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Average Consensus on General Digraphs

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***Abstract*— We study the average consensus problem of multi-agent systems for general network topologies with unidirectional information flow. We propose a linear distributed algorithm which guarantees state averaging on arbitrary strongly connected digraphs. In particular, this graphical condition does not require that the network be balanced or symmetric, thereby extending the previous results in the literature. The novelty of our approach is the augmentation of an additional variable for each agent, called "surplus", whose function is to locally record individual state updates. For convergence analysis, we employ graph-theoretic and nonnegative matrix tools, with the eigenvalue perturbation theory playing a crucial role.**

I. Introduction

This paper presents a novel approach to the design of distributed algorithms for *average consensus*: That is, a system of networked agents reaches an agreement on the average value of their initial states, through merely local interaction among peers. The approach enables multi-agent systems to achieve average consensus on arbitrary strongly connected network topologies with unidirectional information flow, where the state sum of agents need not be time-invariant.

There has been an extensive literature addressing multi-agent consensus problems. Many fundamental distributed algorithms (developed in, e.g., [1]–[5]) are of the *synchronous* type: At an arbitrary specified time, individual agents are assumed to sense and/or communicate with all the neighbors, and then simultaneously execute their local protocols. In particular, Olfati-Saber and Murray [3] studied algorithms of such type to achieve average consensus on digraphs, and justified that a *balanced* and *strongly connected* topology is necessary and sufficient to guarantee convergence.

In this paper, we propose a novel extension of the algorithm in [3], and prove that it guarantees state averaging on *general* strongly connected digraphs. In particular, the balanced topological requirement in [3] is dropped, and hence individual agents need not maintain identical amounts of flow-in and flow-out information. The primary challenge of average consensus on arbitrary strongly connected digraphs lies in that the state sum of the agents cannot be preserved in general, thereby causing shifts in the average value. To handle this problem, the novelty of our approach is to augment an additional variable for each agent, which we call "surplus", whose function is to record every state change of the associated agent; thus in effect, these variables locally maintain the information of the average shift amount.

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The idea of adding surplus variables is indeed a continuation of our own previous work in [6], [7], where the original surplus-based approach is proposed to tackle *quantized* average consensus on general digraphs. There we developed a quantized (thus nonlinear) averaging algorithm, and the convergence analysis is based on finite Markov chains. By contrast, the algorithm designed in this paper is linear, and hence the convergence can be characterized by the spectral properties of the associated matrices. On the other hand, our averaging algorithms differ also from those basic ones [1]–[5] in that the associated matrices contain negative entries. Consequently for our analysis tools, besides nonnegative matrix theory and algebraic graph theory, it is found that the *matrix eigenvalue perturbation theory* is instrumental. Finally, we note in [8], [9] that the approach of using auxiliary variables also to achieve averaging on general digraphs has been independently exploited. In [8], a mechanism similar to surplus is proposed for a broadcast gossip algorithm; however, the convergence of that algorithm is not proved, and is remarked to be difficult. In [9], a nonlinear (division involved) algorithm is designed whose idea is based on computing the stationary distribution for the Markov chain characterized by the agent network, and is thus quite different from consensus-type algorithms [1]–[5]. In contrast with [8], [9], the algorithm we design is linear, and we provide a rigorous justification for convergence which is based on the matrix perturbation theory.

The paper is organized as follows. In Section II we formulate the distributed average consensus problem. In Section III we present our novel solution algorithm, and justify that it guarantees state averaging on general strongly connected digraphs. Further, in Section IV we explore certain special graph topologies and in Section V we provide a numerical example for demonstration. Finally, in Section VI we state our conclusions.

Notation. Let $\mathbf{1} := [1 \cdots 1]^T \in \mathbb{R}^n$ be the vector of all ones. For a complex number λ, denote its real part by $\mathrm{Re}(\lambda)$, imaginary part by $\mathrm{Im}(\lambda)$, conjugate by $\bar{\lambda}$, and modulus by $|\lambda|$. Given a matrix M, $|M|$ denotes its determinant; $||M||_2$ and $||M||_\infty$ denote its 2-norm and infinity-norm respectively; the spectrum $\sigma(M)$ is the set of its eigenvalues; and the spectral radius $\rho(M)$ is the maximum modulus of its eigenvalues.

II. Problem Formulation

Given a network of n (> 1) agents, we model its interconnection structure by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: Each *node* in

$\mathcal{V} = \{1, ..., n\}$ stands for an agent, and each directed *edge* (j, i) in $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes that agent j communicates to agent i (namely, the information flow is from j to i). Selfloop edges are not allowed, i.e., $(i, i) \notin \mathcal{E}$. In $\mathcal{G}$ a node i is *reachable* from a node j if there exists a sequence of directed edges from j to i. We say that $\mathcal{G}$ is *strongly connected* if every node is reachable from every other node.

