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Robust Gain-Scheduled Estimation: A Convex Solution

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***Abstract*— In this paper we present an algorithm for the systematic synthesis of robust gain-scheduled estimators through convex optimization. We consider uncertain linear parameter-varying (LPV) dynamical systems described in the standard LFT form, while the uncertainty and scheduling blocks in the interconnection are described by general dynamic and static full-block IQC-multipliers respectively. It is shown how to unify the recent results on robust $\mathscr{L}_2$-gain estimation with the well-known results on LPV control, resulting in LMI conditions for the existence of robust gain-scheduled estimators that guarantee a given $\mathscr{L}_2$-gain for the closed-loop system.**

I. Introduction

During the last three decades the synthesis of $\mathscr{H}_\infty$ and LPV controllers has received a lot of attention [1], [2], [3], [4], [5], [6]. These methods also very naturally cover the nominal estimator and gain-scheduled estimator synthesis problem. However, despite the fact that the developed synthesis techniques had a major impact and have been used for many applications, they generally can only be employed in a reliable way if the involved LTI/LPV models describe the real system sufficiently well. If, on the other hand, the LTI/LPV models are uncertain (i.e. inaccurate), the problem of synthesizing controllers that are optimally robust to these uncertainties is much harder. Fortunately, in some special cases, additional structure in the problem can be exploited in order to arrive at a convex solution.

A well known framework for the analysis of uncertain systems is the Integral Quadratic Constraint (IQC) approach, which was initially formulated in [7]. IQCs are very useful in capturing a rich class of uncertainties. One could for example think of repeated static nonlinearities such as saturation [8], [9] or smoothly time-varying parametric uncertainties as well as uncertain time-varying time-delays, both with bounds on the rate-of-variation [10], [11], [12], [13]. Until recently, the IQC framework could also be employed for a limited number of synthesis applications, if the corresponding IQC-multipliers were restricted to be static [4], [5], [6], [14], [15]. Preliminary work on synthesis based on dynamic IQC-multipliers has been reported in [16] and [17]. One of the essential difficulties was the characterization of nominal stability of the closed-loop system. This problem has been resolved in [17], by means of a suitable positivity constraint on the LMI-solutions.

Although the latter result also yielded a complete solution for the robust estimator and feed-forward controller synthesis problems [17], [18], it remained an open question how to convexify the more general robust gain-scheduled estimator synthesis problem. Indeed, following the usual procedure of formulating primal and dual matrix inequalities and subsequently eliminating the estimator realization matrices fails, due to the induced nonlinear coupling on the IQC-multipliers [6]. Moreover, if the uncertainties are captured by general dynamic IQCs, it also becomes a non-trivial question how to characterize closed-loop stability and how to formulate the dual matrix inequality that is required in order to eliminate the estimator realization matrices [17], [18], [19].

As the main feature of this paper, it is shown how the techniques presented in [6], [17], [18], [19] can be unified in order to overcome the aforementioned difficulties and to arrive at LMI conditions for the existence of robust gain-scheduled estimators that guarantee a given bound on the $\mathscr{L}_2$-gain for the closed-loop system. We consider uncertain LPV systems described in the standard LFT form, while the uncertainty and scheduling blocks in the system are described by general dynamic and static full-block IQC-multipliers respectively.

The remainder of this paper is organized as follows: After having introduced some preliminaries in Section II, we formally state the problem in Section III. Then in Section IV we give a short recap on IQC analysis based on the results of [7], followed by the main results in Section V. We conclude the paper with a numerical example and some final remarks in Section VI and VII respectively.

II. Notation and Preliminaries

$\mathscr{L}_2$ denotes the space of vector-valued square integrable functions defined on $[0,\infty)$, with the usual inner product given by $\langle .,. \rangle$. $\mathscr{RL}_\infty^{m\times n}$ ($\mathscr{RH}_\infty^{m\times n}$) denotes the space of all real-rational and proper (and stable) matrix functions that have no poles on the extended imaginary axis (in the closed right-half complex-plane). By an operator we mean a map G: $\mathscr{L}_2^a \to \mathscr{L}_2^b$, and for two given operators $G=\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{pmatrix}$ and Δ, the LFT $\Delta \star G$ is defined as $G_{22}+G_{21}\Delta\left(I-G_{11}\Delta\right)^{-1}G_{12}$, assuming $\left(I-G_{11}\Delta\right)^{-1}$ exists. Moreover, if $\left(I-G_{11}\Delta\right)^{-1}$ is causal, the LFT is said to be well-posed. Realizations of LTI systems are denoted by $G=\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right] := C\left(sI-A\right)^{-1}B+D$ and with $G(i\omega)^*$ we mean $G(-i\omega)^T$. If G has no eigenvalues on the extended imaginary axis and P is a symmetric matrix, then, by the KYP-Lemma, the frequency domain inequality (FDI) $G(i\omega)^* P G(i\omega) \prec 0 \ \forall \omega \in \mathbb{R}\cup\{\infty\}$ is equivalent to the existence of a symmetric matrix X, for which the following LMI is feasible:

$$\begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & P \end{pmatrix}}_{\mathcal{M}(X,P)} \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix} \prec 0. \quad (1)$$

It is finally convenient to say that X is a certificate for the FDI and to use the abbreviation $(\star)^* PG$ for $G^* PG$.

III. Problem Formulation

Consider the uncertain LPV plant in Figure 1 where $G \in \mathscr{RH}_\infty$ represents a proper, stable LTI system that admits a minimal realization of the form

$$\begin{pmatrix} z_\mathrm{d} \\ z_\mathrm{s} \\ z_\mathrm{p} \\ y \end{pmatrix} = \underbrace{\left[\begin{array}{c|ccc} A & B_\mathrm{d} & B_\mathrm{s} & B_\mathrm{p} \\ \hline C_\mathrm{d} & D_\mathrm{dd} & D_\mathrm{ds} & D_\mathrm{dp} \\ C_\mathrm{s} & D_\mathrm{sd} & D_\mathrm{ss} & D_\mathrm{sp} \\ C_\mathrm{p} & D_\mathrm{pd} & D_\mathrm{ps} & D_\mathrm{pp} \\ C_\mathrm{y} & D_\mathrm{yd} & D_\mathrm{ys} & D_\mathrm{yp} \end{array}\right]}_{G} \begin{pmatrix} w_\mathrm{d} \\ w_\mathrm{s} \\ w_\mathrm{p} \end{pmatrix}, \quad A \in \mathbb{R}^{n\times n},$$

where respectively $\mathrm{col}(w_\mathrm{d}, w_\mathrm{s}, w_\mathrm{p}) \in \mathscr{L}_2^{n_{w_\mathrm{d}}+n_{w_\mathrm{s}}+n_{w_\mathrm{p}}}$ and $\mathrm{col}(z_\mathrm{d}, z_\mathrm{s}, z_\mathrm{p}, y) \in \mathscr{L}_2^{n_{z_\mathrm{d}}+n_{z_\mathrm{s}}+n_{z_\mathrm{p}}+n_y}$ denote the collection of uncertainty, scheduling and exogenous disturbance input signals and uncertainty, scheduling, performance and measurement output signals.

