \left(Ax + G(\cdot)u^a(t,x)\right) \leq -\alpha_3(\|x\|)$ where

$$\alpha_3(\|x\|) = \varpi_1(\|x\|)\|x\|^2 \quad (8)$$

for some continuous function $\varpi_1(\cdot)$.

iii). $\left\|\frac{\partial V_0}{\partial x}\right\| \leq \alpha_4(\|x\|)$ where for some continuous $\varpi_2(\cdot)$

$$\alpha_4(r) = \varpi_2(r)r \quad (9)$$

where $x := \mathrm{col}(x_1, \cdots, x_n)$, $\frac{\partial V_0}{\partial x} := \begin{bmatrix}\frac{\partial V_0}{\partial x_1} & \cdots & \frac{\partial V_0}{\partial x_n}\end{bmatrix}^T$.

Assumption 4. The Lyapunov function $V_0(\cdot)$ in Assumption 3 satisfies

i). there exists a nonsingular matrix $M(\cdot)$ such that

$$G^T(t,y)\frac{\partial V_0}{\partial x} = M(t,y)y \quad (10)$$

ii). there exists a continuous nondecreasing scalar function $\zeta(r) > r$ for $r > 0$ such that for any $d \in [0,\ \bar{d}]$,

$$\|x_d\| \leq \rho\|x\| \quad \text{if} \quad V_0(x_d) \leq \zeta\left(V_0(x)\right)$$

where ρ is a positive constant.

Without loss of generality, it is assumed throughout the paper that the output matrix C in (1)–(2) has the form

$$C = [\,0 \quad I_m\,] \quad (11)$$

Then, system (1)–(2) can be rewritten as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}G_1(t,y)\\ G_2(t,y)\end{bmatrix}}_{G(\cdot)}u + \underbrace{\begin{bmatrix}\Phi_1(t,x_1,x_2,x_{1d},x_{2d})\\ \Phi_2(t,x_1,x_2,x_{1d},x_{2d})\end{bmatrix}}_{\Phi(\cdot)} + \underbrace{\begin{bmatrix}E_1\\ E_2\end{bmatrix}}_{E}\Delta\Psi(\cdot) \quad (12)$$

$$y = [\,0 \quad I_m\,]\,x \quad (13)$$

where $x = \mathrm{col}(x_1, x_2)$, $x_1 \in \mathcal{R}^{n-m}$, $A_1 \in \mathcal{R}^{(n-m)\times(n-m)}$, $E_1 \in \mathcal{R}^{(n-m)\times p}$ and equation (5) is employed to obtain (12). The terms $G_1(\cdot)$ and $\Phi_1(\cdot)$ are the first $n-m$ components of $G(\cdot)$ and $\Phi(\cdot)$ respectively.

Introduce partitions of P and Q which are conformable with the decomposition in (12)–(13):

$$P = \begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}, \qquad Q = \begin{bmatrix}Q_1 & Q_2\\ Q_2^T & Q_3\end{bmatrix} \quad (14)$$

It is clear from $P > 0$ and $Q > 0$ that $P_1 > 0$, $P_3 > 0$, $Q_1 > 0$ and $Q_3 > 0$. Using the matrix partitions in (14), it follows from (6) and (11) that

$$[0 \quad F] = FC = [\,E_1^T \quad E_2^T\,]\,P$$
$$= [\,\left(P_1(E_1 + P_1^{-1}P_2E_2)\right)^T \quad E_1^TP_2 + E_2^TP_3\,]$$

which implies that

$$P_1(E_1 + P_1^{-1}P_2E_2) = 0 \quad (15)$$

Remark 1. Since $y = x_2$ in equation (12)–(13), the condition that a nonlinear function is Lipschitz w.r.t. x and x_d can be reduced to that the function is Lipschitz w.r.t. x_1 and x_{1d} in their definition domain throughout the paper.

III. SLIDING MODE OBSERVER DESIGN

Firstly introduce a coordinate transformation:

$$z = Tx := \begin{bmatrix}I_{n-m} & P_1^{-1}P_2\\ 0 & I_m\end{bmatrix}x \quad (16)$$

