2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

An output feedback distributed predictive control algorithm

Marcello Farina and Riccardo Scattolini

Abstract— **This paper presents an output feedback Distributed Predictive Control (DPC) algorithm for interconnected systems based on neighbor-to-neighbor communication. The algorithm, based on the joint use of a distributed feedback control law and decoupled Luenberger estimators, does not require an iterative exchange of information among neighbors and can be also used for control of independent systems with coupled constraints. Convergence results are proven and a simulation example is reported to illustrate the performance of the algorithm.**

I. INTRODUCTION

The ever increasing complexity of industrial systems and infrastructures, as well as safety and reliability considerations, make the development of new distributed control algorithms an active field of research. Among the many approaches proposed in the literature, those based on Model Predictive Control (MPC) are particularly promising [1], [6], [11], [12]. A new state feedback Distributed Predictive Control (DPC) algorithm has recently been proposed in [3]: it is based on a non iterative scheme with neighbor-to-neighbor communication among the subsystems, and is inspired to the robust state feedback MPC approach first introduced in [9] and later extended to the output feedback case in [8].

The main rationale behind DPC is to transmit among the neighbors the future reference trajectories and to interpret the difference between these trajectories and the true ones as disturbances to be rejected by a proper robust MPC method. Therefore in DPC it is not necessary for each subsystem to know the dynamical models governing the trajectories of the other subsystems and the transmission of information is limited; moreover joint constraints between the subsystems can be included.

In this paper, the state feedback DPC algorithm presented in [3] is extended to the output feedback case by the use of Luenberger observers for the estimation of the subsystems' states. It is proven that, under standard assumptions in MPC, the subsystems' state trajectories starting from given sets in the state space converge to the origin. This result is achieved by considering the state estimation error as a further disturbance to be rejected by the control system. Notably, the same considerations developed in this paper can be used to show the robustness of the proposed approach also with respect to exogenous unknown (but bounded) disturbances.

The paper is organized as follows. In Section II the partitioned system is introduced, while the output feedback DPC algorithm is defined in Section III. The main convergence results are presented in Section IV, while Section V deals with the decentralized Luenberger observer design problem. Section VI illustrates a simulation example, and some conclusions are drawn in Section VII. All the proofs are postponed to the Appendix.

This research has been supported by the European 7th framework STREP project "Hierarchical and distributed model predictive control (HD-MPC)", contract number INFSO-ICT-223854.

M. Farina and R. Scattolini are with the Dipartimento di Elettronica e Informazione, Politecnico di Milano, Milan, Italy {farina,riccardo.scattolini}@elet.polimi.it

Notation. We say that a matrix is Schur if all its eigenvalues lie in the interior of the unit circle. We use the shorthand $\mathbf{v} = (v_1, \dots, v_s)$ to denote a column vector with s (not necessarily scalar) components $v_1, \dots, v_s$. The symbol $\oplus$ denotes the Minkowski sum, namely $C = A \oplus B$ if and only if $C = \{c : c = a + b, \text{ for all } a \in A, b \in B\}$. We also denote $\bigoplus_{i=1}^M A_i = A_1 \oplus \cdots \oplus A_M$. For a discrete-time signal s_t and $a, b \in \mathbb{N}$, $a \le b$, we denote $(s_a, s_{a+1}, \dots, s_b)$ with $s_{[a:b]}$.

II. PARTITIONED SYSTEMS

Consider a large-scale system model

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \\ \mathbf{y}_t &= \mathbf{C}\mathbf{x}_t \end{aligned} \tag{1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is the state vector, and $\mathbf{u}_t \in \mathbb{R}^m$ and $\mathbf{y}_t \in \mathbb{R}^p$ are the input variable and the output variable, respectively. Let the system (1) be partitioned in M low order interconnected non overlapping subsystems, where a generic submodel has $x_t^{[i]} \in \mathbb{R}^{n_i}$ as state vector, i.e., $\mathbf{x}_t = (x_t^{[1]}, \dots, x_t^{[M]})$ and $\sum_{i=1}^M n_i = n$. According to this decomposition, the state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}, \dots, A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are diagonal blocks of $\mathbf{A}$, whereas the non-diagonal blocks of $\mathbf{A}$ (i.e., A_{ij}, with $i \neq j$) define the dynamic coupling between subsystems. Namely, we say that subsystem j is a *dynamic neighbor* of subsystem i if $A_{ij} \neq 0$. The set of dynamic neighbors of i is denoted $\mathcal{N}_i$ ($i \notin \mathcal{N}_i$).

Furthermore, we assume that the input $\mathbf{u}_t$ and the output $\mathbf{y}_t$ can be partitioned into M input and output vectors $u_t^{[i]} \in \mathbb{R}^{m_i}$ and $y_t^{[i]} \in \mathbb{R}^{p_i}$, respectively, $i = 1, \dots, M$. We assume that $u_t^{[i]}$ directly affects only the state of the i-th subsystem $x_t^{[i]}$ and $y_t^{[i]}$ only depends on $x_t^{[i]}$, for all $i = 1, \dots, M$. This implies that $\mathbf{B} = \text{diag}(B_1, \dots, B_M)$ and $\mathbf{C} = \text{diag}(C_1, \dots, C_M)$, where $B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{p_i \times n_i}$, $i = 1, \dots, M$. It results that the i-th subprocess obeys to the linear dynamics

$$\begin{aligned} x_{t+1}^{[i]} &= A_{ii} x_t^{[i]} + B_i u_t^{[i]} + \textstyle\sum_{j \in \mathcal{N}_i} A_{ij} x_t^{[j]} \\ y_t^{[i]} &= C_i x_t^{[i]} \end{aligned} \tag{2}$$

The local states and inputs are possibly constrained, i.e., $x_t^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u_t^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, where the sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are convex neighborhoods of the origin. Furthermore we define $\mathbb{X} = \prod_{i=1}^M \mathbb{X}_i \subseteq \mathbb{R}^n$ and $\mathbb{U} = \prod_{i=1}^M \mathbb{U}_i$, which are convex by convexity of $\mathbb{X}_i$ and $\mathbb{U}_i$, respectively, for $i = 1, \dots, M$. We also introduce the collective state constraints, involving more than one subsystem's state

$$H_s(\mathbf{x}_t) \le 0 \tag{3}$$

where $s = 1,\dots,n_c$. We say that H_s is a constraint on subsystem i if $x^{[i]}$ is an argument of H_s. We denote by $\mathscr{C}_i = \{s \in \{1,\dots,n_c\}$: H_s is a constraint on $i\}$ the set of constraints on subsystem i. We say that subsystem j is a *constraint neighbor* of subsystem i if there exists $\bar{s} \in \mathscr{C}_i$ such that $x^{[j]}$ is an argument of $H_{\bar{s}}$, and we let $\mathscr{H}_i$ denote the set of the constraint neighbors of subsystem i. Finally we define, for all $s \in \mathscr{C}_i$, a function $h_s^{[i]}(x^{[i]},\mathbf{x}) = H_s(\mathbf{x})$, where $x^{[i]}$, the i-th vector component of $\mathbf{x}$, is not an argument of $H_s(a,\cdot)$. Note that, if subsystems i and j are constraint neighbors, there exists $\bar{s} \in \mathscr{C}_i \cap \mathscr{C}_j$ such that $h_{\bar{s}}^{[i]}$ and $h_{\bar{s}}^{[j]}$ are equivalent, in the sense that they represent the same constraint $H_{\bar{s}}$.