The plant G is subject to perturbations by the bounded and causal operators Δ_d and Δ_s, also referred to as the uncertainty and scheduling block respectively, which interact with G through an LFT, which we assume to be well-posed. The uncertainty block Δ_d belongs to a given star-convex set $\mathbf{\Delta}_\mathrm{d}$ with center zero (*i.e.* $[0,1]\mathbf{\Delta}_\mathrm{d} \subseteq \mathbf{\Delta}_\mathrm{d}$), capturing the properties of the uncertainties and nonlinearities, while the scheduling block $\Delta_\mathrm{s} := \hat{\Delta}_\mathrm{s} \circ \eta$ is assumed to be a linear function of an online measurable time-varying parameter vector $\eta : [0,\infty) \to \Lambda$. Here the map $\hat{\Delta}_\mathrm{s} : \mathbb{R}^k \to \mathbb{R}^{n_{w_\mathrm{s}} \times n_{z_\mathrm{s}}}$ is defined by $\hat{\Delta}_\mathrm{s}(\eta) := \sum_{i=1}^k \eta_i H_i$ for some fixed matrices H_i, $\eta \in \mathbb{R}^k$, and we assume that η takes its values in $\Lambda := \mathrm{co}\{\eta^1, \ldots, \eta^m\} \subseteq \mathbb{R}^k$, where $\eta^j = (\eta_1^j, \ldots, \eta_k^j)$, $j \in \mathbf{J} := \{1, \ldots, m\}$ represent the generator points. Without loss of generality Λ contains the origin. Then the scheduling block Δ_s is contained in the set $\mathbf{\Delta}_\mathrm{s} := \{\hat{\Delta}_\mathrm{s} \circ \eta : \eta \in \mathbf{C}_\mathrm{p}([0,\infty), \Lambda)\}$ (with $\mathbf{C}_\mathrm{p}([0,\infty), \Lambda)$ denoting the space of piecewise continuous functions $[0,\infty) \to \Lambda$) and defines the scheduling operator through $w_\mathrm{s}(t) = \hat{\Delta}_\mathrm{s}(\eta(t)) z_\mathrm{s}(t)$.

The main goal in robust gain-scheduled estimation is the synthesis of a filter $E \star \Delta_\mathrm{c}$ that dynamically and causally processes the measurement y and the scheduling signal η in order to provide an estimate u of the signal z_p in the sense that the $\mathscr{L}_2$-gain from w_p to $z_\mathrm{e} = z_\mathrm{p} - u$ is rendered less than an *a priori* given $\gamma > 0$. Here the operator Δ_c represents the so-called scheduling function that is defined with some to-be-constructed $\hat{\Delta}_\mathrm{c} : \mathbb{R}^{n_{w_\mathrm{s}} \times n_{z_\mathrm{s}}} \to \mathbb{R}^{n_{w_\mathrm{c}} \times n_{z_\mathrm{c}}}$ as $\Delta_\mathrm{c} := \hat{\Delta}_\mathrm{c}(\Delta_\mathrm{s})$. Note that Δ_c defines an operator through $w_\mathrm{c}(t) = \hat{\Delta}_\mathrm{c}(\hat{\Delta}_\mathrm{s}(\eta(t))) z_\mathrm{c}(t)$. Moreover, E is a proper and stable LTI system that admits a realization of the form

$$\begin{pmatrix} u \\ z_\mathrm{c} \end{pmatrix} = \underbrace{\left[\begin{array}{c|cc} A_E & B_\mathrm{y} & B_\mathrm{c} \\ \hline C_\mathrm{u} & D_\mathrm{uy} & D_\mathrm{uc} \\ C_\mathrm{c} & D_\mathrm{cy} & D_\mathrm{cc} \end{array}\right]}_{E} \begin{pmatrix} y \\ w_\mathrm{c} \end{pmatrix}, \tag{2}$$

where A_E is Hurwitz and where $\mathrm{col}(y, w_\mathrm{c}) \in \mathscr{L}_2^{n_y + n_{w_\mathrm{c}}}$ and $\mathrm{col}(u, z_\mathrm{c}) \in \mathscr{L}_2^{n_u + n_{z_\mathrm{c}}}$ denote the collection of measurement and scheduling input and the control and scheduling output signals respectively.

Given G, Λ and $\mathbf{\Delta}_\mathrm{d}$, the goal of this paper can now be formally stated as follows: Design a gain-scheduled estimator $E \star \Delta_\mathrm{c}$ such that, for all $\Delta_\mathrm{d} \in \mathbf{\Delta}_\mathrm{d}$ and $\Delta_\mathrm{s} \in \mathbf{\Delta}_\mathrm{s}$, the interconnection of Figure 1 is well-posed, stable and the $\mathscr{L}_2$-gain from w_p to z_e is rendered less than γ.

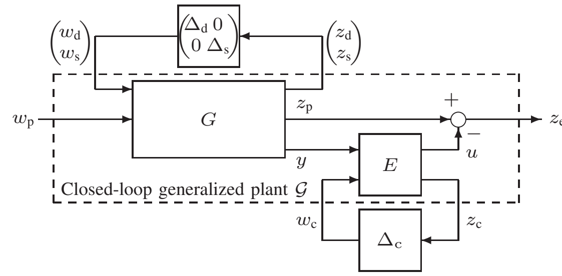

Fig. 1. Robust gain-scheduled estimation problem.

IV. Robust Stability and Performance Analysis

As a preparation, consider the standard input-output setting for robust stability and performance analysis in Figure 2, where $\mathcal{G} \in \mathscr{RH}_\infty^{(n_\kappa + n_{z_\mathrm{e}}) \times (n_\psi + n_{w_\mathrm{p}})}$ represents the nominal and stable closed-loop generalized plant, as represented by the dashed box in Figure 1. Here $\kappa = \mathrm{col}(z_\mathrm{d}, z_\mathrm{s}, z_\mathrm{c}) \in \mathscr{L}_2^{n_{z_\mathrm{d}} + n_{z_\mathrm{s}} + n_{z_\mathrm{c}}}$ and $\psi = \mathrm{col}(w_\mathrm{d}, w_\mathrm{s}, w_\mathrm{c}) \in \mathscr{L}_2^{n_{w_\mathrm{d}} + n_{w_\mathrm{s}} + n_{w_\mathrm{c}}}$ denote the collection of scheduling and uncertainty signals and $\mathcal{G}$ admits the realization

$$\mathcal{G} := \begin{pmatrix} \mathcal{G}_{\kappa\psi} & \mathcal{G}_{\kappa\mathrm{p}} \\ \mathcal{G}_{\mathrm{e}\psi} & \mathcal{G}_\mathrm{ep} \end{pmatrix} = \left(\begin{array}{ccc:c} \mathcal{G}_\mathrm{dd} & \mathcal{G}_\mathrm{ds} & 0 & \mathcal{G}_\mathrm{dp} \\ \mathcal{G}_\mathrm{sd} & \mathcal{G}_\mathrm{ss} & 0 & \mathcal{G}_\mathrm{sp} \\ \mathcal{G}_\mathrm{cd} & \mathcal{G}_\mathrm{cs} & \mathcal{G}_\mathrm{cc} & \mathcal{G}_\mathrm{cp} \\ \hdashline \mathcal{G}_\mathrm{ed} & \mathcal{G}_\mathrm{es} & \mathcal{G}_\mathrm{ec} & \mathcal{G}_\mathrm{ep} \end{array}\right) = \left[\begin{array}{c|ccc:c} \mathcal{A} & \mathcal{B}_\mathrm{d} & \mathcal{B}_\mathrm{s} & \mathcal{B}_\mathrm{c} & \mathcal{B}_\mathrm{p} \\ \hline \mathcal{C}_\mathrm{d} & \mathcal{D}_\mathrm{dd} & \mathcal{D}_\mathrm{ds} & 0 & \mathcal{D}_\mathrm{dp} \\ \mathcal{C}_\mathrm{s} & \mathcal{D}_\mathrm{sd} & \mathcal{D}_\mathrm{ss} & 0 & \mathcal{D}_\mathrm{sp} \\ \mathcal{C}_\mathrm{c} & \mathcal{D}_\mathrm{cd} & \mathcal{D}_\mathrm{cs} & \mathcal{D}_\mathrm{cc} & \mathcal{D}_\mathrm{cp} \\ \hdashline \mathcal{C}_\mathrm{e} & \mathcal{D}_\mathrm{ed} & \mathcal{D}_\mathrm{es} & \mathcal{D}_\mathrm{ec} & \mathcal{D}_\mathrm{ep} \end{array}\right], \tag{3}$$