From (15), system (12)–(13), in the new coordinate system z, can be described by

$$
\begin{aligned}
\dot{z}_1 &= (A_1 + P_1^{-1}P_2A_3)z_1 + \big(A_2 - A_1P_1^{-1}P_2 + \\
&\quad P_1^{-1}P_2(A_4 - A_3P_1^{-1}P_2)\big)z_2 + [I_{n-m} \quad P_1^{-1}P_2]G(\cdot)u \\
&\quad +[I_{n-m} \quad P_1^{-1}P_2]\Phi(t, T^{-1}z, T^{-1}z_d) \qquad (17) \\
\dot{z}_2 &= A_3z_1 + \big(A_4 - A_3P_1^{-1}P_2\big)\, z_2 + G_2(t, y)u + \\
&\quad \Phi_2(t, T^{-1}z, T^{-1}z_d) + E_2\Delta\Psi(t, T^{-1}z, T^{-1}z_d) \qquad (18) \\
y &= z_2 \qquad (19)
\end{aligned}
$$

where $z = \text{col}(z_1, z_2)$ with $z_1 \in \mathcal{R}^{n-m}$. From (3), the initial condition related to the delay is given by

$$z(t) = T\phi(t) := \psi_1(t), \quad t \in [-\bar{d}, 0] \qquad (20)$$

For system (17)–(19), consider a dynamical system

$$
\begin{aligned}
\dot{\hat{z}}_1 &= (A_1 + P_1^{-1}P_2A_3)\hat{z}_1 + \big(A_2 - A_1P_1^{-1}P_2 + P_1^{-1}P_2 \\
&\quad \cdot(A_4 - A_3P_1^{-1}P_2)\big)y + [I_{n-m} \quad P_1^{-1}P_2]G(\cdot)u \\
&\quad +[I_{n-m} \quad P_1^{-1}P_2]\Phi(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) \qquad (21) \\
\dot{\hat{z}}_2 &= A_3\hat{z}_1 + \big(A_4 - A_3P_1^{-1}P_2\big)\,\hat{z}_2 + D(y - \hat{z}_2) \\
&\quad +G_2(t, y)u + \Phi_2(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) + \nu(\cdot) \qquad (22)
\end{aligned}
$$

where

$$\hat{z}_y := \begin{bmatrix} \hat{z}_1 \\ y \end{bmatrix}, \qquad \hat{z}_{yd} := \begin{bmatrix} \hat{z}_{1d} \\ y_d \end{bmatrix} \qquad (23)$$

the matrix D is chosen such that $A_4 - A_3P_1^{-1}P_2 - D$ is Hurwitz stable, and the term $\nu(\cdot)$ is defined by

$$
\begin{aligned}
\nu(\cdot) &= \big(A_4 - A_3P_1^{-1}P_2 - D\big)\,(y - \hat{z}_2) + \Big(\|E_2\|\xi_1(\cdot) \\
&\quad \cdot\xi_2(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) + k(\cdot)\Big)\text{sgn}(y - \hat{z}_2) \qquad (24)
\end{aligned}
$$

where $k(\cdot)$ is to be determined later. The initial condition related to the delay is given by

$$\hat{z}(t) = \psi_2(t), \quad t \in [-\bar{d}, 0] \qquad (25)$$

where $\psi_2(\cdot)$ can be chosen as any continuous function such that

$$\|\psi_1(t) - \psi_2(t)\| \le b_0 \qquad (26)$$

for some constant b_0, where $\psi_1(\cdot)$ is defined in (20).

Let $e_{z_1} = z_1 - \hat{z}_1$, and $e_{z_2} = z_2 - \hat{z}_2$. Then from (17)–(19) and (21)–(22), the error dynamical equation is described by

$$
\begin{aligned}
\dot{e}_{z_1} &= (A_1 + P_1^{-1}P_2A_3)e_{z_1} + [I_{n-m} \quad P_1^{-1}P_2]\delta(\Phi) \quad (27) \\
\dot{e}_{z_2} &= A_3e_{z_1} + \big(A_4 - A_3P_1^{-1}P_2 - D\big)\,e_{z_2} + \delta(\Phi_2) \\
&\quad +E_2\Delta\Psi(t, T^{-1}z, T^{-1}z_d) - \nu(\cdot) \qquad (28)
\end{aligned}
$$

where $\nu(\cdot)$ is defined by (24), and the functional operator $\delta(\cdot)$ is defined by (65) in the Appendix.

For system (27)–(28), consider a sliding surface

$$S := \{(e_{z_1}, e_{z_2}) \mid e_{z_2} = 0\} \qquad (29)$$

Theorem 1. Under Assumptions 1 and 2, the sliding motion of system (27)–(28) associated with the sliding surface (29) is uniformly asymptotically stable if there exists a constant $q_0 > 1$ such that

$$q := \lambda_{\min}(Q_1) - 2\|[P_1 \quad P_2]\|\,\|T^{-1}\|\mathcal{L}_\Phi\left(1 + \sqrt{q_0\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\right) > 0 \qquad (30)$$

Proof: From the definition of the sliding surface in (29), it is clear that system (27) is the sliding mode dynamics which govern the sliding motion, and thus it is only necessary to prove that (27) is uniformly asymptotically stable.