At time $k \in \mathbb{Z}_+$ (nonnegative integers) each node $i \in \mathcal{V}$ has a scalar state $x_i(k) \in \mathbb{R}$; the aggregate state is denoted by $x(k) = [x_1(k) \cdots x_n(k)]^T \in \mathbb{R}^n$. The *average consensus problem* aims at designing distributed algorithms, where individual nodes update their states using only the local information of their neighboring nodes in the digraph $\mathcal{G}$ such that every $x_i(k)$ eventually converges to the initial average $x_a := \mathbf{1}^T x(0)/n$.

To achieve state averaging on general digraphs, the main difficulty is that the state sum $\mathbf{1}^T x$ need not remain invariant, which can result in losing track of the initial average x_a. To deal with this problem, we propose associating to each node i an additional variable $s_i(k) \in \mathbb{R}$, called *surplus*; write $s(k) = [s_1(k) \cdots s_n(k)]^T \in \mathbb{R}^n$ and set $s(0) = 0$. The function of surplus is to locally record the state changes of individual nodes such that $\mathbf{1}^T(x(k) + s(k)) = \mathbf{1}^T x(0)$ for all time k; in other words, surplus keeps the quantity $\mathbf{1}^T(x + s)$ constant over time. The rules of how to utilize and communicate surplus mark the distinctive feature of our averaging algorithms compared to those in the literature [1]–[5], as we will see in detail in Section III.

Definition 1: A network of agents achieves *average consensus* if for every initial condition $(x(0), 0)$, $(x(k), s(k)) \to (x_a\mathbf{1}, 0)$ as $k \to \infty$.

Problem. Design a distributed algorithm such that the agents achieve average consensus on general (strongly connected) digraphs.

To solve this problem, we will propose in Section III a surplus-based distributed algorithm, under which we will justify that average consensus is achieved for general digraphs.

III. Averaging on General Networks

In this section, we first propose a linear distributed algorithm based on surplus, which may be seen as an extension of the standard consensus algorithms in the literature [1]–[5]. Then we prove that the proposed algorithm ensures average consensus for arbitrary strongly connected digraphs.

A. Algorithm Description

Consider a system of n agents represented by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For each node $i \in \mathcal{V}$, let $\mathcal{N}_i^+ := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ denote the set of its "in-neighbors", and $\mathcal{N}_i^- := \{h \in \mathcal{V} : (i, h) \in \mathcal{E}\}$ the set of its "out-neighbors". Note that $\mathcal{N}_i^+ \neq \mathcal{N}_i^-$ in general; and $i \notin \mathcal{N}_i^+$ or $\mathcal{N}_i^-$, for selfloop edges are excluded. There are three operations that every node i performs at time $k \in \mathbb{Z}_+$. First, node i sends its state information $x_i(k)$ and weighted surplus $b_{ih}s_i(k)$ to each out-neighbor $h \in \mathcal{N}_i^-$; here the *sending weight* b_{ih} is assumed to satisfy that $b_{ih} \in (0, 1)$ if $h \in \mathcal{N}_i^-$, $b_{ih} = 0$ if $h \in \mathcal{V} - \mathcal{N}_i^-$, and $\sum_{h \in \mathcal{N}_i^-} b_{ih} < 1$. Second, node i receives state information $x_j(k)$ and weighted surplus $b_{ji}s_j(k)$ from each in-neighbor $j \in \mathcal{N}_i^+$. Third, node i updates its own state $x_i(k)$ and surplus $s_i(k)$ as follows:

$$x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}_i^+} a_{ij}(x_j(k) - x_i(k)) + \epsilon s_i(k), \tag{1}$$

$$\begin{aligned} s_i(k+1) = (1 - \sum_{h \in \mathcal{N}_i^-} b_{ih})s_i(k) + \sum_{j \in \mathcal{N}_i^+} b_{ji}s_j(k) \\ - \Big(x_i(k+1) - x_i(k)\Big), \end{aligned} \tag{2}$$

where the *updating weight* a_{ij} is assumed to satisfy that $a_{ij} \in (0, 1)$ if $j \in \mathcal{N}_i^+$, $a_{ij} = 0$ if $j \in \mathcal{V} - \mathcal{N}_i^+$, and $\sum_{j \in \mathcal{N}_i^+} a_{ij} < 1$; in addition, the parameter ϵ is a positive number which specifies the amount of surplus used to update the state. It is well to note that, for sending surplus information, each agent is required to know the number of its out-neighbors.