where the closed-loop realization matrices are given by

$$\left(\begin{array}{c|ccc:c} \mathcal{A} & \mathcal{B}_\mathrm{d} & \mathcal{B}_\mathrm{s} & \mathcal{B}_\mathrm{c} & \mathcal{B}_\mathrm{p} \\ \hline \mathcal{C}_\mathrm{d} & \mathcal{D}_\mathrm{dd} & \mathcal{D}_\mathrm{ds} & 0 & \mathcal{D}_\mathrm{dp} \\ \mathcal{C}_\mathrm{s} & \mathcal{D}_\mathrm{sd} & \mathcal{D}_\mathrm{ss} & 0 & \mathcal{D}_\mathrm{sp} \\ \mathcal{C}_\mathrm{c} & \mathcal{D}_\mathrm{cd} & \mathcal{D}_\mathrm{cs} & \mathcal{D}_\mathrm{cc} & \mathcal{D}_\mathrm{cp} \\ \hdashline \mathcal{C}_\mathrm{e} & \mathcal{D}_\mathrm{ed} & \mathcal{D}_\mathrm{es} & \mathcal{D}_\mathrm{ec} & \mathcal{D}_\mathrm{ep} \end{array}\right) = \left(\begin{array}{cc|ccc:c} A & 0 & B_\mathrm{d} & B_\mathrm{s} & 0 & B_\mathrm{p} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline C_\mathrm{d} & 0 & D_\mathrm{dd} & D_\mathrm{ds} & 0 & D_\mathrm{dp} \\ C_\mathrm{s} & 0 & D_\mathrm{sd} & D_\mathrm{ss} & 0 & D_\mathrm{sp} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline C_\mathrm{p} & 0 & D_\mathrm{pd} & D_\mathrm{ps} & 0 & D_\mathrm{pp} \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \\ \hdashline 0 & -I & 0 \end{array}\right) \times$$

$$\left(\begin{array}{c|c:c} A_E & B_\mathrm{y} & B_\mathrm{c} \\ \hline C_\mathrm{u} & D_\mathrm{uy} & D_\mathrm{uc} \\ \hdashline C_\mathrm{c} & D_\mathrm{cy} & D_\mathrm{cc} \end{array}\right) \left(\begin{array}{cc|ccc:c} 0 & I & 0 & 0 & 0 & 0 \\ \hline C_\mathrm{y} & 0 & D_\mathrm{yd} & D_\mathrm{ys} & 0 & D_\mathrm{yp} \\ 0 & 0 & 0 & 0 & I & 0 \end{array}\right).$$

As a natural consequence, $\mathcal{G}$ is (i) stable if A and A_E are Hurwitz and (ii) subject to perturbations by the bounded and causal operator $\Delta := \mathrm{diag}\,(\Delta_\mathrm{d}, \Delta_\mathrm{s}, \Delta_\mathrm{c})$ which is contained in the set $\mathbf{\Delta} := \mathrm{diag}\big(\mathbf{\Delta}_\mathrm{d}, \mathbf{\Delta}_\mathrm{s}, \hat{\Delta}_\mathrm{c}(\mathbf{\Delta}_\mathrm{s})\big)$ and represents the collection of uncertainty and scheduling blocks.

The system interconnection of Figure 2 is said to (i) be well-posed if the operator $I - \mathcal{G}_{\kappa\psi}\Delta$ has a causal inverse for all $\Delta \in \mathbf{\Delta}$, (ii) be robustly stable if $I - \mathcal{G}_{\kappa\psi}\Delta$ is well-posed and if its inverse is bounded on $\mathscr{L}_2$ and (iii) have a robust $\mathscr{L}_2$-gain performance of level γ, if it is robustly stable and if for all $\Delta \in \mathbf{\Delta}$ the $\mathscr{L}_2$-gain from w_p to z_e is less than $\gamma > 0$.

Clearly, the structure of (3) reveals that $I - \mathcal{G}_{\kappa\psi}\Delta$ is well-posed if and only if

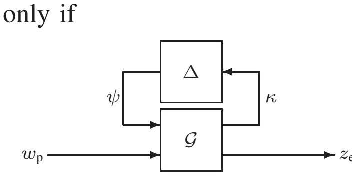

Fig. 2. Standard configuration for robust stability and performance analysis.

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} G_{\rm dd} & G_{\rm ds} \\ G_{\rm sd} & G_{\rm ss} \end{pmatrix}\begin{pmatrix} \Delta_{\rm d} & 0 \\ 0 & \Delta_{\rm s} \end{pmatrix} \tag{4}$$

is well posed for all $\Delta_{\rm d} \in \mathbf{\Delta}_{\rm d}$, $\Delta_{\rm s} \in \mathbf{\Delta}_{\rm s}$ and

$$I - \mathcal{G}_{\rm cc}\Delta_{\rm c} \tag{5}$$

is well-posed for all $\Delta_{\rm c} \in \hat{\Delta}_{\rm c}(\mathbf{\Delta}_{\rm s})$ respectively. By recalling that the interconnection of G with $\Delta_{\rm d}$ and $\Delta_{\rm s}$ and hence (4) is well-posed by assumption, we only require (5) to have a bounded inverse for all $\Delta_{\rm c} \in \hat{\Delta}_{\rm c}(\mathbf{\Delta}_{\rm s})$, which in turn just means that $E \star \Delta_{\rm c}$ must be well-posed.

A. Analysis with IQCs

Recall that the operator Δ is said to satisfy the IQC defined by the multiplier $\Pi = \Pi^* \in \mathscr{RL}_\infty^{(n_\kappa+n_\psi)\times(n_\kappa+n_\psi)}$ if the following condition holds true:

$$\left\langle \begin{pmatrix} \kappa \\ \Delta(\kappa) \end{pmatrix}, \Pi \begin{pmatrix} \kappa \\ \Delta(\kappa) \end{pmatrix} \right\rangle \geq 0 \quad \forall \kappa \in \mathscr{L}_2^{n_\kappa}. \tag{6}$$