Applying matrix block multiplication to equation (4), it follows from the partition (14) that

$$A_1^TP_1 + A_3^TP_2^T + P_1A_1 + P_2A_3 = -Q_1$$

This implies

$$(A_1 + P_1^{-1}P_2A_3)^TP_1 + P_1(A_1 + P_1^{-1}P_2A_3) = -Q_1 \quad (31)$$

From (66) in Lemma 1 in the Appendix,

$$\|\delta(\Phi)\| \le \|T^{-1}\|\mathcal{L}_\Phi(\|e_{z_1}(t)\| + \|e_{z_{1d}}(t)\|) \qquad (32)$$

For system (27), consider the Lyapunov function $V_e = e_{z_1}^TP_1e_{z_1}$. If there is a constant $q_0 > 1$ such that $V_e(e_{z_{1d}}) \le q_0V_e(e_{z_1})$, then, from the definition of V_e,

$$\|e_{z_{1d}}\| \le \sqrt{q_0\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\;\|e_{z_1}\| \qquad (33)$$

and thus, using (31), (32) and (33), the derivative of V_e along the trajectories of the system (27) is described by

$$
\begin{aligned}
&\dot{V}_e = -e_{z_1}^TQ_1e_{z_1} + 2e_{z_1}^TP_1[I_{n-m} \quad P_1^{-1}P_2]\delta(\Phi) \\
&\le -\lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1 \quad P_2]\|\,\|T^{-1}\|\mathcal{L}_\Phi(\|e_{z_1}\| \\
&\quad + \sqrt{q_0\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\;\|e_{z_1}\|) = -q\|e_{z_1}\|^2 \qquad (34)
\end{aligned}
$$

Hence the conclusion follows from $q > 0$. ∇

Remark 2. Theorem 1 has shown that $e_{z_1}(t)$ in the error equation (27) is uniform asymptotic stable. From (34) and the definition of V_e, there exist constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\|e_{z_1}(t)\| \le \beta_1\exp\{-\beta_2t\} =: b_1(t), \qquad t \ge 0 \qquad (35)$$

where β_1 is related to the initial value $z_1(0)$ and $\hat{z}_1(0)$. Both β_1 and β_2 can be calculated using basic matrix theory. From (26) and (35), it follows that

$$
\begin{aligned}
\|e_{z_{1d}}(t)\| &\le \max\Big\{\beta_1\exp\{\beta_2\bar{d}\}\exp\{-\beta_2t\}, b_0\Big\} \\
&=: b_2(t) \qquad (36)
\end{aligned}
$$

where b_0 is given in (26).

Theorem 2. Under Assumptions 1 and 2, system (27)–(28) with $\nu(\cdot)$ given in (24) is driven to the sliding surface (29) in finite time and remains on it thereafter if $k(\cdot)$ is chosen as

$$
\begin{aligned}
k &= \|A_3\|b_1(t) + (\mathcal{L}_{\Phi_2} + \|E_2\|\xi_1(t, y, \|y_d\|)\mathcal{L}_{\xi_2})\,\|T^{-1}\| \\
&\quad \cdot(b_1(t) + b_2(t)) + \eta \qquad (37)
\end{aligned}
$$

where the functions $b_1(\cdot)$ and $b_2(\cdot)$ are determined by (35) and (36) respectively, $\xi_1(\cdot)$ and $\xi_2(\cdot)$ are defined in (7), and η is a positive constant.

Proof: From equation (28)

$$e_{z_2}^T \dot{e}_{z_2} = e_{z_2}^T (A_4 - A_3 P_1^{-1} P_2 - D) e_{z_2} + e_{z_2}^T \Big(A_3 e_{z_1} + \delta(\Phi_2) + E_2 \Delta\Psi(t, T^{-1}z, T^{-1}z_d) \Big) - e_{z_2}^T \nu(\cdot) \quad (38)$$

It is clear that for any vector e_{z_2},

$$e_{z_2}^T \mathrm{sgn}(y - \hat{z}_2) = e_{z_2}^T \mathrm{sgn}(e_{z_2}) \geq \|e_{z_2}\| \quad (39)$$