III. The Output Feedback DPC algorithm

Our aim is to design, for each subsystem i, an algorithm for computing a control input $u_t^{[i]}$ based on the output $y_t^{[i]}$ and some information which is transmitted by its neighbors $\mathscr{N}_i \cup \mathscr{H}_i$, which guarantees closed loop asymptotic convergence to the origin of the state of the large scale system (1), the minimization of a given local cost function and constraint satisfaction. Given (2), for a given subsystem i we define a local Luenberger observer, which provides an estimate $\bar{x}_t^{[i]}$ of the state $x_t^{[i]}$, based on the local measurement $y_t^{[i]}$, and the state estimates provided by i-th dynamic neighbors, i.e., $\bar{x}_t^{[j]}$, $j \in \mathscr{N}_j$. Namely

$$\bar{x}_{t+1}^{[i]} = A_{ii}\bar{x}_t^{[i]} + B_i u_t^{[i]} + \sum_{j\in\mathscr{N}_i} A_{ij}\bar{x}_t^{[j]} - L_i(y_t^{[i]} - C_i\bar{x}_t^{[i]}) \tag{4}$$

Assuming that the estimator (4) enjoys the stability properties specified in the following, given the system state initial conditions $\mathbf{x}_0$ and the observer initial conditions $\bar{\mathbf{x}}_0 = (\bar{x}_0^{[1]},\dots,\bar{x}_0^{[M]})$, we require that there exist, for all $i = 1,\dots,M$, sets $\Sigma_i \subseteq \mathbb{R}^{n_i}$ such that, for all $t \ge 0$, $\boldsymbol{\sigma}_t = \mathbf{x}_t - \bar{\mathbf{x}}_t \in \boldsymbol{\Sigma} = \prod_{i=1}^M \Sigma_i$, i.e., $\sigma_t^{[i]} = x_t^{[i]} - \bar{x}_t^{[i]} \in \Sigma_i$ for all $i = 1,\dots,M$.

Furthermore we set, for each subsystem, a reference trajectory $\tilde{x}_t^{[i]}$ which is transmitted to the subsystems which have i as neighbor. Through suitable constraints, we guarantee that $\bar{x}_t^{[i]}$ lies in a specified time-invariant neighborhood of $\tilde{x}_t^{[i]}$ i.e, $\bar{x}_t^{[i]} \in \tilde{x}_t^{[i]} \oplus \mathscr{E}_i$, where $0 \in \mathscr{E}_i$. This, in turn, implies that $x_t^{[i]}$ is also guaranteed to lie in a given neighborhood of $\tilde{x}_t^{[i]}$, i.e, $x_t^{[i]} \in \tilde{x}_t^{[i]} \oplus \mathscr{E}_i \oplus \Sigma_i$ for all $i = 1,\dots,M$.

Letting $w_t^{[i]} = \sum_{j\in\mathscr{N}_i} A_{ij}(\bar{x}_t^{[j]} - \tilde{x}_t^{[j]}) - L_i(y_t^{[i]} - C_i\bar{x}_t^{[i]})$, the i-th observer equation (4) can be written as follows

$$\bar{x}_{t+1}^{[i]} \quad = \quad A_{ii}\bar{x}_t^{[i]} + B_i u_t^{[i]} + \textstyle\sum_{j\in\mathscr{N}_i} A_{ij}\tilde{x}_t^{[j]} + w_t^{[i]} \tag{5}$$

where the term $w_t^{[i]} \in \mathbb{W}_i = \bigoplus_{j\in\mathscr{N}_i} A_{ij}\mathscr{E}_j \oplus (-L_iC_i)\Sigma_i$ represents a bounded disturbance affecting equation (5) and $\sum_{j\in\mathscr{N}_i} A_{ij}\tilde{x}_t^{[j]}$ is as a known input. Provided that, for all $i = 1,\dots,M$, the constraint $\bar{x}_t^{[i]} - \tilde{x}_t^{[i]} \in \mathscr{E}_i$ is satisfied for all $t \ge 0$, we cast the problem of designing an output-feedback distributed controller for (2) as the problem of designing a robust state-feedback control law for (5), for all $i = 1,\dots,M$. For the statement of the local MPC sub-problems (i-DPC problems) we rely on a robust MPC algorithm presented in [8]. The i-th subsystem nominal model associated to (5) is

$$\hat{x}_{t+1}^{[i]} \quad = \quad A_{ii}\hat{x}_t^{[i]} + B_i\hat{u}_t^{[i]} + \textstyle\sum_{j\in\mathscr{N}_i} A_{ij}\tilde{x}_t^{[j]} \tag{6}$$

The control law, both for the i-th subsystem (2) and for the equation (5) is assigned, for all $t \ge 0$, according to

$$u_t^{[i]} = \hat{u}_t^{[i]} + K_i^{aux}(\bar{x}_t^{[i]} - \hat{x}_t^{[i]}) \tag{7}$$

where K_i^{aux} is a suitable control gain. Letting $z_t^{[i]} = \bar{x}_t^{[i]} - \hat{x}_t^{[i]}$, from (5) and (7) we obtain

$$z_{t+1}^{[i]} \quad = \quad (A_{ii} + B_iK_i^{aux})z_t^{[i]} + w_t^{[i]} \tag{8}$$

where $w_t^{[i]} \in \mathbb{W}_i$. Since $\mathbb{W}_i$ is bounded, if $(A_{ii} + B_iK_i^{aux})$ is Schur, then there exists a robust positively invariant (RPI) set Z_i for (8) such that, for all $z_t^{[i]} \in Z_i$, then $z_{t+1}^{[i]} \in Z_i$. In view of (8), if $u_k^{[i]}$ is computed as in (7) for all $k \ge t$, then

$$\bar{x}_t^{[i]} - \hat{x}_t^{[i]} \in Z_i \tag{9}$$

implies that $\bar{x}_k^{[i]} - \hat{x}_k^{[i]} \in Z_i$ for all $k \ge t$.

Now write $\bar{x}_t^{[i]} - \tilde{x}_t^{[i]} = (\bar{x}_t^{[i]} - \hat{x}_t^{[i]}) + (\hat{x}_t^{[i]} - \tilde{x}_t^{[i]})$ and define the set E_i for all $i = 1,\dots,M$ as a set containing the origin and satisfying $E_i \oplus Z_i \subseteq \mathscr{E}_i$. Since, in view of (9), $\bar{x}_k^{[i]} - \hat{x}_k^{[i]} \in Z_i$ for all $k \ge t$, if we also satisfy the constraint

$$\hat{x}_k^{[i]} - \tilde{x}_k^{[i]} \in E_i \tag{10}$$

for all $k \ge t$, then $\bar{x}_k^{[i]} - \tilde{x}_k^{[i]} \in \mathscr{E}_i$ for all $k \ge t$ as required.