For the correct interpretation of this expression we refer the reader to [20]. In this paper we assume that Π is factorized as $\Phi^* P \Phi$, with P being a symmetric matrix and $\Phi \in \mathscr{RH}_\infty^{n_\psi \times n_\kappa}$ a typically tall transfer matrix. In applications one constructs a whole family of multipliers $\mathbf{\Pi} = \Phi^* \mathbf{P} \Phi$ with a suitable set of symmetric matrices $P \in \mathbf{P}$ such that the IQC holds for all $\Delta \in \mathbf{\Delta}$. We do not make use of any particular structure of Φ and P for the uncertainty $\Delta_{\rm d}$ and restrict our attention to static full-block multipliers for the scheduling block $\Delta_{\rm sc} := \operatorname{diag}(\Delta_{\rm s}, \Delta_{\rm c})$. Hence, let us consider the following two IQCs:

$$\left\langle \begin{pmatrix} z_{\rm d} \\ \Delta_{\rm d}(z_{\rm d}) \end{pmatrix}, \Psi^* P_1 \Psi \begin{pmatrix} z_{\rm d} \\ \Delta_{\rm d}(z_{\rm d}) \end{pmatrix} \right\rangle \geq 0 \quad \forall z_{\rm d} \in \mathscr{L}_2^{n_{z_{\rm d}}}, \tag{7}$$

$$\left\langle \begin{pmatrix} z_{\rm sc} \\ \Delta_{\rm sc} z_{\rm sc} \end{pmatrix}, P_{\rm 2e} \begin{pmatrix} z_{\rm sc} \\ \Delta_{\rm sc} z_{\rm sc} \end{pmatrix} \right\rangle \geq 0 \ \forall z_{\rm sc} := \begin{pmatrix} z_{\rm s} \\ z_{\rm c} \end{pmatrix} \in \mathscr{L}_2^{n_{z_{\rm s}}+n_{z_{\rm c}}}. \tag{8}$$

Here $\Psi := \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} \in \mathscr{RH}_\infty^{n_\psi \times (n_{w_{\rm d}}+n_{z_{\rm d}})}$ is partitioned according to the structure of $\operatorname{col}(z_{\rm d}, \Delta_{\rm d}(z_{\rm d}))$ and $P_1 \in \mathbf{P}_1$ is any suitable (LMIable) set of structured symmetric matrices such that (7) holds. Moreover, if we define the constraints

$$(\star)^T \underbrace{\left(\begin{array}{cc|cc} Q & Q_{12} & S & S_{12} \\ Q_{12}^T & Q_{22} & S_{21} & S_{22} \\ \hline S^T & S_{21}^T & R & R_{12} \\ S_{12}^T & S_{22}^T & R_{12}^T & R_{22} \end{array}\right)}_{P_{\rm 2e}} \begin{pmatrix} I \\ \hat{\Delta}_{\rm sc}(\eta^j) \end{pmatrix} \succ 0, \ \begin{pmatrix} R & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \prec 0 \ \forall j \in \mathbf{J}. \tag{9}$$

where $\hat{\Delta}_{\rm sc}(\eta^j) := \operatorname{diag}(\hat{\Delta}_{\rm s}(\eta^j), \hat{\Delta}_{\rm c}(\hat{\Delta}_{\rm s}(\eta^j))$, then (8) is satisfied if $P_{\rm 2e}$ is confined to $\mathbf{P}_{\rm 2e} := \{P_{\rm 2e} : (9)\}$. Hence, with $P_1 \in \mathbf{P}_1$ and $P_{\rm 2e} \in \mathbf{P}_{\rm 2e}$, we infer that (6) is satisfied for all $\Delta \in \mathbf{\Delta}$ with

$$\Pi = \Phi^* P \Phi = (\star)^* \left(\begin{array}{c|cccc} P_1 & 0 & 0 & 0 & 0 \\ \hline 0 & Q & Q_{12} & S & S_{12} \\ 0 & Q_{12}^T & Q_{22} & S_{21} & S_{22} \\ 0 & S^T & S_{21}^T & R & R_{12} \\ 0 & S_{12}^T & S_{22}^T & R_{12}^T & R_{22} \end{array}\right) \left(\begin{array}{cccccc} \Psi_1 & 0 & 0 & \Psi_2 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right).$$

It is well known from [7] that robust stability and performance of the system interconnection in Figure 2 can now be characterized as follows.

Theorem 1: Suppose that (4) is well-posed and that $\Delta_{\rm d}$ satisfies (7) for all $\Delta_{\rm d} \in \mathbf{\Delta}_{\rm d}$. Then for all $\Delta \in \mathbf{\Delta}$ the system interconnection of Figure 2 is well-posed, robustly stable and has a robust $\mathscr{L}_2$-gain performance level of γ, if there exists $P_1 \in \mathbf{P}_1$ and $P_{\rm 2e} \in \mathbf{P}_{\rm 2e}$ for which the following FDI holds:

$$(\star)^*(\star)^* P_{\rm p} \Phi_{\rm p}(i\omega) \left(\begin{array}{cc} \mathcal{G}_{\kappa\psi}(i\omega) & \mathcal{G}_{\kappa{\rm p}}(i\omega) \\ I & 0 \\ \hline \mathcal{G}_{{\rm e}\psi}(i\omega) & \mathcal{G}_{\rm ep}(i\omega) \\ 0 & I \end{array}\right) \prec 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \tag{10}$$

Here $P_{\rm p} = \operatorname{diag}(P, I, -\gamma^2 I)$ and $\Phi_{\rm p} = \operatorname{diag}(\Phi, I, I)$.

From now on, let us assume that there exists at least one $P_1 \in \mathbf{P}_1$ with

$$\Psi_1(i\omega)^* P_1 \Psi_1(i\omega) \succ 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\} \tag{11}$$

and suppose that Ψ_1 and Ψ_2 respectively admit the minimal realizations $\Psi_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ and $\Psi_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$, with A_1 and A_2 being Hurwitz. Then, by the KYP-Lemma, the FDIs (10) and (11) are equivalent to the existence of some symmetric matrices $\mathcal{X}$ and $\hat{X}$ for which the following LMIs hold:

$$(\star)^T \mathcal{M}(\mathcal{X}, P_{\rm e}) \left(\begin{array}{ccc|cccc} & I & & & 0 & & \\ \hline A_1 & 0 & B_1\mathcal{C}_{\rm d} & B_1\mathcal{D}_{\rm dd} & B_1\mathcal{D}_{\rm ds} & 0 & B_1\mathcal{D}_{\rm dp} \\ 0 & A_2 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A} & \mathcal{B}_{\rm d} & \mathcal{B}_{\rm s} & \mathcal{B}_{\rm c} & \mathcal{B}_{\rm p} \\ \hline C_1 & C_2 & D_1\mathcal{C}_{\rm d} & D_1\mathcal{D}_{\rm dd}+D_2 & D_1\mathcal{D}_{\rm ds} & 0 & D_1\mathcal{D}_{\rm dp} \\ 0 & 0 & \mathcal{C}_{\rm s} & \mathcal{D}_{\rm sd} & \mathcal{D}_{\rm ss} & 0 & \mathcal{D}_{\rm sp} \\ 0 & 0 & \mathcal{C}_{\rm c} & \mathcal{D}_{\rm cd} & \mathcal{D}_{\rm cs} & \mathcal{D}_{\rm cc} & \mathcal{D}_{\rm cp} \\ \hline 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & \mathcal{C}_{\rm p} & \mathcal{D}_{\rm pd} & \mathcal{D}_{\rm ps} & \mathcal{D}_{\rm pc} & \mathcal{D}_{\rm pp} \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right) \prec 0, \tag{12}$$

$$\begin{pmatrix} A_1^T \hat{X} + \hat{X} A_1 + C_1^T P_1 C_1 & \hat{X} B_1 + C_1^T P_1 D_1 \\ B_1^T \hat{X} + D_1^T P_1 C_1 & D_1^T P_1 D_1 \end{pmatrix} \succ 0. \tag{13}$$

Here we recall from (3) that the diagonal blocks of $\mathcal{A}$ are given by A and $A_{\rm E}$ respectively. Hence, we partition $\mathcal{X}$ as

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & \bar{X} \end{pmatrix} = \left(\begin{array}{ccc|c} X_{11} & X_{12} & X_{13} & U_1 \\ X_{12}^T & X_{22} & X_{23} & U_2 \\ X_{13}^T & X_{23}^T & X_{33} & U_3 \\ \hline U_1^T & U_2^T & U_3^T & \bar{X} \end{array}\right),$$

where X_{11}, X_{22}, X_{33} and $\bar{X}$ have compatible dimensions with A_1, A_2, A and $A_{\rm E}$ respectively. It is now possible to state the following result.

