2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A unified approach to the velocity-free consensus algorithms design for double integrator dynamics with input saturations

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Abstract— **This paper considers the consensus problem of double integrator multi-agent systems where each agent is subject to input saturations, and the velocity (second state) of each agent is not available for feedback. We present a unified approach to the consensus algorithms design that extends most of the existent consensus algorithms developed for double integrator multi-agents in ideal situations to handel these two problems simultaneously. To illustrate the effectiveness of the proposed approach, we present solutions to three different second order consensus problems and provide simulation results.**

I. INTRODUCTION

In contrast to multi-agents with first order dynamics, consensus algorithms for double integrators can be naturally extended to design cooperative control strategies for complex physical systems with applications to flocking [1], formation control of unmanned vehicles [2]-[3], rigid body attitude synchronization [4]-[5] and synchronization of networked Euler-Lagrange systems [6]-[7]. The consensus problem of double integrators involves the design of consensus algorithms such that agents can reach an agreement on their states, or on a common objective, using local information exchange. This information exchange is generally restricted to be directed, dynamically changing, and may be delayed.

In the related literature to the second order consensus problem, tools from algebraic graph theory have been successfully applied to establish conditions under which second order consensus is reached. In directed networks, it has been shown that second order consensus will be reached if and only if the communication graph has a spanning tree and the control gains are carefully selected [8], [9]. Within a similar framework, several related problems to consensus have been considered such as the formation control problem, [10], consensus with group reference velocity, [11] and leader-follower problems, [12]. Also, the case of dynamically changing topologies have been discussed in [8] and [13]. The effects of communication delays that are inherently present in communication systems have also been considered in [14]-[16] and references therein. However, the above consensus algorithms are based on the assumption that the full state vector is available for feedback.

In practice, it is sometimes desirable to design consensus algorithms that do not require full states information. If we consider, for example, a group of point masse agents, an important problem is to design consensus algorithms in the case where the velocity information (the second state) is not available for feedback, either because it is not precisely measured or agents are not equipped with velocity sensors. Another important problem arises when the input of each agent is subject to input saturations. Unfortunately, the papers dealing with these two problems are not numerous and only the simple case of fixed and undirected communication topology has been considered. The author in [11] proposed consensus algorithms that account for input saturations in the full state information case. In the same reference, the author presents a second order consensus algorithm that removes the requirement of velocity measurements. Consensus algorithms that take into account the two above problems simultaneously have been proposed in [17]. In this reference, a new approach, based on auxiliary systems, has been proposed to simplify the consensus algorithm design problem in this case. However, it is difficult to show that the results in [17] are applicable under more general communication topologies, that may be directed, time-varying and/or subject to communication delays.

The main contribution of this paper is to provide a unified approach that extends most of the existent consensus algorithms, developed for double integrator dynamics with a certain communication topology, to account for input saturations and remove the requirement of velocity measurements. Instrumental to our approach is the introduction of two second order auxiliary systems that simplify the consensus algorithm design. The first auxiliary system is used to generate an intermediate reference trajectory for each agent, and its input is designed such that all agents reach an agreement on their reference trajectories. The input of the second auxiliary system is designed such that each agent tracks its corresponding intermediate reference trajectory without velocity measurements. With this setting, the control input of each agent is constructed using only the auxiliary states to account for input saturations. As a result, the consensus algorithm design problem with the above mentioned constraints is reduced to the design of a consensus algorithm in ideal situations, *i.e.,* in the full state information case and without input saturations. To show the effectiveness of the obtained results, we consider three different problems related to second order consensus, and extend some consensus algorithms developed for double integrator dynamics to account for input saturations in the partial state feedback case.

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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II. Problem description and preliminaries

Consider a group of n-identical autonomous agents modeled by the following second-order dynamics

$$\boldsymbol{\Sigma}_{\mathbf{p}}: \quad \ddot{\mathbf{p}}_i = \mathbf{u}_i, \quad \text{for} \ \ i \in \mathcal{N}, \tag{1}$$

where $\mathcal{N} \triangleq \{1, ..., n\}$, $\mathbf{p}_i \in \mathbb{R}^m$ and $\dot{\mathbf{p}}_i$ denote respectively the position and velocity states of the i^{th} agent, the vector $\mathbf{u}_i \in \mathbb{R}^m$ is the control input and $\boldsymbol{\Sigma}_{\mathbf{p}}$ is used to designate the multi-agent system (1). The communication topology between agents is represented by a weighted graph $\mathcal{G}_n = (\mathcal{N}, \mathcal{E}, \mathcal{K})$, where $\mathcal{N}$ is the set of nodes or vertices, describing the set of vehicles in the team, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of pairs of nodes, called edges, and $\mathcal{K} = [k_{ij}]$ is a weighted adjacency matrix. An edge $(i, j) \in \mathcal{E}$ indicates that agent i can receive information from agent j, which is designated as its neighbor. The weighted adjacency matrix of a weighted graph is defined such that $k_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$ and $k_{ij} = 0$ if and only if $(i, j) \notin \mathcal{E}$. If the communication topology is bidirectional, then $\mathcal{G}_n$ is undirected, the pairs of nodes in $\mathcal{E}$ are unordered, $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$, and $\mathcal{K}$ is symmetric. In the case of unidirectional communication topology, $\mathcal{G}_n$ is a directed graph, $\mathcal{E}$ contains ordered pairs, and $\mathcal{K}$ is not necessarily symmetric.