Then, by applying (7), (39) and (24) to (38),

$$e_{z_2}^T(t)\dot{e}_{z_2}(t) \leq \|A_3\| \, \|e_{z_1}\| \, \|e_{z_2}\| + \Big(\|E_2\| \xi_1(t, y, \|y_d\|) \cdot \delta(\xi_2) + \delta(\Phi_2) \Big) \|e_{z_2}\| - k(\cdot)\|e_{z_2}\| \quad (40)$$

where $\delta(\cdot)$ is a functional operator defined in (65) in Appendix. From (66) in Lemma 1, (35) and (36),

$$\|\delta(\xi_2)\| \leq \mathcal{L}_{\xi_2} \|T^{-1}\| (b_1(t) + b_2(t)) \quad (41)$$
$$\|\delta(\Phi_2)\| \leq \mathcal{L}_{\Phi_2} \|T^{-1}\| (b_1(t) + b_2(t)) \quad (42)$$

Applying (37), (41) and (42) to (40) yields

$$e_{z_2}^T \dot{e}_{z_2} \leq -\eta \|e_{z_2}\| \quad (43)$$

Hence the conclusion follows. ∇

Theorems 1 and 2 together show that system (27)–(28) is uniform asymptotically stable. Thus, (21)–(22) is a sliding mode observer for system (17)–(19). Clearly , the formula

$$\hat{x} = T\hat{z}_y \quad (44)$$

gives an estimate for the states x of the dynamical system (1), where T is defined in (16) and $\hat{z}_y$ is defined in (23) with $\hat{z}_1$ given by (21)–(22). Actually, from $z = Tx$,

$$\|x - \hat{x}\| = \left\| T^{-1}z - T^{-1}\hat{z}_y \right\| \leq \|T^{-1}\| \, \|e_{z_1}\| \quad (45)$$

and thus $\hat{x}$ defined in (44) gives an estimate for the state x.

IV. OUTPUT FEEDBCAK SYNTHESIS

Introduce a function

$$\mathcal{H}(t, x, x_d) := \varpi_2(\|x\|)\xi_2(t, x, x_d) \quad (46)$$

where $\varpi_2(\cdot)$ and $\xi_2(\cdot)$ are given in (9) and (7) respectively.

Assumption 5. The nonlinear term $\Phi(\cdot)$ satisfies

$$\left(\frac{\partial V_0}{\partial x}\right)^T \Phi(t, x, x_d) = y^T N(t, x, x_d)$$

where $V_0(\cdot)$ is defined in Assumption 3, and $N(\cdot) \in \mathcal{R}^{n\times m}$ is Lipschitz w.r.t. x and x_d in the considered domain.

For system (1)–(2), consider the control law

$$u := u^a(t, \hat{x}) + u^b(t, y, y_d, \hat{x}, \hat{x}_d) + u^c(t, y, \hat{x}, \hat{x}_d) \quad (47)$$

where $u^a(\cdot)$ is given in Assumption 3, and $u^b(\cdot)$ and $u^c(\cdot)$ are, respectively, defined by

$$u^b(\cdot) := \begin{cases} -\frac{\varepsilon_1 M^{-T}(\cdot)y}{2\|y\|^2} \|E\| \xi_1^2(t, y, \|y_d\|)\mathcal{H}^2(t, \hat{x}, \hat{x}_d), & y \neq 0 \\ 0 & y = 0 \end{cases} \quad (48)$$

$$u^c(\cdot) := \begin{cases} -M^{-T}(\cdot)y \left(\frac{\varepsilon_2}{2} + \frac{\|y^T N(t,\hat{x},\hat{x}_d)\|}{\|y\|^2} \right), & y \neq 0 \\ 0 & y = 0 \end{cases} \quad (49)$$

where $\hat{x}$ is given by (44) and (21)–(22), ε_1 and ε_2 are positive constants, $\mathcal{H}(\cdot)$ is defined in (46) and $M(t, y)$ satisfies (10).

Theorem 3. Suppose $\mathcal{H}(\cdot)$ in (46) is Lipschitz w.r.t x and x_d. Then, under Assumptions 1-5, the closed-loop system formed by applying control (47) to system (1)–(2) is uniformly asymptotically stable if the matrix $W(\cdot) := [w_{ij}(\cdot)]_{2\times 2}$ is positive definite with $\inf\{\lambda_{\min}(W(\cdot))\} > 0$ where

$$\begin{aligned} w_{11} &:= \varpi_1(\|x\|) - \tfrac{1}{2\varepsilon_1} \\ w_{22} &:= \lambda_{\min}(Q_1) - \tfrac{1}{2\varepsilon_2}\|T^{-1}\|^2 \mathcal{L}_N^2 (1 + \gamma_2^2) \\ &\quad -2\|[P_1 \;\; P_2]\| \, \|T^{-1}\| \mathcal{L}_\Phi (1 + \gamma_2) \\ w_{12} &= w_{21} := \tfrac{1}{2}\|T^{-1}\| \Big(\varpi_2(\|x\|)\|G(t, y)\|\mathcal{L}_{u^a} \\ &\quad + \xi_1(t, y, \gamma_1\|y\|)\|E\|(1 + \gamma_2)\mathcal{L}_{\mathcal{H}} \Big) \end{aligned}$$

for $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\gamma_1 > 1$ and $\gamma_2 > 1$.