Theorem 2: Suppose that (4) is well-posed and that $\Delta_{\rm d}$ satisfies (7) for all $\Delta_{\rm d} \in \mathbf{\Delta}_{\rm d}$. Then for all $\Delta \in \mathbf{\Delta}$ the system interconnection of Figure 2 is well-posed, robustly stable and has a robust $\mathscr{L}_2$-gain performance level of γ if

$$\exists\ \mathcal{X}, \hat{X}, P_1 \in \mathbf{P}_1, P_{\rm 2e} \in \mathbf{P}_{\rm 2e} \ : \ (12), (13) \text{ hold.} \tag{14}$$

Remark 1: Note that the IQC $\left\langle \begin{pmatrix} z_{\rm s} \\ \Delta_{\rm s} z_{\rm s} \end{pmatrix}, P_2 \begin{pmatrix} z_{\rm s} \\ \Delta_{\rm s} z_{\rm s} \end{pmatrix} \right\rangle \geq 0$ for all $z_{\rm s} \in \mathscr{L}_2^{n_{z_{\rm s}}}$ is satisfied for all $\Delta_{\rm s} \in \mathbf{\Delta}_{\rm s}$, if the symmetric matrix P_2 is confined to

$$\mathbf{P}_2 := \left\{ P_2 = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} : (\star)^T P_2 \begin{pmatrix} I \\ \hat{\Delta}_{\rm s}(\eta^j) \end{pmatrix} \succ 0, R \prec 0 \ \forall j \in \mathbf{J} \right\}. \tag{15}$$

The results of this paper rely on the elimination of the realization matrices of E, and, consequently, the elimination of the scheduling function $\hat{\Delta}_{\rm c}(\Delta_{\rm s})$ from the synthesis problem. Therefore, the so-called extended multiplier $P_{\rm 2e} \in \mathbf{P}_{\rm 2e}$ for analysis results simplifies into the reduced multiplier $P_2 \in \mathbf{P}_2$ appearing in the synthesis conditions.

B. From Analysis to Synthesis

Since the system matrices depend on to-be-designed estimator variables, the conditions in (14) are no longer affine in all variables, such that LMI solvers are unable to handle the synthesis problem. A common procedure to resolve this problem is to eliminate the estimator variables by applying the so-called Elimination Lemma [6, Lemma A.2]. However, there are three main issues appearing in the robust gain-scheduling estimator synthesis problem that need to be resolved.

1) Due to the generality of the multipliers, $\mathcal{X} \succ 0$ is no longer the appropriate condition in order to enforce closed-loop stability [17].
2) In order to eliminate the estimator variables by applying the Elimination Lemma, it is required to formulate a dual solvability condition by applying the Dualization Lemma [6, Lemma A.1]. However, the outer-factors of the IQC-multiplier factorization $\Psi^* P_1 \Psi$ are generally tall and, hence, cannot be inverted. Since the inverse of Ψ is essential in order to explicitly formulate the dual of matrix inequality (12), the primal matrix inequality (12) must be reformulated with a square factorization of $\Psi^* P_1 \Psi$ (*i.e.* $\Psi^* P_1 \Psi = \hat{\Psi}^* \mathrm{diag}(I, -I) \hat{\Psi}$ with $\hat{\Psi}$ being square and invertible) [18].
3) Unlike the standard $\mathscr{H}_\infty$-controller synthesis problem [2], [3], it is not sufficient to only eliminate the estimator variables. The primal and dual solvability conditions induce a non-convex constraint on the multipliers which, in general, cannot be convexified.

C. A Characterization of Nominal Stability

The first issue can be resolved by considering the characterization of nominal stability as presented in [17]. Recall that $P_1 \in \mathbf{P}_1$ is generally indefinite. Therefore, $\mathcal{X} \succ 0$ is no longer an appropriate condition in order to enforce stability on the underlying closed-loop system. The following theorem provides a coupling constraint between (12) and (13), which is equivalent to $\mathcal{A}$ being Hurwitz.

Lemma 1: [17] $\mathcal{A}$ is stable and FDIs (10) as well as (11) hold if and only if there exist solutions $\mathcal{X}$ and $\hat{X}$ of LMIs (12) and (13) which are coupled as

$$\begin{pmatrix} X_{11} - \hat{X} & X_{13} & U_1 \\ X_{13}^T & X_{33} & U_3 \\ U_1^T & U_3^T & \bar{X} \end{pmatrix} \succ 0. \tag{16}$$

It is now possible to exploit Lemma 1 in order to state the following result.

Theorem 3: Suppose that (4) is well-posed and that Δ_d satisfies (7) for all $\Delta_\mathrm{d} \in \mathbf{\Delta}_\mathrm{d}$. Then there exist a stable and causal estimator $E \star \Delta_\mathrm{c}$ such that for all $\Delta \in \mathbf{\Delta}$ the system interconnection of Figure 2 is well-posed, robustly stable and the resulting $\mathscr{L}_2$-gain from w_p to z_e is rendered less than γ, if

$$\exists\ \mathcal{X}, \hat{X}, P_1 \in \mathbf{P}_1, P_{2\mathrm{e}} \in \mathbf{P}_{2\mathrm{e}} : (12), (13), (16) \text{ hold.} \tag{17}$$

D. Reformulation of the Analysis LMIs of Theorem 2

As discussed in Section IV-B it is non-trivial to formulate an explicit dual solvability condition. In order to resolve this problem we will rely on the results of [19] and [21]. Indeed, one can show that it is possible to construct a new factorization of $\Psi^* P_1 \Psi$ that has square and invertible outer-factors. Moreover, one can, subsequently, eliminate the initial multiplier factorization $\Psi^* P_1 \Psi$ appearing in the FDI (10) and replace it by the new one. The key difficulty is, how this can be done by using state-space arguments.

Lemma 2 (IQC Squaring [19]): Let us define the matrices $J = (0\ I)$ and $\hat{J} = (I\ 0)$ and suppose that (14) holds. Then there exist matrices $\hat{\Psi}_j \in \mathscr{RH}_\infty$, $j = 1, 2, 3$ with $\hat{\Psi}_1^{-*}$, $\hat{\Psi}_2^{-1} \in \mathscr{RH}_\infty$ and symmetric matrices

$$Z = \begin{pmatrix} Z_{11} & \hat{J} \\ \hat{J}^T & Z_{22} \end{pmatrix}, \quad Z_\epsilon = \begin{pmatrix} Z_{11\epsilon} & \hat{J} \\ \hat{J}^T & Z_{22\epsilon} \end{pmatrix} \tag{18}$$

with $Z_\epsilon \to Z$ for $\epsilon \to 0$ such that

$$\Psi^* P_1 \Psi = \begin{pmatrix} \hat{\Psi}_1 & \hat{\Psi}_3 \\ 0 & \hat{\Psi}_2 \end{pmatrix}^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \hat{\Psi}_1 & \hat{\Psi}_3 \\ 0 & \hat{\Psi}_2 \end{pmatrix} =: \hat{\Psi}^* \hat{P}_1 \hat{\Psi}, \tag{19}$$

$\hat{\Psi}$ admits the controllable realization

$$\hat{\Psi} = \begin{pmatrix} \hat{\Psi}_1 & \hat{\Psi}_3 \\ 0 & \hat{\Psi}_2 \end{pmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & \hat{A}_2 & 0 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_3 & \hat{D}_1 & \hat{D}_3 \\ 0 & \hat{C}_2 & 0 & \hat{D}_2 \end{array}\right], \tag{20}$$

Z_ϵ satisfies

$$(\star)^T \mathcal{M}\Big(Z_\epsilon, \mathrm{diag}\big(\hat{P}_1, -P_1\big)\Big) \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{I} & \multicolumn{2}{c}{0} \\ \hline A_1 & 0 & B_1 & 0 \\ 0 & \check{A}_2 & 0 & \check{B}_2 \\ \hline \hat{C}_1 & \check{C}_3 & \hat{D}_1 & \check{D}_3 \\ 0 & \check{C}_2 & 0 & \check{D}_2 \\ \hline C_1 & C_2 J & D_1 & D_2 \end{array}\right) \prec 0. \tag{21}$$

for all small $\epsilon > 0$ and equality holds with Z, if $\epsilon = 0$.