We are now in the position to state the local minimization problem for all subsystems at instant t. Given the future reference trajectories of i and its neighbors $\tilde{x}_k^{[j]}$, $k = t,\dots,t+N-1$, $j \in \mathscr{N}_i \cup \mathscr{H}_i \cup \{i\}$, the i-DPC problem consists in

$$\min_{\hat{x}_t^{[i]},\hat{u}_{[t:t+N-1]}^{[i]}} V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) \tag{11}$$

subject to the dynamic and static constraints (6), (4), (7), (9), (10), to the local state and input constraints

$$\hat{x}_k^{[i]} \in \hat{\mathbb{X}}_i \tag{12}$$

$$\hat{u}_k^{[i]} \in \hat{\mathbb{U}}_i \tag{13}$$

where $\hat{\mathbb{X}}_i \oplus Z_i \oplus \Sigma_i \subseteq \mathbb{X}_i$ and $\hat{\mathbb{U}}_i \oplus K_i^{aux}Z_i \subseteq \mathbb{U}_i$ and to the regional state constraints

$$\hat{h}_s^{[i]}(\hat{x}_k^{[i]}, \tilde{\mathbf{x}}_k) \le 0 \tag{14}$$

for $k = t,\dots,t+N-1$, for all $s \in \mathscr{C}_i$, where the function $\hat{h}_s^{[i]}$ is defined in such a way that $\hat{h}_s^{[i]}(\hat{x}_k^{[i]}, \tilde{\mathbf{x}}_k) \le 0$ guarantees that $h_s^{[i]}(x_k^{[i]}, \mathbf{x}_k^*) \le 0$ for all $x_k^{[i]} \in \hat{x}_k^{[i]} \oplus Z_i \oplus \Sigma_i$ and $\mathbf{x}_k^* \in \tilde{\mathbf{x}}_k \oplus \prod_{i=1}^M \mathscr{E}_i \oplus \boldsymbol{\Sigma}$. Furthermore, the nominal state trajectory must satisfy the following terminal constraint

$$\hat{x}_{t+N}^{[i]} \in \hat{\mathbb{X}}_i^F \tag{15}$$

where $\hat{\mathbb{X}}_i^F$ is the i-th nominal subsystem terminal set, whose properties will be specified in the following. The cost V_i^N is

$$V_i^N = \sum_{k=t}^{t+N-1} (\|\hat{x}_k^{[i]}\|_{Q_i}^2 + \|\hat{u}_k^{[i]}\|_{R_i}^2) + \|\hat{x}_{t+N}^{[i]}\|_{P_i}^2 \qquad (16)$$

where Q_i, R_i, and P_i are suitably-defined symmetric and positive definite matrices.

In the stated problem, minimization is performed with respect both to the nominal system state $\hat{x}_t^{[i]}$ and to the nominal input trajectory $\hat{u}_{[t:t+N-1]}^{[i]}$, [9]. Letting the pair $\hat{x}_{t/t}^{[i]}, \hat{u}_{[t:t+N-1]/t}^{[i]}$ be the solution to the i-DPC problem (11) at time t, we set the input to the nominal system (6), at time t, as $\hat{u}_{t/t}^{[i]}$. According to (7), the input to the real system (2), at instant t, is

$$u_t^{[i]} = \hat{u}_{t/t}^{[i]} + K_i^{aux}(\bar{x}_t^{[i]} - \hat{x}_{t/t}^{[i]}) \qquad (17)$$

Furthermore, let us define as $\hat{x}_{k/t}^{[i]}$ the trajectory stemming from $\hat{x}_{t/t}^{[i]}$ and $\hat{u}_{[t:t+N-1]/t}^{[i]}$, in view of equation (6). The value of the reference state variable $\tilde{x}_{t+N}^{[i]}$ is set to

$$\tilde{x}_{t+N}^{[i]} = \hat{x}_{t+N/t}^{[i]} \qquad (18)$$

We stress that we do not define, at each instant t, a new reference trajectory $\tilde{x}_k^{[i]}$, $k = t+1,\dots,t+N$, but we append the value $\tilde{x}_{t+N}^{[i]}$ to the reference trajectory which has been already defined for $k \leq t+N-1$.

At instant t, the information that must be transmitted between neighboring agents consists in the currently computed values of $\bar{x}_t^{[i]}$ and $\tilde{x}_{t+N}^{[i]}$.

IV. Convergence results

The following definitions and assumptions are needed to state the main result of the paper. The sets of admissible initial conditions $\mathbf{x}_0$, $\bar{\mathbf{x}}_0$, and $\tilde{x}_{[0:N-1]}^{[j]}$, for all $j = 1\dots,M$ are defined.

Definition 1: Letting $\mathbf{x} = (x^{[1]},\dots,x^{[M]})$, we denote the feasibility region $\mathbb{X}^N$ for all the i-DPC problems as the set

$$\begin{aligned}\mathbb{X}^N := \quad & \{\mathbf{x}: \text{ if } x_0^{[i]} = x^{[i]} \text{ for all } i=1,\dots,M \\ & \text{then } \exists \bar{\mathbf{x}}_0, (\tilde{x}_{[0:N-1]}^{[1]},\dots,\tilde{x}_{[0:N-1]}^{[M]}), (\hat{x}_{0/0}^{[1]},\dots,\hat{x}_{0/0}^{[M]}), \\ & (\hat{u}_{[0:N-1]}^{[1]},\dots,\hat{u}_{[0:N-1]}^{[M]}) \text{ such that (2), (9), (10),} \\ & \text{(12)-(15) are satisfied for all } i=1,\dots,M\}\end{aligned}$$

We denote, for each $\mathbf{x} \in \mathbb{X}^N$, the region of feasible initial state estimates. Letting $\bar{\mathbf{x}} = (\bar{x}^{[1]},\dots,\bar{x}^{[M]})$

$$\begin{aligned}\bar{\mathbb{X}}_{\mathbf{x}}^N := \quad & \{\bar{\mathbf{x}}: \text{ if } x_0^{[i]} = x^{[i]} \text{ and } \bar{x}_0^{[i]} = \bar{x}^i \text{ for all } i=1,\dots,M \\ & \text{then } \exists (\tilde{x}_{[0:N-1]}^{[1]},\dots,\tilde{x}_{[0:N-1]}^{[M]}), (\hat{x}_{0/0}^{[1]},\dots,\hat{x}_{0/0}^{[M]}), \\ & (\hat{u}_{[0:N-1]}^{[1]},\dots,\hat{u}_{[0:N-1]}^{[M]}) \text{ such that (2), (9), (10),} \\ & \text{(12)-(15) are satisfied for all } i=1,\dots,M\}\end{aligned}$$

Also, given $\mathbf{x} \in \mathbb{X}^N$ and $\bar{\mathbf{x}} \in \bar{\mathbb{X}}_{\mathbf{x}}^N$, the region of feasible initial reference trajectories is

$$\begin{aligned}\tilde{\mathbb{X}}_{\mathbf{x},\bar{\mathbf{x}}}^N := \quad & \{(\tilde{x}_{[0:N-1]}^{[1]},\dots,\tilde{x}_{[0:N-1]}^{[M]}): \text{ if } x_0^{[i]} = x^i \text{ and } \bar{x}_0^{[i]} = \bar{x}^i \\ & \text{for all } i=1,\dots,M \text{ then } \exists (\hat{x}_{0/0}^{[1]},\dots,\hat{x}_{0/0}^{[M]}), \\ & (\hat{u}_{[0:N-1]}^{[1]},\dots,\hat{u}_{[0:N-1]}^{[M]}) \text{ such that (2), (9), (10),} \\ & \text{(12)-(15) are satisfied for all } i=1,\dots,M\}\end{aligned}$$

Assumption 1: Letting $\mathbf{L} = \text{diag}(L_1,\dots,L_M)$, the matrix $\mathbf{A}+\mathbf{LC}$ is Schur. Furthermore, there exist, for all $i=1,\dots,M$, sets $\Sigma_i \subset \mathbb{R}^{n_i}$ such that $\mathbf{\Sigma}$ is a positively invariant set for the system $\boldsymbol{\sigma}_{t+1} = (\mathbf{A}+\mathbf{LC})\boldsymbol{\sigma}_t$.