Proof: By applying the control law in (47) to system (1)–(2), the closed loop system is described by

$$\begin{aligned} \dot{x} &= Ax + G(t, y)(u^a(t, \hat{x}) + u^b(t, y, y_d, \hat{x}, \hat{x}_d) \\ &\quad + u^c(t, y, \hat{x}, \hat{x}_d)) + \Phi(t, x, x_d) + \Delta\Phi(t, x, x_d) \end{aligned} \quad (50)$$

where $\hat{x}$ is given by (44), (21) and (22). In $\mathrm{col}(x, e_{z_1}, e_{z_2})$ coordinates, the closed-loop system can be described by (50), (27) and (28). For the closed-loop system, consider the Lyapunov candidate function

$$V(t, x, e_{z_1}, e_{z_2}) = V_0(t, x) + e_{z_1}^T P_1 e_{z_1} + \tfrac{1}{2} e_{z_2}^T e_{z_2}$$

where $V_0(\cdot)$ satisfies Assumption 3, and P_1 is given in (14). Then, the time derivative of the function $V(\cdot)$ along the trajectories of the closed-loop systems is described by

$$\begin{aligned} \dot{V} &= \tfrac{\partial V_0}{\partial t} + \left(\tfrac{\partial V_0}{\partial x}\right)^T \big(Ax + G(t, y)u^a(t, \hat{x})\big) \\ &\quad + \left(\tfrac{\partial V_0}{\partial x}\right)^T G(t, y)u^b(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T \Delta\Phi(t, x, x_d) \\ &\quad + \left(\tfrac{\partial V_0}{\partial x}\right)^T G(t, y)u^c(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T \Phi(t, x, x_d) \\ &\quad + \dot{e}_{z_1}^T(t)P_1 e_{z_1}(t) + e_{z_1}^T P_1 \dot{e}_{z_1} + e_{z_2}^T(t)\dot{e}_{z_2}(t) \end{aligned} \quad (51)$$

From Assumptions 3 and 4,

$$\begin{aligned} &\tfrac{\partial V_0}{\partial t} + \left(\tfrac{\partial V_0}{\partial x}\right)^T \big(Ax + G(t, y)u^a(t, \hat{x})\big) \\ &\leq -\alpha_3(\|x\|) + \alpha_4(\|x\|)\mathcal{L}_{u^a}\|G(t, y)\| \, \|x - \hat{x}\| \\ &\leq -\varpi_1(\|x\|)\|x\|^2 + \varpi_2(\|x\|)\mathcal{L}_{u^a}\|G(t, y)\| \\ &\quad \cdot \|T^{-1}\| \|e_{z_1}\| \, \|x\| \end{aligned} \quad (52)$$

where (8) and (9) are employed above. From Assumptions 2-4 and Young's inequality $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ for any $\varepsilon > 0$,

$$\begin{aligned}&\left(\tfrac{\partial V_0}{\partial x}\right)^T G(t,y)u^b(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T \Delta\Phi(t,x,x_d)\\ &\quad\le y^T M^T(t,y)u^b(\cdot) + \varpi_2(\|x\|)\|x\|\,\|E\|\xi_1(\cdot)\xi_2(t,x,x_d)\\ &\quad\le y^T M^T(t,y)u^b(\cdot) + \frac{\varepsilon_1}{2}\|E\|^2\xi_1^2(t,y,\|y_d\|)\mathcal{H}^2(t,\hat{x},\hat{x}_d)\\ &\quad+\frac{1}{2\varepsilon_1}\|x\|^2 + \|x\|\,\|E\|\xi_1(t,y,\|y_d\|)\delta(\mathcal{H}) \qquad (53)\end{aligned}$$

where the operator $\delta(\cdot)$ is given in (69) in the Appendix. From the definition of $u^b(\cdot)$ in (48), it is easy to obtain that

$$y^T M^T(\cdot)u^b(\cdot) + \frac{\varepsilon_1}{2}\|E\|^2\xi_1^2(\cdot)\mathcal{H}^2(t,\hat{x},\hat{x}_d) = 0 \qquad (54)$$