It is obvious that the initial multiplier factorization $\Psi^* P_1 \Psi$ appearing in the FDI (10) can now simply be replaced with the factorization $\hat{\Psi}^* \hat{P}_1 \hat{\Psi}$ as in (19). However, it has only recently been shown in [21] how this can be done in state-space, by systematically merging (12) and (21).

Lemma 3 (LMI Gluing): Suppose that (14) holds and that Z_ϵ satisfies (21) for all small $\epsilon > 0$. Then (21) and (12) imply

$$(\star)^T \mathcal{M}(\hat{\mathcal{X}}, T_3) \left(\begin{array}{ccc|cccc} \multicolumn{3}{c|}{I} & \multicolumn{4}{c}{0} \\ \hline A_1 & 0 & B_1 \mathcal{C}_\mathrm{d} & B_1 \mathcal{D}_\mathrm{dd} & B_1 \mathcal{D}_\mathrm{ds} & 0 & B_1 \mathcal{D}_\mathrm{dp} \\ 0 & \check{A}_2 & 0 & \check{B}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A} & \mathcal{B}_\mathrm{d} & \mathcal{B}_\mathrm{s} & \mathcal{B}_\mathrm{c} & \mathcal{B}_\mathrm{p} \\ \hline \hat{C}_1 & \check{C}_3 & \hat{D}_1 \mathcal{C}_\mathrm{d} & \hat{D}_1 \mathcal{D}_\mathrm{ee} & \hat{D}_1 \mathcal{D}_\mathrm{ds} & 0 & \hat{D}_1 \mathcal{D}_\mathrm{dp} \\ 0 & \check{C}_2 & 0 & \check{D}_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathcal{C}_\mathrm{s} & \mathcal{D}_\mathrm{sd} & \mathcal{D}_\mathrm{ss} & 0 & \mathcal{D}_\mathrm{sp} \\ 0 & 0 & \mathcal{C}_\mathrm{c} & \mathcal{D}_\mathrm{cd} & \mathcal{D}_\mathrm{cs} & \mathcal{D}_\mathrm{cc} & \mathcal{D}_\mathrm{cp} \\ \hdashline 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & \mathcal{C}_\mathrm{p} & \mathcal{D}_\mathrm{pd} & \mathcal{D}_\mathrm{ps} & \mathcal{D}_\mathrm{pc} & \mathcal{D}_\mathrm{pp} \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right) \prec 0, \tag{22}$$

where $T_3 = \mathrm{diag}\big(\hat{P}_1, P_{2\mathrm{e}}, I, -\gamma^2 I\big)$, $\mathcal{D}_\mathrm{ee} = \mathcal{D}_\mathrm{dd} + \hat{D}_1^{-1} \check{D}_3$ and

$$\hat{\mathcal{X}} = \begin{pmatrix} \tilde{X} & \tilde{U} \\ \tilde{U}^T & \bar{X} \end{pmatrix} = \left(\begin{array}{ccc|c} X_{11} - Z_{11\epsilon} & X_{12} J - \hat{J} & X_{13} & U_1 \\ J^T X_{12}^T - \hat{J}^T & J^T X_{22} J - Z_{22\epsilon} & J^T X_{23} & J^T U_2 \\ X_{13}^T & X_{23}^T J & X_{33} & U_3 \\ \hline U_1^T & U_2^T J & U_3^T & \bar{X} \end{array}\right). \tag{23}$$

Since the outer-factors of the multipliers are now square and invertible, it is possible to eliminate $\hat{C}_{\Psi 2}$ and $\hat{D}_{\Psi 2}$ by performing a simple congruence transformation. This yields

$$
(\star)^T \mathcal{M}(\hat{\mathcal{X}}, T_3) \left(\begin{array}{c|cccc}
I & 0 & 0 & 0 & 0 \\
\hat{\mathcal{A}} & \hat{\mathcal{B}}_{\rm d} & \hat{\mathcal{B}}_{\rm s} & \hat{\mathcal{B}}_{\rm c} & \hat{\mathcal{B}}_{\rm p} \\ \hline
\hat{\mathcal{C}}_{\rm d} & \hat{\mathcal{D}}_{\rm dd} & \hat{\mathcal{D}}_{\rm ds} & 0 & \hat{\mathcal{D}}_{\rm dp} \\
0 & I & 0 & 0 & 0 \\ \hline
\hat{\mathcal{C}}_{\rm s} & \hat{\mathcal{D}}_{\rm sd} & \mathcal{D}_{\rm ss} & 0 & \mathcal{D}_{\rm sp} \\
\hat{\mathcal{C}}_{\rm c} & \hat{\mathcal{D}}_{\rm cd} & \mathcal{D}_{\rm cs} & \mathcal{D}_{\rm cc} & \mathcal{D}_{\rm cp} \\ \hdashline
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\ \hline
\hat{\mathcal{C}}_{\rm p} & \hat{\mathcal{D}}_{\rm pd} & \mathcal{D}_{\rm ps} & \mathcal{D}_{\rm pc} & \mathcal{D}_{\rm pp} \\
0 & 0 & 0 & 0 & I
\end{array}\right) \prec 0, \tag{24}
$$

where

$$
\left[\begin{array}{c|cccc}
\hat{\mathcal{A}} & \hat{\mathcal{B}}_{\rm d} & \hat{\mathcal{B}}_{\rm s} & \hat{\mathcal{B}}_{\rm c} & \hat{\mathcal{B}}_{\rm p} \\ \hline
\hat{\mathcal{C}}_{\rm d} & \hat{\mathcal{D}}_{\rm d} & \hat{\mathcal{D}}_{\rm ds} & 0 & \hat{\mathcal{D}}_{\rm dp} \\
\hat{\mathcal{C}}_{\rm s} & \hat{\mathcal{D}}_{\rm sd} & \mathcal{D}_{\rm ss} & 0 & \mathcal{D}_{\rm sp} \\
\hat{\mathcal{C}}_{\rm c} & \hat{\mathcal{D}}_{\rm cd} & \mathcal{D}_{\rm cs} & \mathcal{D}_{\rm cc} & \mathcal{D}_{\rm cp} \\
\hat{\mathcal{C}}_{\rm p} & \hat{\mathcal{D}}_{\rm pd} & \mathcal{D}_{\rm ps} & \mathcal{D}_{\rm pc} & \mathcal{D}_{\rm pp}
\end{array}\right] :=
$$