Assumption 2: The matrix $A_{ii}+B_iK_i^{aux}$ is Schur, for all $i=1,\dots,M$.

Assumption 3: Letting $\mathbf{K}^{aux} = \text{diag}(K_1^{aux},\dots,K_M^{aux})$, $\hat{\mathbb{X}} = \prod_{i=1}^M \hat{\mathbb{X}}_i$, $\hat{\mathbb{U}} = \prod_{i=1}^M \hat{\mathbb{U}}_i$ and $\hat{\mathbb{X}}^F = \prod_{i=1}^M \hat{\mathbb{X}}_i^F$, it holds that:

- (i) The matrix $\mathbf{A}+\mathbf{BK}^{aux}$ is Schur;
- (ii) $\hat{H}_s^{[i]}(\hat{\mathbf{x}}) \leq 0$ for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$, for all $s \in \mathscr{C}_i$, for all $i=1,\dots,M$, where $\hat{H}^{[i]}$ is defined in such a way that $\hat{H}_s^{[i]}(\hat{\mathbf{x}}) = \hat{h}_s^{[i]}(\hat{x}^{[i]},\hat{\mathbf{x}})$ for all $s \in \mathscr{C}_i$, for all $i=1,\dots,M$.
- (iii) $\hat{\mathbb{X}}^F \subseteq \hat{\mathbb{X}}$ is an invariant set for $\hat{\mathbf{x}}^+ = (\mathbf{A}+\mathbf{BK}^{aux})\hat{\mathbf{x}}$;
- (iv) $\hat{\mathbf{u}} = \mathbf{K}^{aux}\hat{\mathbf{x}} \in \hat{\mathbb{U}}$ for any $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$;
- (v) for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$

$$\|\hat{\mathbf{x}}^+\|_{\mathbf{P}}^2 - \|\hat{\mathbf{x}}\|_{\mathbf{P}}^2 \leq -(\|\hat{\mathbf{x}}\|_{\mathbf{Q}}^2 + \|\hat{\mathbf{u}}\|_{\mathbf{R}}^2) \qquad (19)$$

where $\mathbf{Q} = \text{diag}(Q_1,\dots,Q_M)$, $\mathbf{R} = \text{diag}(R_1,\dots,R_M)$, $\mathbf{P} = \text{diag}(P_1,\dots,P_M)$.

Assumption 4: Given the sets $\mathscr{E}_i$ and the RPI sets Z_i for equations (8), there exists a real positive constant $\bar{\rho}_E > 0$ such that $Z_i \oplus \mathscr{B}_{\bar{\rho}_E}(0) \subseteq \mathscr{E}_i$ for all $i=1,\dots,M$, where $\mathscr{B}_{\bar{\rho}_E}(0)$ is a ball of radius $\bar{\rho}_E > 0$ centered at the origin.

Proper ways to select the design parameters fulfilling Assumptions 2-4 are discussed in [4]. Conditions for guaranteeing Assumption 1 are provided in the following Section V. Now we are in the position to state the main result.

Theorem 1: Let Assumptions 1-4 be satisfied and let E_i be a neighborhood of the origin satisfying $E_i \oplus Z_i \subseteq \mathscr{E}_i$. Then, for any initial reference trajectory in $\tilde{\mathbb{X}}_{\mathbf{x}_0,\bar{\mathbf{x}}_0}^N$, the trajectory $\mathbf{x}_t$, starting from any initial condition $\mathbf{x}_0 \in \mathbb{X}^N$, $\bar{\mathbf{x}}_0 \in \bar{\mathbb{X}}_{\mathbf{x}_0}^N$, asymptotically converges to the origin.

A main issue of the off-line design phase of DPC is the solution of a suitable reference trajectory. It is a critical tuning parameter, since its choice strongly affects the initial feasibility. Generally speaking, the feasibility region can be enhanced by setting sufficiently high N. An empirical algorithm for the selection of the reference trajectory based on this principle has been proposed in [4].

V. Computation of the positive invariant sets Σ_i

The DPC algorithm requires the computation of the sets $\mathscr{E}_i$, Z_i, E_i, and Σ_i. A viable way to compute $\mathscr{E}_i$, Z_i and E_i has been proposed in [4]. In this section we provide conditions guaranteeing that Assumption 1 is fulfilled and propose a constructive method for setting suitable sets Σ_i.

Define $G_i = A_{ii} + L_iC_i$ and $\tilde{A}_{ij} = A_{ij}$ for all $i,j=1,\dots,M$ such that $i \neq j$, and $\tilde{A}_{ii} = 0$. From (4), we write

$$\sigma_{t+1}^{[i]} = G_i\sigma_t^{[i]} + v_t^{[i]} \qquad (20)$$

where $v_t^{[i]} = \sum_{j=1}^M \tilde{A}_{ij}\sigma_t^{[j]}$. A given zonotope $\mathscr{E}_{\Sigma_i}$, centered at the origin and containing $\sigma^{[i]}$, $i=1\dots,M$, can be equivalently represented in two ways:

$$\begin{aligned}\mathscr{E}_{\Sigma_i} &= \{\sigma^{[i]} \in \mathbb{R}^{n_i} | \sigma^{[i]} = \Xi_{\Sigma i}d_i \text{ where } \|d_i\|_\infty \leq l_{\Sigma i}\} && \text{(21a)} \\ &= \{\sigma^{[i]} \in \mathbb{R}^{n_i} | f_{\Sigma i,r}^T\sigma^{[i]} \leq l_{\Sigma i} \text{ for all } r\} && \text{(21b)}\end{aligned}$$

where $d_i \in \mathbb{R}^{n_{d_i}}$, $\Xi_{\Sigma i} \in \mathbb{R}^{n_i \times n_{d_i}}$, $f_{\Sigma i,r} \in \mathbb{R}^{n_i}$, and $r = 1,\dots,\bar{r}_i$ for all $i = 1,\dots,M$. The constants $l_{\Sigma i} \in \mathbb{R}_+$ can be regarded as scaling factors. Assuming that $\sigma_t^{[j]} \in \mathscr{E}_{\Sigma_j}$ for all $j \in \mathcal{N}_i$, then $v_t^{[i]} \in \mathbb{V}_i = \bigoplus_{j=1}^{M} \tilde{A}_{ij} \mathscr{E}_{\Sigma_i}$. The minimal RPI set (mRPI) [10] Z_{Σ_i} of (20) is given by

$$Z_{\Sigma i} = \bigoplus_{k=0}^{\infty} G_i^k \mathbb{V}_i \tag{22}$$

It is generally impossible to obtain an explicit characterization of $Z_{\Sigma i}$, and also $Z_{\Sigma i}$ is not a polytope [10]. However, given a scalar $\delta > 0$, if $\mathbb{V}_i$ is a neighbor of the origin, there exist $\alpha_i \in \mathbb{R}$, $s_i \in \mathbb{I}$ such that the set

$$Z_{\Sigma i}^{\delta} = (1-\alpha_i)^{-1} \bigoplus_{k=0}^{s_i-1} G_i^k \mathbb{V}_i \tag{23}$$

is a polytopic RPI outer δ-approximation of the mRPI set for (20)[1]. A condition allowing the existence of sets Σ_i satisfying Assumption 1 is that $Z_{\Sigma i}^{\delta} \subset \mathscr{E}_{\Sigma_i}$, for all $i = 1,\dots,M$. If this holds, we set $\Sigma_i = Z_{\Sigma i}^{\delta}$. Consequently, if, for a given $k \geq 0$, $\sigma_k^{[i]} \in Z_{\Sigma i}^{\delta} = \Sigma_i$ for all $i = 1,\dots,M$, then $v_k^{[i]} \in \bigoplus_{j=1}^{M} \tilde{A}_{ij} \Sigma_i \subset \mathbb{V}_i$ and $\sigma_{k+1}^{[i]} \in Z_{\Sigma i}^{\delta}$, being $Z_{\Sigma i}^{\delta}$ RPI for (20). Since we assume that $\sigma_0^{[i]} \in \Sigma_i$ for all $i = 1,\dots,M$, by induction we obtain that $\sigma_t^{[i]} \in \Sigma_i$ for all i and for all $t \geq 0$.