Definition 1: The second order consensus problem consists of the design of a consensus algorithm $\mathbf{u}_i$, such that the solution of (1) satisfies

$$(\mathbf{p}_i - \mathbf{p}_j) \to 0, \quad (\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j) \to 0, \tag{2}$$

for $i, j \in \mathcal{N}$ and for any initial conditions. In this case, multi-agent system (1) is said to achieve second order consensus.

The above problem is generally referred to as the free-consensus problem, and several related problems can be discussed in a similar framework, including consensus with reference trajectory, leader/follower, flocking, rendezvous, and formation control problems.

We assume that all agents are subject to input saturations, such that $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, and the velocity of the agents are not available for feedback. Also, we assume that there exists a consensus algorithm developed for the multi-agent system (1), in the full state information case and without input saturations, expressed as

$$\mathbf{u}_i = \boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n), \tag{3}$$

where $\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n)$ is a protocol designed using the states of the multi-agent system $\boldsymbol{\Sigma}_{\mathbf{p}}$ under the communication topology described by $\mathcal{G}_n$. This protocol is generally constructed based on the position and velocity states of the i^{th} agent and the states of its neighbors, as well as information on a global objective if assigned to the team, and satisfies the following assumptions.

A1. The multi-agent system

$$\boldsymbol{\Sigma}_{\mathbf{p}}: \quad \ddot{\mathbf{p}}_i = \boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n), \quad \text{for} \ \ i \in \mathcal{N}, \tag{4}$$

achieves second order consensus in the sense of Definition 1, where $\mathcal{G}_n$ can be restricted to be directed, time-varying, and/or subject to communication delays.

A2. The protocol $\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n)$ can be written as: $\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n) := \boldsymbol{f}_d + \bar{\boldsymbol{\Psi}}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n)$, with $\boldsymbol{f}_d \in \mathbb{R}^m$ satisfying $\|\boldsymbol{f}_d\|_\infty \leq \boldsymbol{f}_{\max}$, and the solutions of (4) guarantee that $\bar{\boldsymbol{\Psi}}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n)$ is globally bounded and converges asymptotically to zero when the multi-agent system (4) achieves consensus.

It should be mentioned that several consensus algorithms, written as in (3), have been proposed in the literature for the multi-agent system (1) in the full state information case and without input saturations. It has been shown that with an appropriate design of this type of protocols, second order consensus will be reached under some conditions on $\mathcal{G}_n$ and/or the control parameters. In addition, the vector $\mathbf{f}_d$ in Assumption A2 can be considered as the only *a priori* bounded term that may not converge to zero when the multi-agent system (4) reaches consensus. This vector is generally defined if a global objective is assigned to the team, such as a desired trajectory.

With the above assumptions, our objective in this work is to provide a unified approach to the consensus algorithm design for multi-agent system (1), under a certain communication topology described by $\mathcal{G}_n$, such that second order consensus is achieved.

Before we proceed, we give some definitions and a preliminary result that will be used to prove our results. We define for any vector $\mathbf{x} = (x^1, ..., x^m)^\top \in \mathbb{R}^m$ the function

$$\boldsymbol{\chi}(\mathbf{x}) = \text{col}[\sigma(x^j)] \in \mathbb{R}^m, \quad \text{for} \quad j = 1, ..., m, \tag{5}$$

with $\sigma : \mathbb{R} \to \mathbb{R}$, is a strictly increasing continuously differentiable function satisfying the following properties:

P1. $\sigma(0) = 0$ and $x\sigma(x) > 0$ for $x \neq 0$,

P2. $|\sigma(x)| \leq \sigma_b$, for σ_b is a strictly positive constant.

P3. $\frac{\partial \sigma(x)}{\partial x}$ is bounded.

Note that property P3 can be verified from P1 and P2. Examples of the function $\sigma(x)$ include: $\tanh(x)$ and $\frac{x}{\sqrt{1+x^2}}$. We state in the following lemma a preliminary result that will be used in the proof of our results.

Lemma 1: Consider the second order system

$$\ddot{\boldsymbol{\zeta}}_i = -L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i) - L_i^d \boldsymbol{\chi}(\dot{\boldsymbol{\zeta}}_i) + \boldsymbol{\varepsilon}_i, \tag{6}$$

where $\boldsymbol{\zeta}_i \in \mathbb{R}^m$, the function $\boldsymbol{\chi}$ is defined in (5), and L_i^p and L_i^d are positive scalars. If $\boldsymbol{\varepsilon}_i$ is bounded for all time and $\boldsymbol{\varepsilon}_i \to 0$, then $\boldsymbol{\zeta}_i$ and $\dot{\boldsymbol{\zeta}}_i$ are bounded and $\boldsymbol{\zeta}_i \to \dot{\boldsymbol{\zeta}}_i \to 0$.