$$
\left[\begin{array}{ccc|cccc}
A_1 & -B_1\mathcal{D}_{\rm dd}\check{D}_2^{-1}\check{C}_2 & B_1\mathcal{C}_{\rm d} & B_1\mathcal{D}_{\rm dd}\check{D}_2^{-1} & B_1\mathcal{D}_{\rm ds} & 0 & B_1\mathcal{D}_{\rm dp} \\
0 & \check{A}_2-\check{B}_2\check{D}_2^{-1}\check{C}_2 & 0 & \check{B}_2\check{D}_2^{-1} & 0 & 0 & 0 \\
0 & \mathcal{B}_{\rm d}\check{D}_2^{-1}\check{C}_2 & \mathcal{A} & \mathcal{B}_{\rm d}\check{D}_2^{-1} & \mathcal{B}_{\rm s} & \mathcal{B}_{\rm c} & \mathcal{B}_{\rm p} \\ \hline
\hat{C}_1 & \check{C}_3-\hat{D}_1\mathcal{D}_{\rm ee}\check{D}_2^{-1}\check{C}_2 & \hat{D}_1\mathcal{C}_{\rm d} & \hat{D}_1\mathcal{D}_{\rm ee}\check{D}_2^{-1} & \hat{D}_1\mathcal{D}_{\rm ds} & 0 & \hat{D}_1\mathcal{D}_{\rm dp} \\
0 & -\mathcal{D}_{\rm sd}\check{D}_2^{-1}\check{C}_2 & \mathcal{C}_{\rm s} & \mathcal{D}_{\rm sd}\check{D}_2^{-1} & \mathcal{D}_{\rm ss} & 0 & \mathcal{D}_{\rm sp} \\
0 & -\mathcal{D}_{\rm cd}\check{D}_2^{-1}\check{C}_2 & \mathcal{C}_{\rm c} & \mathcal{D}_{\rm cd}\check{D}_2^{-1} & \mathcal{D}_{\rm cs} & \mathcal{D}_{\rm cc} & \mathcal{D}_{\rm cp} \\
0 & -\mathcal{D}_{\rm pd}\check{D}_2^{-1}\check{C}_2 & \mathcal{C}_{\rm p} & \mathcal{D}_{\rm pd}\check{D}_2^{-1} & \mathcal{D}_{\rm ps} & \mathcal{D}_{\rm pc} & \mathcal{D}_{\rm pp}
\end{array}\right]. \tag{25}
$$

It is now crucial to observe that we have appropriately reformulated (12) in order to apply the Dualization Lemma [6]:

$$
(\star)^T \mathcal{M}(\hat{\mathcal{X}}, T_3)^{-1} \left(\begin{array}{c:cccc}
-\hat{\mathcal{A}}^T & -\hat{\mathcal{C}}_{\rm d}^T & -\hat{\mathcal{C}}_{\rm s}^T & -\hat{\mathcal{C}}_{\rm c}^T & -\hat{\mathcal{C}}_{\rm e}^T \\
I & 0 & 0 & 0 & 0 \\ \hdashline
0 & I & 0 & 0 & 0 \\
-\hat{\mathcal{B}}_{\rm d}^T & -\hat{\mathcal{D}}_{\rm dd}^T & -\hat{\mathcal{D}}_{\rm sd}^T & -\hat{\mathcal{D}}_{\rm cd}^T & -\hat{\mathcal{D}}_{\rm ed}^T \\ \hdashline
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\ \hdashline
-\hat{\mathcal{B}}_{\rm s}^T & -\hat{\mathcal{D}}_{\rm ds}^T & -\mathcal{D}_{\rm ss}^T & -\mathcal{D}_{\rm cs}^T & -\mathcal{D}_{\rm es}^T \\
-\hat{\mathcal{B}}_{\rm c}^T & 0 & 0 & -\mathcal{D}_{\rm cc}^T & -\mathcal{D}_{\rm ec}^T \\ \hdashline
0 & 0 & 0 & 0 & I \\
-\hat{\mathcal{B}}_{\rm p}^T & -\hat{\mathcal{D}}_{\rm dp}^T & -\mathcal{D}_{\rm sp}^T & -\mathcal{D}_{\rm cp}^T & -\mathcal{D}_{\rm ep}^T
\end{array}\right) \succ 0. \tag{26}
$$

Moreover, we can eliminate the unknown realization matrices of E from (22) and (26) by applying the Elimination Lemma [6]. This resolves the second issue discussed in Section IV-B.

V. Main Results

Before we state the main result, let us define the symmetric matrices P_2 and $\tilde{P}_2$ which take their values from the set $\mathbf{P}_2$ as well as the matrices $T_1 := \mathrm{diag}(P_1, P_2, I, -\gamma^2 I)$ and $T_2 := \mathrm{diag}(P_1, \tilde{P}_2, 0, -\gamma^2 I)$. Let us also define the arbitrary basis matrix Γ of the kernel of $\left(0\, C_{\rm y}\, D_{\rm yd}\, D_{\rm ys}\, D_{\rm yp}\right)$. Then we have introduced all the necessary ingredients in order to provide a finite-dimensional convex feasibility test for the existence of robust gain-scheduled estimators that guarantee a given $\mathscr{L}_2$-gain for the system interconnection of Figure 1.

Theorem 4: Statement (17) is valid if and only if there exist matrices X, Y, $\hat{X}$, $P_1 \in \mathbf{P}_1$, $P_2, \tilde{P}_2 \in \mathbf{P}_2$ for which the following LMIs hold:

$$
\Gamma^T \mathcal{O}^T \mathcal{M}(X, T_1)\, \mathcal{O}\Gamma \prec 0, \quad \mathcal{O}^T \mathcal{M}(Y, T_2)\, \mathcal{O} \prec 0 \tag{27}
$$

$$
\mathcal{O} = \left(\begin{array}{ccc|ccc}
 & I & & & 0 & \\ \hline
A_1 & 0 & B_1 C_{\rm d} & B_1 D_{\rm dd} & B_1 D_{\rm ds} & B_1 D_{\rm dp} \\
0 & A_2 & 0 & B_2 & 0 & 0 \\
0 & 0 & A & B_{\rm d} & B_{\rm s} & B_{\rm p} \\ \hline
C_1 & C_2 & D_1 C_{\rm d} & D_1 D_{\rm dd} + D_2 & D_1 D_{\rm ds} & D_1 D_{\rm dp} \\ \hline
0 & 0 & C_{\rm s} & D_{\rm sd} & D_{\rm ss} & D_{\rm sp} \\
0 & 0 & 0 & 0 & I & 0 \\ \hline
0 & 0 & C_{\rm p} & D_{\rm pd} & D_{\rm ps} & D_{\rm pp} \\
0 & 0 & 0 & 0 & 0 & I
\end{array}\right)
$$

$$
\begin{pmatrix} A_1^T \hat{X} + \hat{X} A_1 + C_1^T P_1 C_1 & \hat{X} B_1 + C_1^T P_1 D_1 \\ B_1^T \hat{X} + D_1^T P_1 C_1 & D_1^T P_1 D_1 \end{pmatrix} \succ 0 \tag{28}
$$

$$
\begin{pmatrix} I\,0\,0 \\ 0\,0\,I \end{pmatrix} Y \begin{pmatrix} I\,0\,0 \\ 0\,0\,I \end{pmatrix}^T - \begin{pmatrix} \hat{X} & 0 \\ 0 & 0 \end{pmatrix} \succ 0, \;\; X - Y \succ 0. \tag{29}
$$

Here $Y = Y^T$ has a block structure identical to that of X.