In the remainder of the section we provide a sufficient condition for guaranteeing that $Z_{\Sigma i}^{\delta} \subset \mathscr{E}_{\Sigma_i}$ is verified for at least a combination of values of l_{Σ_i}, once the "shape" of $\mathscr{E}_{\Sigma_i}$ is defined i.e., Ξ_{Σ_i} and $f_{\Sigma_i,r}$ are given.

We define the matrices $T_{\Sigma ij}^{(k)}$, $i,j = 1,\dots,M$, $j \neq i$, as

$$(T_{\Sigma ij}^{(k)})^T = \begin{bmatrix} f_{\Sigma i,1}^T \\ \vdots \\ f_{\Sigma i,\bar{r}_i}^T \end{bmatrix} G_i^k \tilde{A}_{ij}$$

and $\mathscr{M}^{\Sigma} \in \mathbb{R}^{M \times M}$ such that its entries μ_{ij}^{Σ} are

$$\mu_{ii}^{\Sigma} = -1,\ i = 1,\dots,M \tag{24a}$$

$$\mu_{ij}^{\Sigma} = \sum_{k=0}^{\infty} \|(T_{\Sigma ij}^{(k)})^T \Xi_{\Sigma j}\|_{\infty},\ i,j = 1,\dots,M \text{ with } i \neq j \tag{24b}$$

Proposition 1: If G_i are Schur, and if $\mathscr{M}^{\Sigma}$ is Hurwitz, then Assumption 1 is satisfied for a sufficiently small value of δ if $l_{\Sigma i}$, $i = 1,\dots,M$, are the entries of the strictly positive vector $\mathbf{l}_{\Sigma}$ satisfying $\mathscr{M}^{\Sigma}\mathbf{l}_{\Sigma} < 0$.

If $\mathscr{M}^{\Sigma}$ is Hurwitz, the existence of $\mathbf{l}_{\Sigma}$ satisfying $\mathscr{M}^{\Sigma}\mathbf{l}_{\Sigma} < 0$ is guaranteed [7]. If the system is irreducible [2], $\mathbf{l}_{\Sigma}$ is the Frobenius eigenvector of matrix $\mathscr{M}^{\Sigma}$, otherwise see [4].

VI. Example

Consider the example illustrated in Figure 1 consisting in four trucks with masses $m_1 = 12$, $m_2 = 10$, $m_3 = 8$, $m_4 = 6$, each endowed with an engine (exerting the force $100u_t^{[i]}$, $i = 1,\dots,4$). Trucks 1 and 2 (resp. 3 and 4) are dynamically coupled through a spring and a damper with coefficients $k_{12} = 0.5$ and $h_{12} = 0.2$ ($k_{34} = 1$ and $h_{34} = 0.3$), respectively. The components of the 2-dimensional state vector $x_t^{[i]}$ of truck

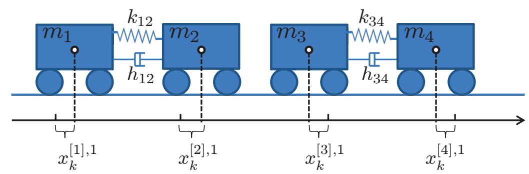

Fig. 1. Illustration of the example.

i are its displacement with respect to a given equilibrium position (i.e., $x_k^{[i],1}$) and the absolute velocity of the truck. For all $i = 1,\dots,4$, positions are measured, i.e., $y_k^{[i]} = x_k^{[i],1}$. Constraints to the inputs $|u_k^{[i]}| \leq 2$ and coupled constraints $|x_k^{[i],1} - x_k^{[i-1],1}| \leq 14$ are set for for all i. The model is discretized with sampling interval $\tau = 0.1$ s. We set the observer's initial conditions to $\bar{x}_0^{[i]} = [50, 0]^T$ and the real system initial conditions are randomly generated in such a way that $x_0^{[i]} - \bar{x}_0^{[i]} \in \Sigma_i$, where Σ_i satisfy Assumption 1. Gains L_i and K_i are defined by pole assignment such that both G_i and F_i have eigenvalues 0.5 and 0.6 $\forall i$. We properly

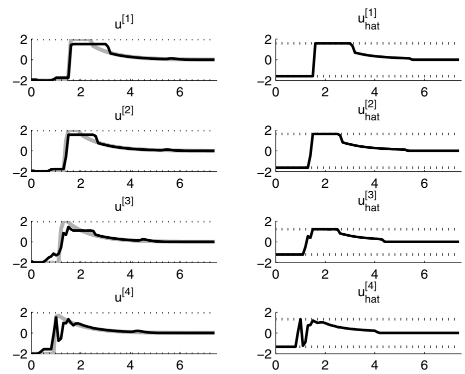

Fig. 2. Left: $u_t^{[i]}$ (black line: DPC; grey line: centralized MPC), thresholds for $u_t^{[i]}$ (dotted lines). Right: $\hat{u}_t^{[i]}$ (solid line), thresholds for $\hat{u}_{t/t}^{[i]}$ (dotted lines).

define quadratic weighting functions and we set sets $\mathscr{E}_i$, Z_i and E_i (for details see [4]). The initial reference trajectories are defined through an iterative procedure, [4], with $N = 59$. In Fig. 2 the plots of the optimal input trajectories obtained with DPC are shown, in Fig. 3 we show the obtained optimal trajectories of the state, and in Fig. 4 we show the trajectory $x_k^{[4],1} - x_k^{[3],1}$, to emphasize the effect and the conservativeness of the coupling constraints (14).

VII. Conclusions

The output feedback DPC algorithm presented in this paper has many features which make it suited for practical applications, such as the limited mutual knowledge and exchange of information among neighbors, the possibility to handle local and global state and control constraints, and guaranteed

[1]if $\mathbb{V}_i$ is not a neighbor of the origin, an outer δ_o-approximation of $\mathbb{V}_i$ can be considered for the computation of the outer δ-approximation of $Z_{\Sigma i}$

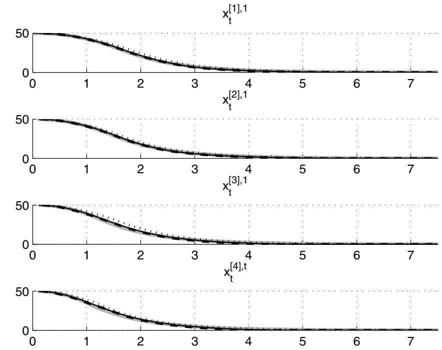

Fig. 3. Positions $x_t^{[i],1}$ (black solid lines: DPC; grey lines: centralized MPC), $\tilde{x}_t^{[i],1}$ (dotted lines), and $\hat{x}_{t/t}^{[i],1}$ (dashed lines).