Proof: See [17] for a similar proof with $\sigma(x) = \tanh(x)$. ■

III. Velocity-free consensus algorithms with input saturations

In this section, we present consensus algorithms for the multi-agent system (1) that account for input saturations and remove the requirement of velocity measurements. To this end, we associate to each agent in the team the following two second order auxiliary systems

$$\boldsymbol{\Sigma}_{\boldsymbol{\zeta}}: \quad \ddot{\boldsymbol{\zeta}}_i = \mathbf{u}_i - \boldsymbol{\Gamma}_i, \quad \text{for} \ \ i \in \mathcal{N}, \tag{7}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{\xi}}: \quad \ddot{\boldsymbol{\xi}}_i = \mathbf{u}_i^v, \quad \text{for} \ \ i \in \mathcal{N}, \tag{8}$$

where $\boldsymbol{\zeta}_i \in \mathbb{R}^m$, $\boldsymbol{\xi}_i \in \mathbb{R}^m$, $\mathbf{u}_i$ is the control input of (1), $\boldsymbol{\Gamma}_i$ and $\mathbf{u}_i^v$ are auxiliary input vectors to be designed. The initial states of (7)-(8), *i.e.,* $\boldsymbol{\zeta}_i(0)$, $\dot{\boldsymbol{\zeta}}_i(0)$, $\boldsymbol{\xi}_i(0)$, and $\dot{\boldsymbol{\xi}}_i(0)$, can be selected arbitrarily. Also, define the vector $\mathbf{r}_i = (\mathbf{p}_i - \boldsymbol{\zeta}_i)$, which is governed, in view of (1) and (7), by the dynamics

$$\boldsymbol{\Sigma}_{\mathbf{r}}: \quad \ddot{\mathbf{r}}_i = \boldsymbol{\Gamma}_i, \quad \text{for} \ \ i \in \mathcal{N}. \tag{9}$$

Under the assumption that each agent can transmit the states of its auxiliary system (8), *i.e.,* $\boldsymbol{\xi}_i$ and $\dot{\boldsymbol{\xi}}_i$, the following result holds.

Theorem 1: Consider the multi-agent system (1) with a communication topology described by $\mathcal{G}_n$. Suppose that there exists a protocol $\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n)$ satisfying Assumption A1 and Assumption A2, with $\boldsymbol{f}_{\max} < \mathbf{u}_{\max}$. Let the input vectors in (1), (7) and (8) be given as

$$\mathbf{u}_i = \boldsymbol{f}_d - L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i) - L_i^d \boldsymbol{\chi}(\dot{\boldsymbol{\zeta}}_i), \tag{10}$$

$$\boldsymbol{\Gamma}_i = \mathbf{u}_i^v - k_i^p(\mathbf{r}_i - \boldsymbol{\xi}_i) - k_i^d(\mathbf{r}_i - \boldsymbol{\xi}_i - \boldsymbol{\psi}_i), \tag{11}$$

$$\mathbf{u}_i^v = \boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\boldsymbol{\xi}}, \mathcal{G}_n), \tag{12}$$

$$\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{r}_i - \boldsymbol{\xi}_i - \boldsymbol{\psi}_i), \tag{13}$$

where $\mathbf{r}_i = (\mathbf{p}_i - \boldsymbol{\zeta}_i)$, L_i^p, L_i^d, k_i^p, k_i^d and k_i^{ψ} are positive scalar gains, $\boldsymbol{\Sigma}_{\boldsymbol{\xi}}$ designates the system (8) with (12), and the function $\boldsymbol{\chi}$ is defined in (5). The vector $\boldsymbol{\psi}_i \in \mathbb{R}^m$ is the output of system (13) and can take arbitrary initial values. If the control gains are selected such that

$$\sigma_b(L_i^p + L_i^d) \leq \mathbf{u}_{\max} - \boldsymbol{f}_{\max}, \tag{14}$$

with σ_b being defined in P2, then the control input is guaranteed to be bounded as: $\|\mathbf{u}_i\|_\infty \leq \mathbf{u}_{\max}$, for $i \in \mathcal{N}$, and the multi-agent system (1) with the control input (10), (7)-(8), and (11)-(13), achieves second order consensus.

Proof: First, from the definition of the function $\boldsymbol{\chi}$ in (5), we can verify that the input law in (10) can be upper bounded independently from the states as:

$$\|\mathbf{u}_i\|_\infty \leq \boldsymbol{f}_{\max} + \sigma_b(L_i^p + L_i^d). \tag{15}$$

Therefore, if the control gains satisfy (14), the upper bound of the control input given in the theorem is obtained.

It can be seen that the auxiliary systems (8) with (12) describes the dynamics of a second order multi-agent system with available states, $\boldsymbol{\xi}_i$ and $\dot{\boldsymbol{\xi}}_i$. Since $\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\boldsymbol{\xi}}, \mathcal{G}_n)$ is designed using only these available states, the convergence of $\boldsymbol{\xi}_i$ and $\dot{\boldsymbol{\xi}}_i$ is completely independent from the trajectories of states of the agents in the team. Therefore, if there exists a protocol $\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n)$ satisfying Assumptions A1 and A2, we can conclude that the multi-agent system (8), with (12), achieves second order consensus, *i.e.,* $(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) \to 0$ and $(\dot{\boldsymbol{\xi}}_i - \dot{\boldsymbol{\xi}}_j) \to 0$, for all $i, j \in \mathcal{N}$. In addition, we know that $\bar{\boldsymbol{\Psi}}_i(\boldsymbol{\Sigma}_{\boldsymbol{\xi}}, \mathcal{G}_n)$ is globally bounded and converges asymptotically to zero once this result is obtained (*i.e.,* the multi-agent (8) with (12) achieves consensus).