Further, from (66) in the Appendix

$$\begin{aligned}&\|x\|\,\|E\|\xi_1(t,y,\|y_d\|)\delta(\mathcal{H})\\ &\quad\le \xi_1(\cdot)\|E\|\,\|T^{-1}\|\mathcal{L}_{\mathcal{H}}\big(\|e_{z_1}\| + \|e_{z_{1d}}\|\big)\|x\| \qquad (55)\end{aligned}$$

Substituting (54) and (55) into (53) yields

$$\begin{aligned}&\left(\tfrac{\partial V_0}{\partial x}\right)^T G(t,y)u^b(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T \Delta\Phi(t,x,x_d)\\ &\quad\le \frac{\|x\|^2}{2\varepsilon_1} + \xi_1(\cdot)\|E\|\,\|T^{-1}\|\mathcal{L}_{\mathcal{H}}\big(\|e_{z_1}\| + \|e_{z_{1d}}\|\big)\|x\| \quad (56)\end{aligned}$$

From Assumption 5, (45) and Young's inequality, it follows that for any $\varepsilon_2 > 0$

$$\begin{aligned}&\left(\tfrac{\partial V_0}{\partial x}\right)^T \Phi(t,x,x_d)\\ &= y^T\left(N(t,x,x_d) - N(t,\hat{x},\hat{x}_d)\right) + y^T N(t,\hat{x},\hat{x}_d)\\ &\le \frac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_{1d}}\|^2\right) + \frac{\varepsilon_2}{2}\|y\|^2\\ &\quad+\|y^T N(t,\hat{x},\hat{x}_d)\| \qquad (57)\end{aligned}$$

where (67) in Lemma 1 in Appendix is employed above. From (57), the definition of $u^c(\cdot)$ in (49), and by the similar reasoning as for (54), it follows that

$$\begin{aligned}&\left(\tfrac{\partial V_0}{\partial x}\right)^T G(t,y)u^c(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T \Phi(t,x,x_d)\\ &\quad\le \frac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_{1d}}\|^2\right) \qquad (58)\end{aligned}$$

From (27), (31) and (32),

$$\begin{aligned}&\dot{e}_{z_1}^T(t)P_1 e_{z_1}(t) + e_{z_1}^T(t)P_1\dot{e}_{z_1}(t)\\ &\quad\le -\lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1\quad P_2]\|\,\|T^{-1}\|\mathcal{L}_\Phi\\ &\qquad\cdot(\|e_{z_1}\| + \|e_{z_{1d}}\|) \qquad (59)\end{aligned}$$

Substituting (52), (56), (58) and (59) into (51), and using (43) yields

$$\begin{aligned}\dot{V} &\le -\varpi_1(\|x\|)\|x\|^2 + \varpi_2(\|x\|)\mathcal{L}_{u^a}\|G(t,y)\|\,\|T^{-1}\|\\ &\quad\cdot\|e_{z_1}\|\,\|x\| + \frac{\|x\|^2}{2\varepsilon_1} + \xi_1(\cdot)\|E\|\,\|T^{-1}\|\mathcal{L}_{\mathcal{H}}\big(\|e_{z_1}\|\\ &\quad+\|e_{z_{1d}}\|\big)\|x\| + \frac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_{1d}}\|^2\right)\\ &\quad-\lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1\quad P_2]\|\,\|T^{-1}\|\\ &\quad\cdot\mathcal{L}_\Phi(\|e_{z_1}\| + \|e_{z_{1d}}\|) - \eta\|e_{z_2}\| \qquad (60)\end{aligned}$$

From the condition ii) in Assumption 4, and the fact that the variables x, e_{z_1} and e_{z_2} are independent of each other, it is straightforward to see the fact that there exists a function $\zeta(r) > r$ such that for any $d \in [0,\ \bar{d}]$

$$V(t-d, x, e_{z_{1d}}, e_{z_{2d}}) \le \zeta(V(t,x,e_{z_1},e_{z_2})) \qquad (61)$$

implies that there exists $\gamma_i > 1$ for $i = 1, 2$ such that

$$\|y_d\| \le \gamma_1\|y\|, \qquad \|e_{z_{1d}}\| \le \gamma_2\|e_{z_1}\| \qquad (62)$$