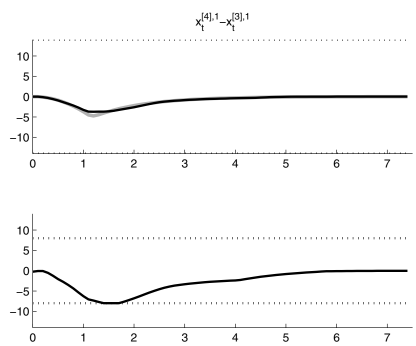

Fig. 4. Above: $x_k^{[4],1}-x_k^{[3],1}$ (solid black line: DPC; grey line: centralized MPC) and constraint (3) (dotted lines); below: $\hat{x}_k^{[4],1}-\tilde{x}_k^{[3],1}$ (solid line) and constraint (14) (dotted lines).

convergence properties.
However, a number of significant developments are required to completely exploit the potentialities of the approach in many significant practical cases. Among them, the solution of the tracking problem for constant reference signals and the possibility to include in the problem formulation cooperative goals for the subsystems will be considered.

APPENDIX

A. Proof of Theorem 1

The collective problem. Define the collective vectors $\hat{\mathbf{x}}_t=(\hat{x}_t^{[1]},\dots,\hat{x}_t^{[M]})$, $\tilde{\mathbf{x}}_t=(\tilde{x}_t^{[1]},\dots,\tilde{x}_t^{[M]})$, $\hat{\mathbf{u}}_t=(\hat{u}_t^{[1]},\dots,\hat{u}_t^{[M]})$, $\mathbf{w}_t=(w_t^{[1]},\dots,w_t^{[M]})$ and $\mathbf{z}_t=(z_t^{[1]},\dots,z_t^{[M]})$. Furthermore, define the matrices $\mathbf{A}^*=$diag$(A_{11},\dots,A_{MM})$, $\tilde{\mathbf{A}}=\mathbf{A}-\mathbf{A}^*$. Collectively, we write equations (4), (5), (6), (7), and (8) as

$$\bar{\mathbf{x}}_{t+1}=\mathbf{A}\bar{\mathbf{x}}_t+\mathbf{B}\mathbf{u}_t-\mathbf{L}(\mathbf{y}_t-\mathbf{C}\bar{\mathbf{x}}_t) \quad (25)$$

$$=\mathbf{A}^*\bar{\mathbf{x}}_t+\mathbf{B}\mathbf{u}_t+\tilde{\mathbf{A}}\tilde{\mathbf{x}}_t+\mathbf{w}_t \quad (26)$$

$$\hat{\mathbf{x}}_{t+1}=\mathbf{A}^*\hat{\mathbf{x}}_t+\mathbf{B}\hat{\mathbf{u}}_t+\tilde{\mathbf{A}}\tilde{\mathbf{x}}_t \quad (27)$$

$$\mathbf{u}_t=\hat{\mathbf{u}}_t+\mathbf{K}^{aux}(\bar{\mathbf{x}}_t-\hat{\mathbf{x}}_t) \quad (28)$$

$$\mathbf{z}_{t+1}=(\mathbf{A}^*+\mathbf{B}\mathbf{K}^{aux})\mathbf{z}_t+\mathbf{w}_t \quad (29)$$

Since each i-DPC problem depends upon local variables (the coupling terms $\tilde{x}_k^{[i]}$ are fixed for $k=t,\dots,t+N-1$), minimizing (11) for all $i=1,\dots,M$ is equivalent to minimize

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_t)=\min_{\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_{[t:t+N-1]}}\mathbf{V}^N(\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_{[t:t+N-1]}) \quad (30)$$

subject to the constraints (27), (25), (28),

$$\bar{\mathbf{x}}_t-\hat{\mathbf{x}}_t\in\mathbb{Z}=\prod_{i=1}^M Z_i \quad (31a)$$

$$\hat{\mathbf{x}}_k-\tilde{\mathbf{x}}_k\in\mathbb{E}=\prod_{i=1}^M E_i \quad (31b)$$

$$\hat{\mathbf{x}}_k\in\hat{\mathbb{X}} \quad (31c)$$

$$\hat{\mathbf{u}}_k\in\hat{\mathbb{U}} \quad (31d)$$

$$\mathbf{H}(\hat{\mathbf{x}}_k,\tilde{\mathbf{x}}_k)\le 0 \quad (31e)$$

for $k=t,\dots,t+N-1$, and the terminal constraint

$$\hat{\mathbf{x}}_{t+N}\in\hat{\mathbb{X}}^F \quad (32)$$

In (31), $\mathbf{H}$ collects all the constraints (14) and note that, by (ii) in Assumption 3, $\mathbf{H}(\hat{\mathbf{x}},\hat{\mathbf{x}})\le 0$ for all $\hat{\mathbf{x}}\in\hat{\mathbb{X}}^F$. The collective cost function $\mathbf{V}^N$ is

$$\mathbf{V}^N(\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_{[t:t+N-1]})=\sum_{k=t}^{t+N-1}(\|\hat{\mathbf{x}}_k\|^2_{\mathbf{Q}}+\|\hat{\mathbf{u}}_k\|^2_{\mathbf{R}})+\|\hat{\mathbf{x}}_{t+N}\|^2_{\mathbf{P}} \quad (33)$$

We also define

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_t)=\min_{\hat{\mathbf{u}}_{[t:t+N-1]}}\mathbf{V}^N(\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_{[t:t+N-1]}) \quad (34)$$

subject to the constraints (27), (31b)-(32).

Feasibility. From Definition 1, it collectively holds that

$$\mathbb{X}^N=\{\mathbf{x}:\ \text{if } \mathbf{x}_0=\mathbf{x} \text{ then } \exists\bar{\mathbf{x}}_0,\tilde{\mathbf{x}}_{[0:N-1]},\hat{\mathbf{x}}_{0/0},\hat{\mathbf{u}}_{[0,N-1]} \text{ such that (27), (31) and (32) are satisfied}\}$$

For each point of the feasibility set $\mathbf{x}\in\mathbb{X}^N$

$$\bar{\mathbb{X}}^N_{\mathbf{x}}:=\{\bar{\mathbf{x}}_0:\ \text{if } \mathbf{x}_0=\mathbf{x} \text{ and } \bar{\mathbf{x}}_0=\bar{\mathbf{x}} \text{ then } \exists\tilde{\mathbf{x}}_{[0:N-1]},\hat{\mathbf{x}}_{0/0},\hat{\mathbf{u}}_{[0,N-1]} \text{ such that (27), (31), and (32) are satisfied}\}$$

Finally, if $\mathbf{x}\in\mathbb{X}^N$, $\bar{\mathbf{x}}\in\bar{\mathbb{X}}^N_{\mathbf{x}}$

$$\tilde{\mathbb{X}}^N_{\mathbf{x},\bar{\mathbf{x}}}:=\{\tilde{\mathbf{x}}_{[0:N-1]}:\ \text{if } \mathbf{x}_0=\mathbf{x} \text{ and } \bar{\mathbf{x}}_0=\bar{\mathbf{x}} \text{ then } \exists\hat{\mathbf{x}}_{0/0},\hat{\mathbf{u}}_{[0,N-1]} \text{ such that (27), (31), and (32) are satisfied}\}$$