Define the error vector $\mathbf{e}_i = \mathbf{r}_i - \boldsymbol{\xi}_i$, which is, in view of (8), (9) and (11), governed by:

$$\ddot{\mathbf{e}}_i = -k_i^p \mathbf{e}_i - k_i^d(\mathbf{e}_i - \boldsymbol{\psi}_i), \tag{16}$$

with $\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i)$. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{n}\left(\dot{\mathbf{e}}_i^T \dot{\mathbf{e}}_i + k_i^p \mathbf{e}_i^T \mathbf{e}_i + k_i^d(\mathbf{e}_i - \boldsymbol{\psi}_i)^T(\mathbf{e}_i - \boldsymbol{\psi}_i)\right).$$

The time-derivative of V evaluated along the dynamics (16) is obtained as: $\dot{V} = -\sum_{i=1}^n k_i^d k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i)^T(\mathbf{e}_i - \boldsymbol{\psi}_i)$, which is negative semi-definite. Hence, we conclude that $\mathbf{e}_i$, $\dot{\mathbf{e}}_i$, $\boldsymbol{\psi}_i$ and $\dot{\boldsymbol{\psi}}_i$ are bounded, which leads us to conclude that $\ddot{V}$ is bounded. Invoking Barbălat Lemma, we conclude that $\dot{\boldsymbol{\psi}}_i = k_i^{\psi}(\mathbf{e}_i - \boldsymbol{\psi}_i) \to 0$. Furthermore, we can verify that $(\ddot{\mathbf{e}}_i - \ddot{\boldsymbol{\psi}}_i)$ is bounded. Invoking barbălat Lemma, we conclude that $(\dot{\mathbf{e}}_i - \dot{\boldsymbol{\psi}}_i) \to 0$, and hence we know that $\dot{\mathbf{e}}_i \to 0$. In addition, we can show that $\dddot{\mathbf{e}}_i$ is bounded. Invoking Barbălat Lemma, we conclude that $\ddot{\mathbf{e}}_i \to 0$, which leads us using (16) to the result $\mathbf{e}_i \to 0$ for all $i \in \mathcal{N}$. Therefore, we conclude that $(\mathbf{r}_i - \boldsymbol{\xi}_i) \to 0$ and $(\dot{\mathbf{r}}_i - \dot{\boldsymbol{\xi}}_i) \to 0$, for $i \in \mathcal{N}$. This, and since we have shown that $(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) \to 0$ and $(\dot{\boldsymbol{\xi}}_i - \dot{\boldsymbol{\xi}}_j) \to 0$, for all $i, j \in \mathcal{N}$, we conclude that $(\mathbf{r}_i - \mathbf{r}_j) \to 0$ and $(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \to 0$, for all $i, j \in \mathcal{N}$.

To this point, the dynamics of the auxiliary systems (7) can be rewritten as

$$\ddot{\boldsymbol{\zeta}}_i = -L_i^p \boldsymbol{\chi}(\boldsymbol{\zeta}_i) - L_i^d \boldsymbol{\chi}(\dot{\boldsymbol{\zeta}}_i) + \boldsymbol{\varepsilon}_i, \tag{17}$$

with $\boldsymbol{\varepsilon}_i = \left(-\bar{\boldsymbol{\Psi}}_i(\boldsymbol{\Sigma}_{\mathbf{r}}, \mathcal{G}_n) + k_i^p \mathbf{e}_i + k_i^d(\mathbf{e}_i - \boldsymbol{\psi}_i)\right)$, which is guaranteed to be bounded and converges asymptotically to zero in view of the above results. Invoking Lemma 1, we conclude that $\boldsymbol{\zeta}_i$, $\dot{\boldsymbol{\zeta}}_i$ are globally bounded and converge asymptotically to zero. As a result, we conclude form the definition of the vector $\mathbf{r}_i$ that $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, and $(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j) \to 0$, for all $i, j \in \mathcal{N}$. ■

The main idea in the proposed approach is to associate to each agent in the team the two second order systems given in (7) and (8). The dynamic system (8) is introduced to generate a first reference trajectory for each agent, defined by $\boldsymbol{\xi}_i$. The input of this system is designed using the states of (8), without consideration of the input constraints, such that all agents agree on their first reference trajectories. The dynamic system (7) is implemented to generate the vector $\boldsymbol{\zeta}_i$, which can be considered as the error between each agent position and the time-varying vector $\mathbf{r}_i$, which is defined above for analysis purposes. Note that $\mathbf{r}_i$ can be seen as a second reference trajectory defined for each agent, and $\dot{\mathbf{r}}_i = (\dot{\mathbf{p}}_i - \dot{\zeta}_i)$ is not available for feedback. This way, the input $\boldsymbol{\Gamma}_i$ in (9) is designed, without velocity measurements as in (11), such that the error between the two reference trajectories converges to zero. Once this is achieved, and all agents agree on their first reference trajectories, the states of the auxiliary systems (7), with (10)-(13), are driven to zero asymptotically, in view of the result of Lemma 1. This indicates that each agent tracks its second reference trajectory, guiding hence all agents to achieve consensus on their states.