Therefore, whenever (61) holds, it follows from (62) that (60) can be described by

$$\begin{aligned}\dot{V} \le &-\left(\varpi_1(\cdot) - \tfrac{1}{2\varepsilon_1}\right)\|x\|^2 - \Big(\lambda_{\min}(Q_1) - \tfrac{1}{2\varepsilon_2}\|T^{-1}\|^2\\ &\cdot\mathcal{L}_N^2(1+\gamma_2^2) - 2\|[P_1\quad P_2]\|\,\|T^{-1}\|\mathcal{L}_\Phi(1+\gamma_2)\Big)\|e_{z_1}\|^2\\ &+\Big(\varpi_2(\|x\|)\mathcal{L}_{u^a}\|G(t,y)\| + \xi_1(t,y,\gamma_1\|y\|)\|E\|\\ &\cdot\mathcal{L}_{\mathcal{H}}(1+\gamma_2)\Big)\|T^{-1}\|\|e_{z_1}\|\,\|x\|\\ =&-[\|x\|\quad\|e_{z_1}\|]\,W(\cdot)\begin{bmatrix}\|x\|\\ \|e_{z_1}\|\end{bmatrix} - \eta\|e_{z_2}\|\end{aligned}$$

Since the matrix function W is positive definite with $\inf\lambda_{\min}(W(\cdot)) > 0$, and $\eta > 0$, the conclusion follows from Razumikhin Theorem (see, e.g. [1]). ∇

V. NUMERICAL SIMULATION

Consider a nonlinear time varying delay system

$$\begin{aligned}\dot{x} &= \underbrace{\begin{bmatrix}-5 & 0\\ 1 & 1\end{bmatrix}}_{A}x + \underbrace{\begin{bmatrix}0\\ \frac{1}{1+\sin^2(t+x_2)}\end{bmatrix}}_{G(\cdot)}u(t)\\ &\quad+\underbrace{\begin{bmatrix}0.2x_2x_{2d}\\ 0.2x_{1d}x_2\exp\{-t\}\end{bmatrix}}_{\Phi(\cdot)} + \underbrace{\begin{bmatrix}1\\ -5\end{bmatrix}\Delta\Psi(\cdot)}_{\Delta\Phi(\cdot)} \qquad (63)\\ y &= \underbrace{[0\quad 1]}_{C}x \qquad (64)\end{aligned}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathcal{R}^2$, $u \in \mathcal{R}$ and $y \in \mathcal{R}$ are respectively the states, input and output of the system. The term $\Delta\Psi(\cdot)$ includes all uncertainties which satisfy

$$\|\Delta\Psi(\cdot)\| \le |x_{2d}|(|x_{1d}| + |x_1|)\exp\{-2-t\}\sin^2 x_2$$

The domain considered here is

$$\Omega = \{(x_1, x_2) \mid x_1 \in \mathcal{R}, |x_2| < 10.15\}$$

Clearly system (63)–(64) has the form in (12)–(13). Also (A, C) is observable, and $\Phi(\cdot)$ is Lipshitz w.r.t. x_1 and x_{1d} with generalised Lipschitz constant $\mathcal{L}_\Phi = 0.2|y|\exp\{-t\}$, and thus from Remark 1, Assumption 1 holds. Let

$$\begin{aligned}E &= [1\quad -5]^T, & L &= [-1\quad 6]^T\\ \xi_1 &= |y_d|\sin^2 y, & \xi_2 &= (|x_{1d}| + |x_1|)\exp\{-2-t\},\\ F &= -5, & Q &= 10I_2\end{aligned}$$

Then, $\mathcal{L}_{\xi_2} = \sqrt{2}\exp\{-2-t\}$,

$$P = \begin{bmatrix} 1.041667 & 0.208333 \\ 0.208333 & 1.041667 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

and Assumption 2 is satisfied. Let

$$u^a = -(1+\sin^2(t+y))(x_1+6y), \quad V_0 = 0.1(x_1^2+x_2^2)$$

It follows that Assumption 3 holds with

$$\begin{array}{lll} \alpha_1(r) = 0.1r^2, & \alpha_2(r) = 0.1r^2, & \alpha_3(r) = r^2, \\ \alpha_4(r) = 0.2r, & \varpi_1(r) = 1, & \varpi_2(r) = 0.2. \end{array}$$

Clearly, $\mathcal{H}(\cdot) = 0.2(|x_{1d}| + |x_1|)\exp\{-2-t\}$. Let

$$\begin{array}{ll} M(\cdot) = \frac{0.2}{1+\sin^2(t+y)}, & \zeta(r) = 1.0201r \\ N(\cdot) = 0.04(x_1 x_{2d} + x_{1d} x_2 \exp\{-t\}), & \rho = 1.01 \end{array}$$

It is straightforward to check that Assumptions 4 and 5 hold. Choose $q_0 = 1.01$. By computation directly,

$$\begin{array}{ll} \mathcal{L}_N = 0.04\sqrt{1+\gamma_1^2}|y|, & \mathcal{L}_{u^a} = 1+\sin^2(t+y), \\ \mathcal{L}_{\mathcal{H}} = 0.0383\exp\{-t\}, & q > 10 - 0.8541|y| \end{array}$$

and all the conditions in Theorems 1–3 are satisfied in the domain Ω with $\gamma_1 = \gamma_2 = 1.01$. Therefore, the observer (21)-(22) and the controller (48) and (49) are well defined, and can be obtained directly. For implementation purposes, let $d(t) = 2 + \sin t$ and $\phi(t) = \mathrm{col}(\cos(t), 1-\sin(t))$. The simulation results shown in Figure 1 confirms that the proposed approach is effective.