Assume that, at instant t, $\mathbf{x}_t\in\mathbb{X}^N$, $\bar{\mathbf{x}}_t\in\bar{\mathbb{X}}^N_{\mathbf{x}_t}$, and that $\tilde{\mathbf{x}}_{[t:t+N-1]}\in\tilde{\mathbb{X}}^N_{\mathbf{x}_t,\bar{\mathbf{x}}_t}$. The optimal nominal input and state sequences obtained by minimizing the collective MPC problem are $\hat{\mathbf{u}}_{[t:t+N-1]/t}=\{\hat{\mathbf{u}}_{t/t},\dots,\hat{\mathbf{u}}_{t+N-1/t}\}$ and $\hat{\mathbf{x}}_{[t:t+N]/t}=\{\hat{\mathbf{x}}_{t/t},\dots,\hat{\mathbf{x}}_{t+N/t}\}$, respectively. Finally, recall that it is set $\tilde{\mathbf{x}}_{t+N}=\hat{\mathbf{x}}_{t+N/t}$. Denote $\hat{\mathbf{u}}_{t+N/t}=\mathbf{K}^{aux}\hat{\mathbf{x}}_{t+N/t}$ and $\hat{\mathbf{x}}_{t+N+1/t}=\mathbf{A}^*\hat{\mathbf{x}}_{t+N/t}+\mathbf{B}\hat{\mathbf{u}}_{t+N/t}+\tilde{\mathbf{A}}\tilde{\mathbf{x}}_{t+N}$. Since $\tilde{\mathbf{x}}_{t+N}=\hat{\mathbf{x}}_{t+N/t}$, the latter is equivalent to $\hat{\mathbf{x}}_{t+N+1/t}=(\mathbf{A}+\mathbf{B}\mathbf{K}^{aux})\hat{\mathbf{x}}_{t+N/t}$. Note that, in view of constraint (32) and Assumption 3, $\hat{\mathbf{u}}_{t+N/t}\in\hat{\mathbb{U}}$ and $\hat{\mathbf{x}}_{t+N+1/t}\in\hat{\mathbb{X}}^F$. Therefore, they satisfy constraints (31c), (31d) and (32). Also, according to Assumption 3, (19) holds. We also define the input sequence

$$\hat{\mathbf{u}}_{[t+1:t+N]/t} = \{\hat{\mathbf{u}}_{t+1/t}, \dots, \hat{\mathbf{u}}_{t+N-1/t}, \hat{\mathbf{u}}_{t+N/t}\}$$

and the state sequence stemming from the initial condition $\hat{\mathbf{x}}_{t+1/t}$ and the input sequence $\hat{\mathbf{u}}_{[t+1:t+N]/t}$ i.e.,

$$\hat{\mathbf{x}}_{[t+1:t+N+1]/t} = \{\hat{\mathbf{x}}_{t+1/t}, \dots, \hat{\mathbf{x}}_{t+N/t}, \hat{\mathbf{x}}_{t+N+1/t}\}$$

Notice that $\mathbf{x}_k - \bar{\mathbf{x}}_k \in \mathbf{\Sigma}$ for all $k = t, \dots, t+N-1$ from Assumption 1, and that, in view of (31a)-(31b), $\mathbf{w}_k \in \prod_{i=1}^M \mathbb{W}_i$ for all $k = t, \dots, t+N-1$. In view of the feasibility of the i-DPC problem at time t, we have that $\bar{\mathbf{x}}_{t+1} - \hat{\mathbf{x}}_{t+1/t} \in \mathbb{Z}$ and $\hat{\mathbf{x}}_{k/t} - \tilde{\mathbf{x}}_k \in \prod_{i=1}^M \mathbb{E}$ for all $k = t+1, \dots, t+N-1$. Note also that $\hat{\mathbf{x}}_{t+N/t} - \tilde{\mathbf{x}}_{t+N} = 0 \in \mathbb{E}$ by (18). Furthermore, since $\tilde{\mathbf{x}}_{t+N} = \hat{\mathbf{x}}_{t+N/t}$ and $\hat{\mathbf{x}}_{t+N} \in \hat{\mathbb{X}}^F$, from (32) it holds that $\mathbf{H}(\hat{\mathbf{x}}_{t+N/t}, \tilde{\mathbf{x}}_{t+N}) \leq 0$ from (*ii*) of Assumption 3. Therefore, we can conclude that the state and the input sequences $\hat{\mathbf{x}}_{[t+1:t+N+1]/t}$ and $\hat{\mathbf{u}}_{[t+1:t+N]/t}$ are feasible at $t+1$, since constraints (31) and (32) are satisfied. This proves that $\mathbf{x}_t \in \mathbb{X}^N$, $\bar{\mathbf{x}}_t \in \bar{\mathbb{X}}^N_{\mathbf{x}_t}$ and $\tilde{\mathbf{x}}_{[t:t+N-1]} \in \tilde{\mathbb{X}}^N_{\mathbf{x}_t, \bar{\mathbf{x}}_t}$ implies that $\mathbf{x}_{t+1} \in \mathbb{X}^N$, $\bar{\mathbf{x}}_t \in \bar{\mathbb{X}}^N_{\mathbf{x}_{t+1}}$ and $\tilde{\mathbf{x}}_{[t+1:t+N]} \in \tilde{\mathbb{X}}^N_{\mathbf{x}_{t+1}, \bar{\mathbf{x}}_{t+1}}$.

Convergence. By optimality, $\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) \leq \mathbf{V}^N(\hat{\mathbf{x}}_{t+1/t}, \hat{\mathbf{u}}_{[t+1:t+N]/t})$. Recalling (33)

$$\begin{aligned} &\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) - \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t}) \leq -(\|\hat{\mathbf{x}}_{t/t}\|^2_{\mathbf{Q}} + \|\hat{\mathbf{u}}_{t/t}\|^2_{\mathbf{R}}) + \\ &+ \|\hat{\mathbf{x}}_{t+N/t}\|^2_{\mathbf{Q}} + \|\hat{\mathbf{u}}_{t+N/t}\|^2_{\mathbf{R}} + \|\hat{\mathbf{x}}_{t+N+1/t}\|^2_{\mathbf{P}} - \|\hat{\mathbf{x}}_{t+N/t}\|^2_{\mathbf{P}} \end{aligned} \quad (35)$$

In view of (19)

$$\|\hat{\mathbf{x}}_{t+N+1/t}\|^2_{\mathbf{P}} \quad - \|\hat{\mathbf{x}}_{t+N/t}\|^2_{\mathbf{P}} + \|\hat{\mathbf{x}}_{t+N/t}\|^2_{\mathbf{Q}} + \|\hat{\mathbf{u}}_{t+N/t}\|^2_{\mathbf{R}} \leq 0$$

and so, from (35), it follows that

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) \leq \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t}) - (\|\hat{\mathbf{x}}_{t/t}\|^2_{\mathbf{Q}} + \|\hat{\mathbf{u}}_{t/t}\|^2_{\mathbf{R}}) \quad (36)$$

Now we analyze the properties of the cost function $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t)$ defined in (30). First, note that, by definition of $\hat{\mathbf{x}}_{t/t}$, we have that $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) = \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t})$. By optimality, we have that

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_{t+1}) = \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t+1}) \leq \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t})$$