It should be noted that the input of each agent, $\mathbf{u}_i$, is guaranteed to be *a priori* bounded, as in (15), independently from the number of neighbors of each agent. Therefore, an upper bound of the input of each agent can be set by an appropriate choice of the control gains, according to

(14), without an *a priori* knowledge on the communication topology between agents. This introduces flexibility in the tuning of the controller gains especially in the case where $\mathbf{u}_{\max}$ is small and the number of neighbors of each agent may be large. In addition, to use the result of Theorem 1, one only needs to design a consensus algorithm for (1) in ideal situations, *i.e.,* in the full state information case without input saturations, such that Assumptions A1 and A2 are satisfied. If this algorithm exists, it can be used as input of the auxiliary system (8) using the states $\boldsymbol{\xi}_i$ and $\dot{\boldsymbol{\xi}}_i$, for $i \in \mathcal{N}$. Therefore, the proposed approach can be used to extend existent consensus algorithms developed for the multi-agent system (1) in ideal situations to account for input saturations without velocity measurements. This will be illustrated in the next section by some application examples.

IV. Application examples

In this section, we propose solutions to three problems related to second order consensus in the partial state information case with input saturations.

A. Case I: Consensus with communication delays

The first example considered in this section involves the design of consensus algorithms for the multi-agent system (1) in the presence of time-varying communication delays. We assume that the information exchange between agents is fixed and undirected and is represented by $\mathcal{G}_n$. Also, we assume that the communication between the i^{th} and j^{th} agents is delayed by $\tau_{ij}(t)$, with $\tau_{ij}(t)$ is not necessarily equal to $\tau_{ji}(t)$.

First, we consider the case where no input constraints are imposed to the agents and the velocity states are available for feedback, and propose the following result.

Proposition 1: Consider the multi-agent system (4) with

$$\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n) = -k_i^v \dot{\mathbf{p}}_i - \textstyle\sum_{j=1}^n k_{ij} \bar{\mathbf{p}}_{ij},$$

where $\bar{\mathbf{p}}_{ij} = (\mathbf{p}_i - \mathbf{p}_j(t - \tau_{ij}(t)))$, k_i^v is a strictly positive scalar gain, k_{ij} is the $(i,j)^{th}$ entry of the adjacency matrix of the undirected graph $\mathcal{G}_n$. Assume that the time-varying communication delays are bounded such that $\tau_{ij}(t) \leq \tau$, for $(i,j) \in \mathcal{E}$, and the control gains satisfy $k_i^z := k_i^v - \frac{1}{2}\sum_{j=1}^n k_{ij}\left(\epsilon + \frac{\tau^2}{\epsilon}\right) > 0$, for some strictly positive ϵ. If the communication graph $\mathcal{G}_n$ is connected, then $\dot{\mathbf{p}}_i$ and $(\mathbf{p}_i - \mathbf{p}_j)$ are bounded and $\dot{\mathbf{p}}_i \to 0$ for $i \in \mathcal{N}$, $(\mathbf{p}_i - \mathbf{p}_j) \to 0$ for all $i, j \in \mathcal{N}$.

Sketch of proof: The result of the proposition can be shown using the Lyapunov-Krasovskii functional

$$V = \frac{1}{2}\sum_{i=1}^n \dot{\mathbf{p}}_i^\top \dot{\mathbf{p}}_i + \frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n k_{ij}(\mathbf{p}_i - \mathbf{p}_j)^\top(\mathbf{p}_i - \mathbf{p}_j) + \sum_{i=1}^n\sum_{j=1}^n \frac{k_{ij}\tau}{2\epsilon}\int_{-\tau}^0\int_{t+s}^t \dot{\mathbf{p}}_i^\top(\varrho)\dot{\mathbf{p}}_i(\varrho)d\varrho ds, \tag{18}$$

with $\tau_{ij}(t) \leq \tau$ and $\epsilon > 0$, leading to the negative semi-definite time-derivative that can be upper bounded as: $\dot{V} \leq -\sum_{i=1}^n k_i^z \dot{\mathbf{p}}_i^\top \dot{\mathbf{p}}_i$, with k_i^z given above. This allows us to conclude that $\dot{\mathbf{p}}_i$ and $(\mathbf{p}_i - \mathbf{p}_j)$ are bounded. Then, with the help of Berabălat Lemma, we show that $\dot{\mathbf{p}}_i \to 0$ and $\ddot{\mathbf{p}}_i \to 0$, which leads us to conclude that $(\mathbf{p}_i - \mathbf{p}_j) \to 0$ for $i, j \in \mathcal{N}$, if the communication graph is connected. A detailed proof of the proposition can be obtained following the above lines and the proof of Theorem 1 in [18]. □