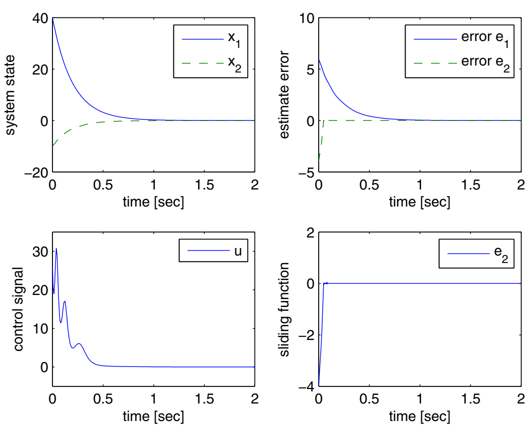

Fig. 1. The time responses of the system states and the estimation error (upper) and the control signal and the sliding function (bottom)

VI. CONCLUSIONS

A sliding mode observer-based control design approach has been proposed for a class of nonlinear time delay systems. The sliding mode observer can estimate the system state uniformly asymptotically and is insensitive to the uncertainty. Sufficient conditions have been derived using the Lyapunov Razuminkin approach under which the observer-based control law can stabilize the corresponding closed-loop system uniformly asymptotically. There is no limitation to the rate of change of the time delay. The accessible parts have been separated from the nonlinear terms and employed in the control design to reduce conservatism.

APPENDIX

Lemma 1 Assume that the function $\Theta(t,z,z_d)$ is Lipschitz w.r.t. z and z_d in their definition domain, and

$$\delta(\Theta) := \Theta(t, T^{-1}z, T^{-1}z_d) - \Theta(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) \quad (65)$$

where T is defined in (16), and $\hat{z}_y$ and $\hat{z}_{yd}$ are defined by (23). Then

$$\|\delta(\Theta)\| \leq \|T^{-1}\|\mathcal{L}_\Theta(\|e_{z1}\| + \|e_{z_{1d}}\|) \quad (66)$$

$$\|\delta(\Theta)\|^2 \leq \|T^{-1}\|^2\mathcal{L}_\Theta^2\left(\|e_{z1}\|^2 + \|e_{z_{1d}}\|^2\right) \quad (67)$$

where $e_{z1} := z_1 - \hat{z}_1$ and $e_{z_{1d}} := \hat{z}_{1d} - \hat{z}_{1d}$.

Proof: Since $\Theta(t,z,z_d)$ is Lipschitz w.r.t. variables z and z_d, it follows from the structure of T in (16) that

$$\begin{aligned} \|\delta(\Theta)\| &= \left\|\Theta(t,T^{-1}z,T^{-1}z_d) - \Theta(t,T^{-1}\hat{z}_y,T^{-1}\hat{z}_{yd})\right\| \\ &\leq \mathcal{L}_\Theta \left\| \mathrm{diag}\{T^{-1},T^{-1}\} \begin{bmatrix} z-\hat{z}_y \\ z_d - \hat{z}_{yd} \end{bmatrix} \right\| \\ &\leq \mathcal{L}_\Theta \|T^{-1}\| \Bigg\| \underbrace{\begin{bmatrix} z_1 - \hat{z}_1 \\ 0 \\ z_{1d} - \hat{z}_{1d} \\ 0 \end{bmatrix}}_{Y} \Bigg\| \end{aligned} \quad (68)$$

It is clear that

$$\|Y\| \leq \|z_1 - \hat{z}_1\| + \|z_{1d} - \hat{z}_{1d}\| = \|e_{z1}\| + \|e_{z_{1d}}\|$$

$$\|Y\|^2 = \|z_1 - \hat{z}_1\|^2 + \|z_{1d} - \hat{z}_{1d}\|^2 = \|e_{z1}\|^2 + \|e_{z_{1d}}\|^2$$

Hence the conclusion follows. ∇

From (44) and definition of operator $\delta(\cdot)$ in (65),

$$\delta(\Theta) = \Theta(t,x,x_d) - \Theta(t,\hat{x},\hat{x}_d) \quad (69)$$

which satisfies both (66) and (67).

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