Let the *adjacency matrix* A of the digraph $\mathcal{G}$ be given by $A := [a_{ij}] \in \mathbb{R}^{n \times n}$, where the entries are the updating weights. Then the *Laplacian matrix* L is defined as $L := D - A$, where $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i = \sum_{j=1}^n a_{ij}$. Thus L has nonnegative diagonal entries, nonpositive off-diagonal entries, and zero row sums. Then the matrix $I - L$ is nonnegative (by $\sum_{j \in \mathcal{N}_i^+} a_{ij} < 1$), and every row sums up to one; namely $I - L$ is *row stochastic*. Also let $B := [b_{ih}]^T \in \mathbb{R}^{n \times n}$, where the entries are the sending weights (note that the transpose in the notation is needed because $h \in \mathcal{N}_i^-$ for b_{ih}). Define the matrix $S := (I - \tilde{D}) + B$, where $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$ with $\tilde{d}_i = \sum_{h=1}^n b_{ih}$. Then S is nonnegative (by $\sum_{h \in \mathcal{N}_i^-} b_{ih} < 1$), and every column sums up to one; i.e., S is *column stochastic*. As can be observed from (2), the matrix S captures the part of update induced by sending and receiving surplus.

With the above matrices, the iteration of states (1) and surpluses (2) can be written in a matrix form as

$$\begin{bmatrix} x(k+1) \\ s(k+1) \end{bmatrix} = M \begin{bmatrix} x(k) \\ s(k) \end{bmatrix}, \text{ where } M := \begin{bmatrix} I - L & \epsilon I \\ L & S - \epsilon I \end{bmatrix}. \tag{3}$$

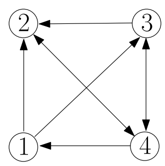
Notice that (i) the matrix M has negative entries due to the presence of the Laplacian matrix L in the $(2, 1)$-block; (ii) the column sums of M are equal to one, which implies that the quantity $x(k) + s(k)$ is a constant for all $k \geq 0$; and (iii) the state evolution specified by the $(1, 1)$-block of M, i.e.,

$$x(k+1) = (I - L)x(k), \tag{4}$$

is that of the standard consensus algorithm studied in [1]–[5]. For an illustration of the distributed algorithm (3) see Fig. 1. We will analyze its convergence properties in the next subsection.

B. Convergence Result

We present the main result of this section.



$$\mathcal{N}_1^+ = \{4\} \qquad \mathcal{N}_1^- = \{2,3\}$$
$$\mathcal{N}_2^+ = \{1,3,4\} \qquad \mathcal{N}_2^- = \{4\}$$
$$\mathcal{N}_3^+ = \{1,4\} \qquad \mathcal{N}_3^- = \{2,4\}$$
$$\mathcal{N}_4^+ = \{2,3\} \qquad \mathcal{N}_4^- = \{1,2,3\}$$

$$M = \left[\begin{array}{cccc|cccc} 1/2 & 0 & 0 & 1/2 & \epsilon/2 & 0 & 0 & \epsilon/2 \\ 1/4 & 1/4 & 1/4 & 1/4 & \epsilon/4 & 3\epsilon/4 & \epsilon/4 & \epsilon/4 \\ 1/3 & 0 & 1/3 & 1/3 & \epsilon/3 & 0 & 2\epsilon/3 & \epsilon/3 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & \epsilon/3 & \epsilon/3 & 2\epsilon/3 \\ \hline 1/2 & 0 & 0 & -1/2 & 1/3-\epsilon/2 & 0 & 0 & 1/4-\epsilon/2 \\ -1/4 & 3/4 & -1/4 & -1/4 & 1/3-\epsilon/4 & 1/2-3\epsilon/4 & 1/3-\epsilon/4 & 1/4-\epsilon/4 \\ -1/3 & 0 & 2/3 & -1/3 & 1/3-\epsilon/3 & 0 & 1/3-2\epsilon/3 & 1/4-\epsilon/3 \\ 0 & -1/3 & -1/3 & 2/3 & 0 & 1/2-\epsilon/3 & 1/3-\epsilon/3 & 1/4-2\epsilon/3 \end{array}\right].$$

Fig. 1. Example of distributed algorithm (3): four agents in a strongly connected but unbalanced digraph. For $i \in [1,4]$ let the updating weights be $a_{ij} = 1/(|\mathcal{N}_i^+|+1)$, $j \in \mathcal{N}_i^+$ and the sending weights $b_{ih} = 1/(|\mathcal{N}_i^-|+1)$, $h \in \mathcal{N}_i^-$. Observe that the matrix M has some negative entries.

Theorem 1: Using the distributed algorithm (3) with the parameter $\epsilon > 0$ sufficiently small, the agents achieve average consensus if and only if the digraph $\mathcal{G}$ is strongly connected.

Without augmenting surplus variables, it is well known [3] that a necessary and sufficient graphical condition for state averaging is that the digraph $\mathcal{G}$ is both strongly connected and *balanced* (i.e., the adjacency matrix A is such that $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$ for all i). A balanced structure can be restrictive, especially when all the weights a_{ij} are identical, it requires the number of incoming and outgoing edges at each node in the digraph to be the same. By contrast, the algorithm (3), with surplus augmented, ensures average consensus for arbitrary strongly connected digraphs including the ones that are unbalanced.