Proposition 1 provides a consensus algorithm that satisfies Assumptions A1 and A2, with $\boldsymbol{f}_d = 0$, under an undirected communication topology with time-varying communication delays. Therefore, under the same conditions reported in Proposition 1, the velocity-free consensus algorithm presented in Theorem 1 with

$$\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\boldsymbol{\xi}}, \mathcal{G}_n) = -k_i^v \dot{\boldsymbol{\xi}}_i - k_i^v \textstyle\sum_{j=1}^n k_{ij} \bar{\boldsymbol{\xi}}_{ij}, \tag{19}$$

with $\bar{\boldsymbol{\xi}}_{ij} = (\boldsymbol{\xi}_i - \boldsymbol{\xi}_j(t - \tau_{ij}(t)))$, guarantees that: $\dot{\mathbf{p}}_i \to 0$ for $i \in \mathcal{N}$, $(\mathbf{p}_i - \mathbf{p}_j) \to 0$, for $i, j \in \mathcal{N}$, under an undirected interconnection topology and with time-varying delays. Also, the control input for each agent is guaranteed to be *a priori* bounded as in (15) with $\boldsymbol{f}_{\max} = 0$.

B. Case II: Free-consensus in directed networks

One of the fundamental algorithms developed for multi-agent system (1) to solve the free-consensus problem can be written as in (3) with

$$\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n) = -\sum_{j=1}^n k_{ij}\left(\alpha(\mathbf{p}_i - \mathbf{p}_j) + \beta(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j)\right), \tag{20}$$

where α and β are positive scalar gains, k_{ij} is the $(i,j)^{th}$ entry of the adjacency matrix of the directed graph $\mathcal{G}_n$ describing the directed communication topology between agents. The above consensus algorithm has been considered in [9], where it has been shown that the multi-agent system (4) with the above algorithm achieves second order consensus if and only if the communication graph contains a directed spanning tree and: $\frac{\beta^2}{\alpha} > \max_{2\leq i\leq n} \frac{\Im^2(\mu_i)}{\Re(\mu_i)[\Re^2(\mu_i)+\Im^2(\mu_i)]}$,where $\Re(\cdot)$ and $\Im(\cdot)$ denote respectively the real and imaginary parts of a number, μ_i, for $i = 2, ..., n$, are the nonzero eigenvalues of the Laplacian matrix L defined as $L = [l_{ij}] \in \mathbb{R}^{n\times n}$, with $l_{ij} = -k_{ij}$, for $i \neq j$, and $l_{ii} = -\sum_{j=1, j\neq i}^n l_{ij}$. In addition, $(\mathbf{p}_i - \mathbf{p}_j)$ and $(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j)$ are globally bounded. Furthermore, if second order consensus is reached, then $\|\dot{\mathbf{p}}_i(t) - \sum_{j=1}^n q_j \dot{\mathbf{p}}_j(0)\| \to 0$ and $\|\mathbf{p}_i(t) - \sum_{j=1}^n q_j \mathbf{p}_j(0) - \sum_{j=1}^n q_j \dot{\mathbf{p}}_j(0)t\| \to 0$, with $\mathbf{q} = (q_1, ..., q_n)^\top$ is the non-negative left eigenvector of L associated with eigenvalue 0 satisfying $\mathbf{q}^\top \mathbf{1}_n = 1$, with $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of all ones elements (See Theorem 1 and Lemma 2 in [9] for more details).

Hence, the above consensus algorithm, which is developed for (1) in ideal situations, satisfies Assumptions A1 and A2, with $\boldsymbol{f}_d = 0$, under the conditions described above. Therefore, we conclude that the multi-agent system (1) when the consensus algorithm given in Theorem 1 is implemented with

$$\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\boldsymbol{\xi}}, \mathcal{G}_n) = -\sum_{j=1}^n k_{ij}\left(\alpha(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) + \beta(\dot{\boldsymbol{\xi}}_i - \dot{\boldsymbol{\xi}}_j)\right), \tag{21}$$

achieves second order consensus without velocity measurements if the directed graph contains a directed spanning tree and the gains α and β satisfy the above mentioned condition. Furthermore, the upper bound of the control input of each agent can be determined *a priori* and is given in (15) with $\boldsymbol{f}_{\max}$ set to zero.

Moreover, in view of the dynamics of the auxiliary systems (8) with (21), we conclude that if consensus is reached, then $\|\dot{\boldsymbol{\xi}}_i(t) - \sum_{j=1}^n q_j \dot{\boldsymbol{\xi}}_j(0)\| \to 0$ and $\|\boldsymbol{\xi}_i(t) - \sum_{j=1}^n q_j \boldsymbol{\xi}_j(0) - \sum_{j=1}^n q_j \dot{\boldsymbol{\xi}}_j(0)t\| \to 0$. This specifies the consensus value of all agents, since the states of each agent will converge to the states of its corresponding auxiliary system (8). Note that the initial states of multi-agent system (1) do not affect the final consensus value, which presents an important advantage since the velocities of agents are not measurable. As a result, the consensus value of the multi-agent system (1) can be set by setting the initial states of the dynamic systems (8).