Once the LMIs (27)-(29) in the variables $X, Y, \hat{X}, P_1, P_2, \tilde{P}_2$ and γ^2 are feasible, the estimator E and the scheduling function $\hat{\Delta}_{\rm c}(\Delta_{\rm s})$ can be constructed according to the steps taken in the proof which is found in the Appendix.

Remark 2: It is also straightforward to derive a convex feasibility test for the synthesis of robust gain-scheduled feed-forward controllers by working with the dual FDI (10).

VI. Numerical Example

In order to illustrate our results, let us consider the uncertain LPV system

$$
\left(\begin{array}{c} \dot{x}(t) \\ \hdashline z_{\rm e}(t) \\ y(t) \end{array}\right) = \left(\begin{array}{cc:cc}
0 & -1+0.95\delta_{\rm d} & -2 & 0 \\
1 & -0.5+0.25\eta(t) & 1 & 0 \\ \hdashline
1 & \eta(t) & 0 & 0 \\
1 & 0 & 0 & 0.01
\end{array}\right) \left(\begin{array}{c} x(t) \\ \hdashline w_{\rm p}(t) \end{array}\right), \tag{30}
$$

where $x(t)$ is the state, $\eta(t) = \sin\frac{1}{10}t$ an on-line measurable scheduling variable, $\delta_{\rm d} \in [-1, 1]$ a time-invariant parametric uncertainty and $w_{\rm p} = \mathrm{col}(w_{\rm p1}, w_{\rm p2})$. Then the operators $\Delta_{\rm s}, \Delta_{\rm d} : \mathscr{L}_2 \to \mathscr{L}_2$ are defined by $w_{\rm s}(t) = (\Delta_{\rm s} z_{\rm s})(t) = \eta(t) z_{\rm d}(t)$ and $w_{\rm d}(t) = (\Delta_{\rm d} z_{\rm d})(t) = \delta_{\rm d} z_{\rm s}(t)$ respectively. In complete analogy to Section III, the goal is to design a robust gain-scheduled estimator that dynamically processes the measurement y and the scheduling signal η in order to provide an estimate u of the signal $z_{\rm p}$ while the $\mathscr{L}_2$-gain from $w_{\rm p}$ to $z_{\rm e} = z_{\rm p} - u$ is rendered less than γ.

For reasons of comparison we have designed three estimators: a nominal LTI estimator $E_{\rm lti}$, a nominal gain-scheduled estimator $E_{\rm nom} \star \Delta_{\rm c}$ and a robust gain-scheduled estimator $E_{\rm rob} \star \Delta_{\rm c}$. Figure 3 shows the estimation error for $\delta_{\rm d} = 0$ (left) and $\delta_{\rm d} = 1$ (right) for a random wave disturbance input $w_{\rm p1}$ (*i.e.* a sinusoid with a random uniformly distributed frequency) and a random uniformly distributed noise input $w_{\rm p2}$ with an amplitude of 1 and 0.0001 respectively.

It is very satisfactory to see that the gain-scheduled estimators $E_{\rm nom} \star \Delta_{\rm c}$ and $E_{\rm rob} \star \Delta_{\rm c}$ outperform the LTI estimator $E_{\rm lti}$ (in the sense that the estimation error is rendered small). This reveals that gain-scheduled estimation can be preferable over robust estimation in practice. Nevertheless, if we simulate the system with $\delta_{\rm d} = 1$, the estimators $E_{\rm nom}$, $E_{\rm rob}$ and $E_{\rm nomgs}$ show a drastic performance degradation. However, it is again very nice to see that the robust gain-scheduled

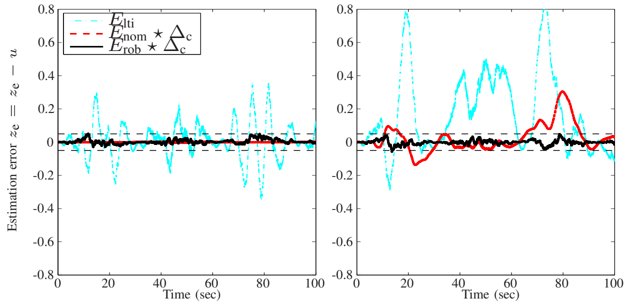

Fig. 3. Left: estimation error for $\delta_{\rm d} = 0$. Right: estimation error for $\delta_{\rm d} = 1$.

estimator $E_{\rm robgs}$ keeps the estimation error small, despite the uncertainty in the system.

We finally remark that the robust gain-scheduled estimator design has been obtained by using dynamic DG-scalings. Using static DG-scalings (*i.e.* allowing for arbitrarily fast variations of $\delta_{\rm d}$) did not lead to a feasible solution. This very nice illustrates that allowing for dynamics in the multipliers can lead to feasible and/or less conservative designs.

VII. Concluding Remarks

In this paper we have shown that the robust gain-scheduled estimator synthesis problem can be turned into a semi-definite program. We have given LMI conditions for the existence of robust gain-scheduled estimators that guarantee a given $\mathscr{L}_2$-gain for the closed-loop system. We considered uncertain dynamical LPV systems described in the standard LFT form, while the uncertainty and scheduling blocks in the system are described by general dynamic and static full-block IQC-multipliers respectively. We finally demonstrated the effectiveness of our results through a numerical example.

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VIII. Appendix

A. Sketch of Proof of Theorem 4: Necessity

Suppose that statement (17) is true. Then there exist matrices $\mathcal{X}$, $\hat{X}$, $P_1 \in \mathbf{P}_1$, $P_{2\rm e} \in \mathbf{P}_{2\rm e}$ for which (12), (13) and (16) are feasible. Now let us reformulate (12) as (22) and eliminate the realization matrices of E from (22) and (26) respectively. This yields two inequalities: $I_{\rm primal}$ and $I_{\rm dual}$. The key observation that resolves the third issue described in Section IV-B is to observe that inequality $I_{\rm dual}$ has the right structure in order to (again) apply the Dualization Lemma [6]. Moreover, after subsequently applying a simple congruence transformation and defining the new variables Y and $\tilde{P}_2$ one can apply the steps described in Section IV-D in the reverse direction. This yields (27). Finally, the stability enforcing positivity conditions in (29) come along very naturally, thanks to the Schur Complement.

B. Sketch of Proof of Theorem 4: Sufficiency

Suppose that all conditions of Theorem 4 hold true. Then there exist matrices X, Y, $\hat{X}$ $P_1 \in \mathbf{P}_1$, $P_2, \tilde{P}_2 \in \mathbf{P}_2$ that render (27)-(29) feasible. Now consider the non-singular matrices X and $X - Y$ (slightly perturb if necessary). Then by defining $\mathcal{X} = \begin{pmatrix} X & X-Y \\ X-Y & X-Y \end{pmatrix}$, we can infer (29) thanks to the Schur complement. For the extension of the scalings we recall from the literature [6] that, given the matrices P_2 and $\tilde{P}_2$, it is always possible (slightly perturb if necessary) to find a non-singular matrix N and a simple permutation matrix $\hat{N}$ such that $\mathbf{P}_{2\rm e} \ni P_{2\rm e} = \hat{N}^T \begin{pmatrix} P_2 & N \\ N^T & N^T(P_2 - \tilde{P}_2)^{-1}N \end{pmatrix} \hat{N}$. This also yields an explicit scheduling function $\hat{\Delta}_{\rm c}(\Delta_{\rm s})$ which depends smoothly on $\Delta_{\rm s}$ and satisfies (9). The realization matrices of E are finally obtained by substituting the constructed matrices $\mathcal{X}$, P_1 and $P_{2\rm e}$ in (12) and solving the resulting LMI which is, after applying the Schur complement, affine in the realization matrix variables of E.