A surplus-based averaging algorithm was initially proposed in [6], [7] for a quantized consensus problem. It guarantees that the integer-valued states converge to either the floor $\lfloor x_a \rfloor$ or the ceiling $\lceil x_a \rceil$ on general digraphs; however, the steady-state surpluses are nonzero in general. There, the set of states and surpluses is finite, and thus arguments of finite Markov chain type are employed to prove the convergence. Distinct from [6], [7], with the algorithm (3) the states converge to the exact average x_a and the steady-state surpluses are zero. Moreover, since (3) is linear, its convergence can be analyzed using tools from matrix theory, as detailed below.

The choice of the parameter ϵ depends on the graph structure and the number of agents. We will obtain bounds on ϵ for general networks in Subsection III-C and for some specific graphs in Section IV.

We now proceed to the proof of Theorem 1. For the necessity argument see [6, Theorem 2]; indeed, the class of strongly connected digraphs characterizes the existence of any distributed algorithm that can solve average consensus. For the sufficiency part, we need the following necessary and sufficient condition for average consensus in terms of the spectrum of the matrix M.

Proposition 1: The distributed algorithm (3) achieves average consensus if and only if 1 is a simple eigenvalue of M, and the other eigenvalues have moduli smaller than one.

For the proof, see [10]. Now let

$$M_0 := \begin{bmatrix} I-L & 0 \\ L & S \end{bmatrix} \quad \text{and} \quad E := \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix}. \qquad (5)$$

Then $M = M_0 + \epsilon E$, and we view M as being obtained by "perturbing" M_0 via the term ϵE. Concretely, we show that the eigenvalues λ_i of the unperturbed matrix M_0 satisfy

$$1 = \lambda_1 = \lambda_2 > |\lambda_3| \geq \cdots \geq |\lambda_{2n}|; \qquad (6)$$

and that after a small perturbation ϵE, the obtained matrix M has only a simple eigenvalue 1 and all the other eigenvalues have moduli smaller than one. Hence by Proposition 1, average consensus is achieved. We point out that, unlike the standard consensus algorithm (4), the tools from nonnegative matrix theory cannot be directly used to analyze the spectrum of M due to the existence of some negative entries.

Sufficiency proof of Theorem 1. First, we prove the assertion (6). Since M_0 is block (lower) triangular, its spectrum is $\sigma(M_0) = \sigma(I-L) \cup \sigma(S)$. Recall that the matrices $I-L$ and S are row and column stochastic, respectively; so their spectral radii satisfy $\rho(I-L) = \rho(S) = 1$. Now owing to that $\mathcal{G}$ is strongly connected, $I-L$ and S are both *irreducible* [11]; thus by the Perron-Frobenius Theorem (see, e.g., [11, Chapter 8]) $\rho(I-L)$ (resp. $\rho(S)$) is a simple eigenvalue of $I-L$ (resp. S). This implies (6). Moreover, for $\lambda_1 = \lambda_2 = 1$, one derives that the corresponding geometric multiplicity equals two by verifying rank$(M_0 - I) = 2n-2$. Hence the eigenvalue 1 has linearly independent right eigenvectors y_1 and y_2 and left eigenvectors z_1 and z_2 as

$$Y := \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1} \\ v_2 & -nv_2 \end{bmatrix}, \quad Z := \begin{bmatrix} z_1^T \\ z_2^T \end{bmatrix} = \begin{bmatrix} \mathbf{1}^T & \mathbf{1}^T \\ v_1^T & 0 \end{bmatrix}.$$

Here $v_1 \in \mathbb{R}^n$ is a left eigenvector of $I-L$ with respect to $\rho(I-L)$ such that it is positive and scaled to satisfy $v_1^T \mathbf{1} = 1$; and $v_2 \in \mathbb{R}^n$ is a right eigenvector of S corresponding to $\rho(S)$ such that it is positive and scaled to satisfy $\mathbf{1}^T v_2 = 1$. The fact that positive eigenvectors v_1 and v_2 exist follows again from the Perron-Frobenius Theorem. In addition, one may check that $ZY = I$.