C. Case III: Consensus with a group reference velocity

The control objective in this example is to design a consensus algorithm such that multi-agent system (1) achieves second order consensus and each member of the team tracks a desired velocity given by $\dot{\mathbf{p}}_d(t)$, which satisfies: $\|\ddot{\mathbf{p}}_d(t)\|_\infty \leq \boldsymbol{a}_{\max} < \mathbf{u}_{\max}$. In the full state information case, a possible protocol that solves this problem in the case of directed communication topology is given as [11]

$$\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\mathbf{p}}, \mathcal{G}_n) = \ddot{\mathbf{p}}_d - k_i^v(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_d) - \sum_{j=1}^n k_{ij}(\mathbf{p}_i - \mathbf{p}_j), \quad (22)$$

where k_i^v is a positive scalar gain and k_{ij} is the $(i,j)^{th}$ entry of the adjacency matrix of the directed graph $\mathcal{G}_n$. It has been shown in [11] that $(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_d)$ and $(\mathbf{p}_i - \mathbf{p}_j)$ are globally bounded and converge asymptotically to zero if the directed communication graph has a directed spanning tree and $k_i^v > \bar{k}$, with $\bar{k}$ depends on the eigenvalues of the Laplacian matrix of $\mathcal{G}_n$ (See Theorem 5.1 in [11] for more details).

Therefore, Assumptions A1 and A2 are satisfied, with $\boldsymbol{f}_d = \ddot{\mathbf{p}}_d$, and the velocity-free consensus algorithm proposed in Theorem 1 with

$$\boldsymbol{\Psi}_i(\boldsymbol{\Sigma}_{\boldsymbol{\xi}}, \mathcal{G}_n) = \boldsymbol{f}_d - k_i^v(\dot{\boldsymbol{\xi}}_i - \dot{\mathbf{p}}_d) - \sum_{j=1}^n k_{ij}(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j), \quad (23)$$

and $\boldsymbol{f}_d = \ddot{\mathbf{p}}_d$, achieves second order consensus under the condition that the digraph $\mathcal{G}_n$ has a directed spanning tree and $k_i^v > \bar{k}$. In addition, the control input for each agent is guaranteed to be a priori bounded as in (15) with $\boldsymbol{f}_{\max} := \boldsymbol{a}_{\max}$.

Remark 1: The proposed consensus algorithm in this example can be considered as the extension of our work in [17] to the case of directed interconnection between agents.

V. SIMULATION RESULTS

We provide in the following simulation results to demonstrate the effectiveness of the proposed consensus algorithms in the two first cases discussed in Section IV. For this purpose, we consider a group of four agents modeled as in (1), with $m = 1$, and with initial conditions: $\mathbf{P}(0) = (1, 1.5, 2, 3)^\top$ and $\dot{\mathbf{P}}(0) = (0.1, 0.02, -0.08, 0.05)^\top$, where $\mathbf{P}(t)$ and $\dot{\mathbf{P}}(t)$ are the vectors containing respectively the positions and velocities of agents, $\mathbf{p}_i(t)$ and $\dot{\mathbf{p}}_i(t)$ for $i \in \mathcal{N} := \{1, 2, 3, 4\}$. We assume that all agents are constrained such that $\mathbf{u}_{\max} = 2$, and the information flow between agents is represented by one of the communication graphs, given in Fig.1, as will be specified in each case. The initial conditions of the auxiliary systems (7), the virtual systems (8) and the first-order system (13) are selected as: $\boldsymbol{\zeta}_i(0) = \dot{\boldsymbol{\zeta}}_i(0) = \dot{\boldsymbol{\xi}}_i(0) = \boldsymbol{\psi}_i(0) = 0$ and $\boldsymbol{\xi}_i(0) = \mathbf{p}_i(0)$, for $i \in \mathcal{N}$. In addition, we consider the function $\boldsymbol{\chi}$ defined in (5) with $\sigma = \tanh$ and $\sigma_b = 1$.

Fig. 1: Interaction graphs

Case I: we consider the consensus problem with time-varying communication delays under fixed and undirected communication topologies. We implement the consensus algorithm proposed in Theorem 1 with (19) and $\boldsymbol{f}_d = 0$, where the bidirectional information exchange between agents is described by the undirected graph $\tilde{\mathcal{G}}_4$ given in Fig.1. The control gains are selected as: $k_{ij} = 1$, for $(i,j) \in \mathcal{E}$, $k_i^v = 2$, and $(k_i^p, k_i^d, k_i^\psi, L_i^p, L_i^d) = (1, 15, 5, 0.5, 1.5)$, for $i \in \mathcal{N}$. The time-varying communication delays are assumed to satisfy: $\tau_{ij}(t) = \tilde{\tau}_{ij}|\sin(0.5t)|$ sec, with $\tilde{\tau}_{12} = \tilde{\tau}_{13} = \tilde{\tau}_{14} = 0.1$, $\tilde{\tau}_{21} = \tilde{\tau}_{23} = 0.15$, $\tilde{\tau}_{31} = \tilde{\tau}_{32} = 0.2$, $\tilde{\tau}_{41} = 0.3$, for $i \in \mathcal{N}$. Note that the choice of the gains satisfy the condition in Proposition 1 with $\tau = 0.3$ sec and $\epsilon = 1$. The obtained results are shown in Fig.2 where we can see that second order consensus is achieved without velocity measurements in the presence of time-varying communication delays, and the control input for each agent satisfies $|\mathbf{u}_i| \leq \mathbf{u}_{\max}$.