Considering (36), we obtain that

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_{t+1}) \leq \mathbf{V}^{N*}(\bar{\mathbf{x}}_t) - (\|\hat{\mathbf{x}}_{t/t}\|^2_{\mathbf{Q}} + \|\hat{\mathbf{u}}_{t/t}\|^2_{\mathbf{R}}) \quad (37)$$

for all $\bar{\mathbf{x}}_t \in \bar{\mathbb{X}}^N_{\mathbf{x}_t}$, being $\mathbf{x}_t \in \mathbb{X}^N$, and for all $\tilde{\mathbf{x}}_{[t:t+N-1]} \in \tilde{\mathbf{X}}^N_{\mathbf{x}_t, \bar{\mathbf{x}}_t}$. Since $\mathbf{Q} > 0$ and $\mathbf{R} > 0$ by definition, $\|\hat{\mathbf{x}}_{t/t}\| \to 0$ and $\|\hat{\mathbf{u}}_{t/t}\| \to 0$ as $t \to +\infty$. From (1), (25), and (28)

$$\begin{cases} \boldsymbol{\sigma}_{t+1} &= (\mathbf{A} + \mathbf{LC})\boldsymbol{\sigma}_t \\ \bar{\mathbf{x}}_{t+1} &= (\mathbf{A} + \mathbf{BK}^{aux})\bar{\mathbf{x}}_t - \mathbf{LC}\boldsymbol{\sigma}_t + \mathbf{B}\left(\hat{\mathbf{u}}_{t/t} - \mathbf{K}^{aux}\hat{\mathbf{x}}_{t/t}\right) \end{cases}$$

By asymptotic convergence to zero of the nominal state and input signals $\hat{\mathbf{x}}_{t/t}$ and $\hat{\mathbf{u}}_{t/t}$ respectively, we obtain that $\mathbf{B}\left(\hat{\mathbf{u}}_{t/t} - \mathbf{K}^{aux}\hat{\mathbf{x}}_{t/t}\right)$ is an asymptotically vanishing term. Since $(\mathbf{A} + \mathbf{BK}^{aux})$ and $(\mathbf{A} + \mathbf{LC})$ are Schur by Assumption 3 and 1, we obtain that $\boldsymbol{\sigma}_t \to 0$ and $\bar{\mathbf{x}}_t \to 0$ as $t \to +\infty$, from which it follows that $\mathbf{x}_t = \bar{\mathbf{x}}_t + \boldsymbol{\sigma}_t \to 0$ as $t \to +\infty$.

B. Proof of Proposition 1 First note that $Z_{\Sigma i}$ can be defined only if G_i is Schur, for all $i = 1, \dots, M$. A condition guaranteeing that Assumption 1 is verified is that, for all $i = 1, \dots, M$, $Z^\delta_{\Sigma i} \subset \Sigma_i$. This condition is satisfied [5] if

$$\sup_{z^{[i]}_\Sigma \in Z^\delta_{\Sigma_i}} f^T_{\Sigma i,r} z^{[i]}_\Sigma < l_{\Sigma i} \quad (38)$$

where $Z^\delta_{\Sigma i}$ is defined in (23). Note that $\sup_{z^{[i]}_\Sigma \in Z^\delta_{\Sigma_i}} f^T_{\Sigma i,r} z^{[i]}_\Sigma \leq \sup_{z^{[i]}_\Sigma \in Z_{\Sigma_i}} f^T_{\Sigma i,r} z^{[i]}_\Sigma + \|f^T_{\Sigma i,r}\|_\infty \delta$ where $Z_{\Sigma i}$ is defined in (22). In view of (22) and of the definition of set $\mathbb{V}_i$ it holds that

$$\begin{aligned} \sup_{z^{[i]}_\Sigma \in Z_{\Sigma_i}} f^T_{\Sigma i,r} z^{[i]}_\Sigma &\leq \textstyle\sum_{k=0}^\infty \sup_{v^{[i]} \in \mathbb{V}_i} f^T_{\Sigma i,r} G_i^k v^{[i]} \\ &\leq \textstyle\sum_{k=0}^\infty \sum_{j=1}^M \sup_{\sigma^{[j]} \in \mathscr{E}_{\Sigma_j}} f^T_{\Sigma i,r} G_i^k \tilde{A}_{ij} \sigma^{[j]} \end{aligned} \quad (39)$$

Recalling (21), the latter quantity is smaller or equal than

$$\begin{aligned} &\textstyle\sum_{j=1}^M \sum_{k=0}^\infty \sup_{\|d_j\|_\infty \leq l_{\Sigma_j}} f^T_{\Sigma i,r} G_i^k \tilde{A}_{ij} \Xi_{\Sigma_j} d_j = \\ &= \textstyle\sum_{j=1}^M \sum_{k=0}^\infty \|f^T_{\Sigma i,r} G_i^k \tilde{A}_{ij} \Xi_{\Sigma_j}\|_\infty l_{\Sigma_j} \end{aligned} \quad (40)$$

Therefore, condition (38) is satisfied for all i, r if, for all i

$$\textstyle\sum_{j=1}^M \sum_{k=0}^\infty \|(T^{(k)}_{\Sigma ij})^T \Xi_{\Sigma_j}\|_\infty l_{\Sigma_j} + \sup_r \|f^T_{\Sigma i,r}\|_\infty \delta \quad < \quad l_{\Sigma_i}$$

The latter is fulfilled, for a sufficiently small value of δ, if

$$\textstyle\sum_{j=1}^M \sum_{k=0}^\infty \|(T^{(k)}_{\Sigma ij})^T \Xi_{\Sigma_j}\|_\infty l_{\Sigma_j} \quad < \quad l_{\Sigma_i} \quad (41)$$

If we define the vector $\mathbf{l}_\Sigma = (l_{\Sigma 1}, \dots, l_{\Sigma M})$, condition (41) is equivalent to $[\mu^\Sigma_{i1} \dots \mu^\Sigma_{iM}]\mathbf{l}_\Sigma < 0$, for all $i = 1, \dots, M$, and hence it corresponds to $\mathscr{M}^\Sigma \mathbf{l}_\Sigma < 0$. This holds, from [7], if $\mathscr{M}^\Sigma$ is Hurwitz.

References

[1] W.B. Dunbar. Distributed receding horizon control of dynamically coupled nonlinear systems. *IEEE Trans. on Automatic Control*, 52:1249–1263, 2007.

[2] L. Farina and S. Rinaldi. *Positive Linear Systems*. J. Wiley and Sons, 2000.

[3] M. Farina and R. Scattolini. Distributed non-cooperative mpc with neighbor-to-neighbor communication. pages 404 – 409.

[4] M. Farina and R. Scattolini. Distributed predictive control: a non-cooperative algorithm with neighbor-to-neighbor communication. 2010. Submitted.

[5] I. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariance sets for discrete-time linear systems. *Mathematical problems in engineering*, 5:317 – 367, 1998.

[6] J. Liu, D. Muñoz de la Peña, and P.D. Christofides. Distributed model predictive control of nonlinear process systems. *AIChE J.*, 55(9):1171–1184, 2009.

[7] O. Mason and R. Shorten. On linear copositive Lyapunov functions and the stability of switched positive linear systems. *IEEE Trans. on Automatic Control*, 52(7):1346 – 1349, 2007.

[8] D.Q. Mayne, S.V. Rakovic, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems: Time varying case. *Automatica*, 43:2082–2087, 2009.

[9] D.Q. Mayne, M.M. Seron, and S. V. Rakovic. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41:219–224, 2005.

[10] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D.Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Trans. Aut. Cont*, 50(3):406 – 410, 2005.

[11] R. Scattolini. Architectures for distributed and hierarchical model predictive control. *Journal of Process Control*, 19:723–731, 2009.

[12] B. T. Stewart, A. N. Venkat, J. B. Rawlings, S. J. Wright, and G. Pannocchia. Cooperative distributed model predictive control. *System & Control Letters*, 59:460–469, 2010.