Next, we will qualify the changes of the two eigenvalues $\lambda_1 = \lambda_2 = 1$ of M_0 under a small perturbation ϵE. We do this by computing the derivatives $d\lambda_1(\epsilon)/d\epsilon$ and $d\lambda_2(\epsilon)/d\epsilon$, both evaluated when ϵ is set zero; here $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ are the eigenvalues of M corresponding respectively to λ_1 and λ_2. By [12, Sections 2.8] we know that when $\epsilon > 0$ is small, the above two derivatives exist, and they are the two eigenvalues of the following matrix:

$$\begin{bmatrix} z_1^T E y_1 & z_1^T E y_2 \\ z_2^T E y_1 & z_2^T E y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ v_1^T v_2 & -n v_1^T v_2 \end{bmatrix}.$$

Therefore, $d\lambda_1(\epsilon)/d\epsilon = 0$, and $d\lambda_2(\epsilon)/d\epsilon = -n v_1^T v_2 < 0$. This implies that when $\epsilon > 0$ is small, $\lambda_1(\epsilon)$ stays put while $\lambda_2(\epsilon)$ moves to the left along the real axis. Then by continuity, there must exist a positive δ_1 such that $\lambda_1(\delta_1) = 1$ and $\lambda_2(\delta_1) < 1$. On the other hand, since eigenvalues are continuous functions of matrix entries (e.g., [13], [14]), there must exist a positive δ_2 such that $|\lambda_i(\delta_2)| < 1$ for all $i \in [3, 2n]$. Thus for any sufficiently small $\epsilon \in (0, \min\{\delta_1, \delta_2\})$, the matrix M has a simple eigenvalue 1 and all other eigenvalues have moduli smaller than one. Therefore, from Proposition 1, the conclusion that average consensus is achieved follows. ∎

Remark 1: Assuming that with the algorithm (3), states converge to the initial average and surpluses converge to zero, the speed of convergence is governed by the second largest (in modulus) eigenvalue of the matrix M. We denote this particular eigenvalue by λ_2^*, and refer to it as the *convergence factor* of algorithm (3) (cf. [4]). Note that λ_2^* depends not only on the graph topology but also on the parameter ϵ; and $\lambda_2^* < 1$ is equivalent to average consensus.

C. Bound on Parameter ϵ

Having shown that the distributed algorithm (3) solves average consensus for sufficiently small parameter ϵ, we present in this subsection an upper bound on ϵ. For this, we borrow a fact from matrix perturbation theory (e.g., [13], [14]) which relates the size of ϵ to the distance between perturbed and unperturbed eigenvalues. Below is the main result of this investigation.

Proposition 2: Suppose that the digraph $\mathcal{G}$ is strongly connected. The distributed algorithm (3) achieves average consensus if the parameter ϵ satisfies $\epsilon \in (0, \bar{\epsilon})$, where

$$\bar{\epsilon} := \frac{1}{(20+8n)^n} \, (1 - |\lambda_3|)^n, \quad \text{with } \lambda_3 \text{ in (6).} \tag{7}$$

We stress that the above bound $\bar{\epsilon}$ ensures average consensus for arbitrary strongly connected topologies. Due to the power n, however, the bound is rather conservative in general. This power is in fact unavoidable for any perturbation bound result with respect to general matrices, as is well known in matrix perturbation literature [13], [14]. In Section IV, we will exploit structures of some special topologies, which yield less conservative bounds on ϵ.

Some preliminaries will be presented first, based on which Proposition 2 follows immediately. Henceforth in this subsection, the digraph $\mathcal{G}$ is assumed to be strongly connected. We begin by introducing a metric for the distance between the spectrums of M_0 and M; here $M = M_0 + \epsilon E$, with M_0 and E in (5). Let $\sigma(M_0) := \{\lambda_1, \ldots, \lambda_{2n}\}$ (where the numbering is the same as that in (6)) and $\sigma(M) := \{\lambda_1(\epsilon), \ldots, \lambda_{2n}(\epsilon)\}$. The *optimal matching distance* $d(\sigma(M_0), \sigma(M))$ [13], [14] is defined by

$$d(\sigma(M_0), \sigma(M)) := \min_{\pi} \max_{i \in [1,2n]} |\lambda_i - \lambda_{\pi(i)}(\epsilon)|, \tag{8}$$

where π is taken over all permutations of $\{1, \ldots, 2n\}$. This means that if we draw $2n$ identical circles centered respectively at $\lambda_1, \ldots, \lambda_{2n}$, then $d(\sigma(M_0), \sigma(M))$ is the smallest radius such that these circles include all $\lambda_1(\epsilon), \ldots, \lambda_{2n}(\epsilon)$. Here is an upper bound on the optimal matching distance [13, Theorem VIII.1.5].

Lemma 1: An upper bound on $d(\sigma(M_0), \sigma(M))$ is $d(\sigma(M_0), \sigma(M)) \le 4\, (||M_0||_\infty + ||M||_\infty)^{1-1/n}\, ||\epsilon E||_\infty^{1/n}$.

Next, we consider the eigenvalues $\lambda_3(\epsilon), \ldots, \lambda_{2n}(\epsilon)$ of M, whose corresponding unperturbed counterparts $\lambda_3, \ldots, \lambda_{2n}$ of M_0 lie strictly inside the unit circle (see Theorem 1).