Moreover, it should be mentioned that the proposed approach offers a flexibility in the tuning of the controller gains. In fact, the weights in the adjacency matrix and k_i^v can be selected independently from the input constraints such that the all auxiliary systems (8) with (12) achieve consensus along with communication delays, *i.e.,* all agents reach an agreement of their first reference states $\boldsymbol{\xi}_i$ and $\dot{\boldsymbol{\xi}}_i$. The gains (k_i^p, k_i^d, k_i^ψ) are also selected independently such that the signals $(\mathbf{r}_i - \boldsymbol{\xi}_i)$ and $(\dot{\mathbf{r}}_i - \dot{\boldsymbol{\xi}}_i)$ converge to zero as fast as possible without using the agents velocities. Finally, the gains (L_i^p, L_i^d) are selected according to (14) to guarantee the upper bounds of the control inputs. Also, note that the transient response of agents presents no oscillations, which indicates that sufficient dumping is introduced to the system via the dynamics of the two auxiliary systems with the missing velocity states of agents.

Case II: We implement the consensus algorithm proposed in Theorem 1 with (21) to solve the free-consensus problem in directed networks discussed in Section IV-B. To this end,

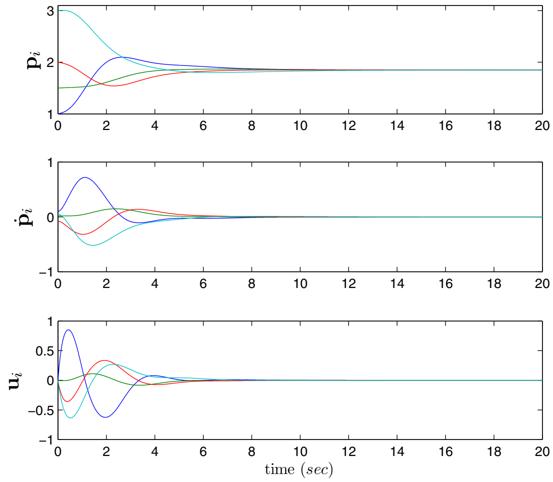

Fig. 2: Case I, Theorem 1 with (19).

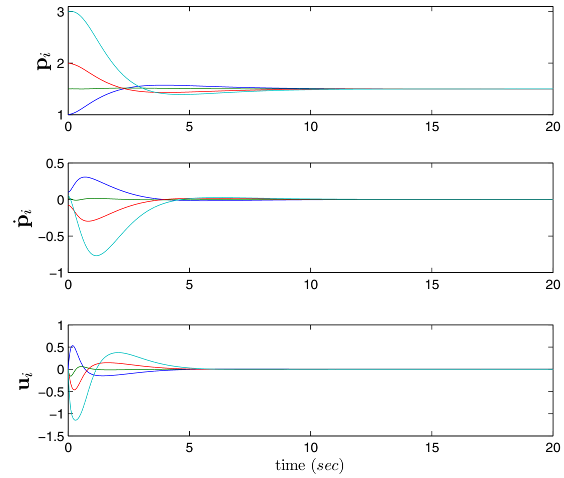

Fig. 3: Case II, Theorem 1 with (21).

we consider that the communication topology between the four agents is represented by $\mathcal{G}_4$ shown in Fig.1. The control gains are selected as: $k_{ij} = 5$, for $(i,j) \in \mathcal{E}$, $(\alpha, \beta) = (1,1)$, and $(k_i^p, k_i^d, k_i^\psi, L_i^p, L_i^d) = (1, 15, 5, 0.5, 1.5)$, for $i \in \mathcal{N}$. Note that the eigenvalues of the Laplacian matrix for the graph $\mathcal{G}_4$ in view of the weights k_{ij} are: $0,\ 5,\ 7.5 \pm 4.3301\varsigma$, with $\varsigma^2 = -1$, and therefore, the conditions on the gains β and α are satisfied. Fig.3 shows the obtained results in this case, where we can see that second order consensus is achieved without velocity measurements, and the control input for each agent satisfies $|\mathbf{u}_i| \le \mathbf{u}_{\max}$. Note that the final position of all agents is constant and equal to 1.5. This corresponds to the expected consensus value in this case in view of the initial conditions $\boldsymbol{\xi}_i(0)$ and $\dot{\boldsymbol{\xi}}_i(0)$, and knowing that the left eigenvector of the Laplacian matrix of the directed graph $\mathcal{G}_4$ in Fig.1 is obtained as: $\mathbf{q} = \frac{1}{3}(1,1,1,0)^\top$.

VI. Conclusion

We considered the consensus problem for double integrator dynamics in the partial state information case with input saturations. Based on dynamic extensions, we proposed a new approach that effectively reduces the second order consensus algorithms design in this case to the design of consensus algorithms in ideal situations, *i.e.,* in the full state information case with no input constraints. With the existence of many consensus algorithms designed in ideal situations, the proposed approach can be applied in a straightforward manner to solve open problems related to the second order consensus in the partial state feedback case and with input saturations. This has been illustrated by three examples and simulation results have been provided to validate our theoretical results.

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