Lemma 2: If the parameter $\epsilon \in (0, \bar{\epsilon})$ with $\bar{\epsilon}$ in (7), then $|\lambda_3(\epsilon)|, \ldots, |\lambda_{2n}(\epsilon)| < 1$.

Proof. Since $L = D - A$ and $S = (I - \tilde{D}) + B$, one can compute $||L||_\infty = 2\max_{i\in[1,n]} d_i < 2$ and $||S||_\infty < n$. Then $||M_0||_\infty \le ||L||_\infty + ||S||_\infty < 2 + n$ and $||E||_\infty \le 1$. By Lemma 1,

$$\begin{aligned} d(\sigma(M_0), \sigma(M)) &\le 4\, (2||M_0||_\infty + \epsilon||E||_\infty)^{1-\frac{1}{n}}\, (\epsilon||E||_\infty)^{\frac{1}{n}} \\ &< 4\, (4 + 2n + \epsilon)^{1-\frac{1}{n}}\, \epsilon^{\frac{1}{n}} \\ &< 4\, (4 + 2n + \epsilon)\, \epsilon^{\frac{1}{n}} < 1 - |\lambda_3|. \end{aligned}$$

The last inequality is due to $\epsilon < \bar{\epsilon}$ in (7). Now recall from the proof of Theorem 1 that the unperturbed eigenvalues $\lambda_3, \ldots, \lambda_{2n}$ of M_0 lie strictly inside the unit circle. Therefore, perturbing the eigenvalues $\lambda_3, \ldots, \lambda_{2n}$ by an amount less than $\bar{\epsilon}$, the resulting eigenvalues $\lambda_3(\epsilon), \ldots, \lambda_{2n}(\epsilon)$ will remain inside the unit circle. ∎

It is left to consider the eigenvalues $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ of M. Since every column sum of M equals one for an arbitrary ϵ, we obtain that 1 is always an eigenvalue of M. Hence $\lambda_1(\epsilon)$ must be equal to 1 for any ϵ. On the other hand, for $\lambda_2(\epsilon)$ the following is true.

Lemma 3: If the parameter $\epsilon \in (0, \bar{\epsilon})$ with $\bar{\epsilon}$ in (7), then $|\lambda_2(\epsilon)| < 1$.

For the proof, see [10]. Now summarizing Lemmas 2 and 3, we obtain that if the parameter $\epsilon \in (0, \bar{\epsilon})$ with $\bar{\epsilon}$ in (7), then $\lambda_1(\epsilon) = 1$ and $|\lambda_2(\epsilon)|$, $|\lambda_3(\epsilon)|$, …, $|\lambda_{2n}(\epsilon)| < 1$. Therefore, by Proposition 1 the distributed algorithm (3) achieves average consensus; this establishes Proposition 2.

IV. Special Topologies

We turn now to a special class of topologies – strongly connected and balanced digraphs – and investigate the required upper bound on the parameter ϵ. Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, its *degree* d is defined by $d := \max_{i\in\mathcal{V}} \mathrm{card}(\mathcal{N}_i^+)$, where $\mathrm{card}(\cdot)$ denotes set cardinality. In the algorithm (3)

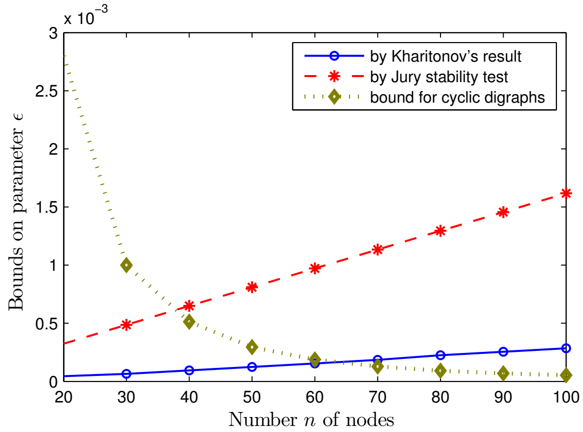


Fig. 2. Upper bounds on parameter ϵ such that the distributed algorithm (3) achieves average consensus on general strongly connected regular digraphs (solid and dashed curves) and cyclic digraphs (dotted curve).

choose the updating and sending weights to be respectively $a_{ij} = 1/(2dn)$ and $b_{ij} = 1/(dn)$, for every $(j,i) \in \mathcal{E}$.

Proposition 3: Suppose that the parameter ϵ satisfies $\epsilon \in (0, 2)$, and the zeros of the following polynomial for every $\mu \neq 0$ with $|\mu - 1/(2n)| \leq 1/(2n)$ lie inside the unit circle:

$$p(\lambda) := \lambda^2 + \alpha_1 \lambda + \alpha_0, \tag{9}$$

where $\alpha_0 := 2\mu^2 - 3\mu - \epsilon + 1$, $\alpha_1 := 3\mu + \epsilon - 2$. Then the distributed algorithm (3) achieves average consensus on strongly connected and balanced digraphs.

For the proof, see [10]. Now we investigate the values of ϵ that ensure the zeros of the polynomial $p(\lambda)$ in (9) inside the unit circle, which in turn guarantee average consensus on strongly connected regular digraphs. For this, we view the polynomial $p(\lambda)$ as *interval polynomials* [15] by letting μ take any value in the square: $0 \leq \operatorname{Re}(\mu) \leq 1/n$, $-1/(2n) \leq \operatorname{Im}(\mu) \leq 1/(2n)$. Applying the bilinear transformation we obtain a new family of interval polynomials:

$$\begin{aligned}\tilde{p}(\gamma) &:= (\gamma - 1)^2 p\left(\frac{\gamma+1}{\gamma-1}\right) \\ &= (1 + \alpha_0 + \alpha_1)\gamma^2 + (2 - 2\alpha_0)\gamma + (1 + \alpha_0 - \alpha_1).\end{aligned}$$

Then by Kharitonov's result for the complex-coefficient case, the stability of $\tilde{p}(\gamma)$ (its zeros have negative real parts) is equivalent to the stability of eight extreme polynomials [15, Section 6.9], which in turn suffices to guarantee that the zeros of $p(\lambda)$ lie strictly inside the unit circle. Checking the stability of eight extreme polynomials results in upper bounds on ϵ in terms of n. This is displayed in Fig. 2 as the solid curve. We see that the bounds grow linearly, which is in contrast with the general bound $\bar{\epsilon}$ in Proposition 2 that decays exponentially and is known to be conservative. This is due to that, from the robust control viewpoint, the uncertainty of μ in the polynomial coefficients becomes smaller as n increases.

Alternatively, we employ the Jury stability test [16] to derive that the zeros of the polynomial $p(\lambda)$ are strictly inside the unit circle if and only if

$$\begin{aligned}\beta_0 &:= \begin{vmatrix} 1 & \alpha_0 \\ \bar{\alpha}_0 & 1 \end{vmatrix} > 0, \\ \beta_1 &:= \begin{vmatrix} \begin{vmatrix} 1 & \alpha_0 \\ \bar{\alpha}_0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & \alpha_1 \\ \bar{\alpha}_0 & \bar{\alpha}_1 \end{vmatrix} \\ \begin{vmatrix} 1 & \bar{\alpha}_1 \\ \alpha_0 & \alpha_1 \end{vmatrix} & \begin{vmatrix} 1 & \bar{\alpha}_0 \\ \alpha_0 & 1 \end{vmatrix} \end{vmatrix} > 0.\end{aligned} \tag{10}$$

Here β_0 and β_1 turn out to be polynomials in ϵ of second and fourth order, respectively; the corresponding coefficients are functions of μ and n. Thus selecting μ such that $\mu \neq 0$ and $|\mu - 1/(2n)| \leq 1/(2n)$, we can solve the inequalities in (10) for ϵ in terms of n. Thereby we obtain the dashed curve in Fig. 2, each plotted point being the minimum value of ϵ over 1000 random samples such that the inequalities in (10) hold. This simulation confirms that the true bound on ϵ for the zeros of $p(\lambda)$ to be inside the unit circle is between the solid and dashed curves. It is suggested that our previous analysis based on Kharitonov's result may not be very conservative.

Finally, we obtain two results by further specializing the balanced digraph $\mathcal{G}$ to be symmetric or cyclic, respectively, and provide analytic ϵ bounds less conservative than (7) for the general case. In particular, the exponent n is not involved. For the proofs, see [10].

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *symmetric* if $(j,i) \in \mathcal{E}$ implies $(i,j) \in \mathcal{E}$. That is, $\mathcal{G}$ is undirected.

Proposition 4: Consider a general connected undirected graph $\mathcal{G}$. Then the algorithm (3) achieves average consensus if the parameter ϵ satisfies $\epsilon \in (0, (1 - (1/n))(2 - (1/n))$.

It is noted that for connected undirected graphs, the upper bound on ϵ ensuring average consensus grows as n increases. This characteristic is in agreement with that of the bounds for the more general class of balanced digraphs as we observed in Fig. 2.

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *cyclic* if $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} = \{(1,2), (2,3), \ldots, (n-1, n), (n, 1)\}$. So a cyclic digraph is strongly connected.

Proposition 5: Suppose that the digraph $\mathcal{G}$ is cyclic. Then the algorithm (3) achieves average consensus if the parameter ϵ satisfies

$$\epsilon \in \left(0, \frac{\sqrt{2}}{3 + \sqrt{5}}(1 - |\lambda_3|)\right), \quad \text{with } \lambda_3 \text{ as in (6).} \tag{11}$$

Further, in this case $|\lambda_3| =$

$$\sqrt{1 - (1/n) + (1/(2n^2)) + (1/n)(1 - 1/(2n))\cos 2\pi/n}.$$

The proof of Proposition 5 is based on the Bauer-Fike perturbation result (e.g., [11, Section 6.3]). In Fig. 2 we plot the upper bound on ϵ in (11) for the class of cyclic digraphs. We see that this bound decays as the number n of nodes increases, which contrasts with the bound characteristic of the more general class of balanced digraphs. This may indicate the conservativeness of our approach based on perturbation theory. Nevertheless, since the Bauer-Fike Theorem is specific only to diagonalizable matrices, the derived upper bound in (11) is less conservative than the general one in (7).

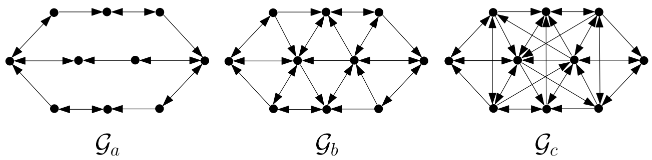


Fig. 3. Three examples of strongly connected but unbalanced digraphs.

TABLE I
CONVERGENCE FACTOR λ_2^* WITH RESPECT TO THREE DIFFERENT VALUES OF PARAMETER ϵ.

	$\epsilon = 0.2$	$\epsilon = 0.7$	$\epsilon = 2.15$
$\mathcal{G}_a$	0.9963	0.9993	1.0003
$\mathcal{G}_b$	0.9951	0.9969	0.9985
$\mathcal{G}_c$	0.9883	0.9930	0.9966

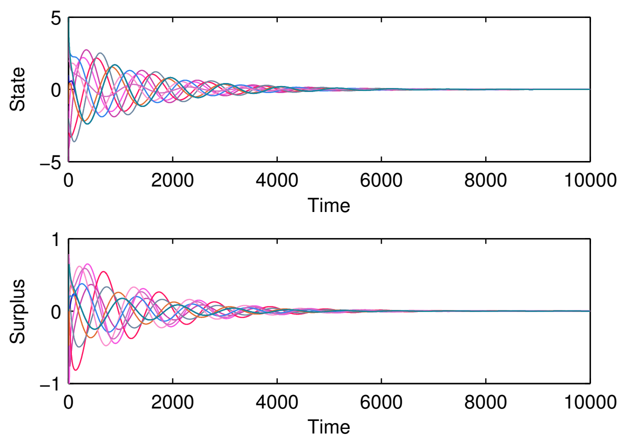


Fig. 4. Convergence paths of states and surpluses: obtained by applying the distributed algorithm (3) with parameter $\epsilon = 0.7$ on digraph $\mathcal{G}_a$.

V. NUMERICAL EXAMPLE

We provide a numerical example to illustrate the convergence properties of the distributed algorithm (3). Consider the three digraphs displayed in Fig. 3, with 10 nodes and respectively 17, 29, and 38 edges. Note that all the digraphs are strongly connected, and in the case of uniform weights they are unbalanced (indeed, no single node is balanced). We apply the distributed algorithm (3) with uniform weights $a = 1/(2\text{card}(\mathcal{E}))$ and $b = 1/\text{card}(\mathcal{E})$.

The convergence factor λ_2^* (see Remark 1) for three different values of the parameter ϵ are summarized in Table I. We see that small ϵ ensures convergence of the algorithm (3) ($\lambda_2^* < 1$), whereas large ϵ can lead to instability. Moreover, in those converging cases the factor λ_2^* decreases as the number of edges increases from $\mathcal{G}_a$ to $\mathcal{G}_c$, which indicates faster convergence when there are more communication channels available for information exchange. We also see that the algorithm (3) is more robust on digraphs with more edges, in the sense that a larger range of values of ϵ is allowed.

For a random initial state $x(0)$ with the average $x_a = 0$ and the initial surplus $s(0) = 0$, we display in Fig. 4 the trajectories of both states and surpluses when the distributed algorithm (3) is applied on digraph $\mathcal{G}_a$ with parameter $\epsilon = 0.7$. Observe that asymptotically, state averaging is achieved and surplus vanishes.

VI. CONCLUSIONS

We have proposed a new distributed algorithm which enables networks of agents to achieve average consensus on arbitrary strongly connected digraphs. To emphasize, the derived graphical condition is more general than those previously reported in the literature, in the sense that it does not require balanced network structure. Moreover, special topologies have been investigated to give less conservative bounds on the parameter ϵ, and a numerical example has been provided to illustrate the convergence results. One interesting future direction would be to extend the current results to general time-varying topologies.

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