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Compensation of State-Dependent Input Delay for Nonlinear Systems

Nikolaos Bekiaris-Liberis and Miroslav Krstic

Abstract— **We introduce and solve stabilization problems for linear and nonlinear systems with state-dependent input delay. Since the state-dependence of the delay makes the prediction horizon dependent on the future value of the state, which means that it is impossible to know a priori how far in the future the prediction is needed, the key design challenge is how to determine the predictor state. We resolve this challenge and establish closed-loop stability of the resulting infinite-dimensional nonlinear system for general nonnegative-valued delay functions of the state. Due to an inherent limitation on the allowable delay rate in stabilization of systems with time-varying delays, in the case of state-dependent delay, where the delay rate becomes dependent on the gradient of the delay function and on the state and control input, only regional stability results are achievable. For forward-complete nonlinear systems we establish global asymptotic stability results and for linear systems we prove exponential stability. Global stability is established under a restrictive but a priori verifiable Lyapunov-like condition that the delay rate be bounded by unity irrespective of the values of the state and input. Several illustrative examples are provided, including unicycle stabilization subject to input delay that grows with the distance from the reference position.**

I. INTRODUCTION

State-dependent delays are common in real world. For example, in control over networks, it makes sense to send control signals less frequently when the state is small and more frequently when the state is large [7]. In control of mobile robots the magnitude of the delay depends on the distance of the robot with the operator interface [17]. A priori known functions of time are used to model state-dependent delays in transmission channels of communication networks, used for the remote stabilization of unstable systems [24]. In supply networks, state-dependent delays appear due to transportation of materials [23]. In milling process, speed-dependent delays arise due to the deformation of the cutting tool [1]. The reaction time of a driver is often modeled as a pure [19] or distributed [21] delay. However, the delay depends on the intensity of the disturbance, the size of the tracking error to which the driver is reacting, the speed of the vehicle, the physical situation of the driver, etc [6]. In irrigation channels the dynamics of a reach are accurately represented by a time-varying delayed-integrator model developed, in [13]. Finally, in population dynamics, the time required for the maturation level of a cell to achieve a certain threshold can be modeled as a state-dependent delay [14].

Compensation of constant input delays in unstable linear plants is achieved using predictor-based (finite spectrum assignment) techniques [3], [16], [25]. Extensions of these designs to linear systems with simultaneous input and state delay can be found in [8]. In addition, predictor-based techniques are developed for linear systems with both unknown plant parameters and delays [5]. Control schemes for nonlinear systems with input delay are developed in [12] whereas nonlinear systems with state delays are considered in [9], [15].

Although there are numerous results concerning plants with constant input delays, the problem of *compensation* of long time-varying input delays, even for linear systems, is tackled in only a few references [3], [11], [18]. Even more rare are papers that deal with the compensation of time-varying input delays in nonlinear systems [9]. No results exist for compensation of a state-dependent input delay, even for linear plants.

We present a methodology for compensating state-dependent input delays for both linear and nonlinear systems. For nonlinear systems with state-dependent input delay and under the assumption of forward-completeness and global stabilizability (by a possibly time-varying control law) in the absence of the input delay, we design a predictor-based compensator (Section II). Our controller uses predictions of future values of the state on appropriate predictor intervals that depend on the current values of the state. Due to the physical restriction on the magnitude of the delay function's gradient (the controller never reaches the system if the delay rate is larger than one), we obtain only a regional stability result. We give an estimate of the region of attraction for our control scheme based on the construction of a strict, time-varying Lyapunov function (Section III). We present a global result for forward-complete systems under a restrictive but a priori verifiable Lyapunov-like condition that the delay rate be bounded by unity irrespective of the values of the state and input (Section V). We also deal with linear systems, treating them as a special case of the design for nonlinear systems, for which we prove exponential stability (Section IV). Finally, three detailed examples are presented to demonstrate the capabilities of the present methodology (Section VI).

Notation: We use the common definition of class $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{K}\mathscr{L}$ functions from [10]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. We say that a function ρ : $\mathbb{R}_+ \times (0,1) \mapsto \mathbb{R}_+$ belongs to class $\mathscr{K}\mathscr{C}$ if it is of class $\mathscr{K}$ with respect to its first argument for each value of its second argument and continuous with respect to its second argument. It belongs to class $\mathscr{K}\mathscr{C}_\infty$ if it is of class $\mathscr{K}_\infty$ with respect to its first argument for each value of its second argument and continuous with respect to its second argument.

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II. Problem Formulation and Controller Design

We consider the following system

$$\dot{X}(t) = f\left(X(t), U\left(t - D(X(t))\right)\right), \tag{1}$$

where $X \in \mathbb{R}^n$, $U \in \mathbb{R}$, $t \in \mathbb{R}_+$, $D \in C^1\left(\mathbb{R}^n; \mathbb{R}_+\right)$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$ and it holds

$$|f(X,\omega)| \leq \alpha_1\left(|X| + |\omega|\right), \tag{2}$$

for a class $\mathscr{K}_\infty$ function α_1. The results of this paper can be extended to the multi-input case when the delay functions are identical in each individual input channel. If this is not the case, one has to follow a different methodology than the one considered here, which is related to [4].

Assumption 1: The plant $\dot{X} = f(X,\omega)$ is strongly forward complete, that is, there exist a smooth positive definite function R and class $\mathscr{K}_\infty$ functions α_2, α_3 and α_4 such that for all $X \in \mathbb{R}^n$ and for all $\omega \in \mathbb{R}$ the following holds

$$\begin{aligned} \alpha_2(|X|) &\leq R(X) \leq \alpha_3(|X|) & (3)\\ \frac{\partial R(X)}{\partial X} f(X,\omega) &\leq R(X) + \alpha_4(|\omega|). & (4) \end{aligned}$$

This property differs from the standard forward completeness [2] in the sense that we assume $f(0,0) = 0$ and, in accordance with that, also assume that $R(\cdot)$ is positive definite. Assumption 1 guarantees that system (1) does not exhibit finite escape time, that is, for every initial condition and every bounded input the solution is defined for all $t \geq 0$.

Assumption 2: The plant $\dot{X}(t) = f\left(X(t), \kappa(t, X(t)) + \omega(t)\right)$ satisfies the uniform in time input-to-state stability property with respect to ω and the function κ is uniformly bounded with respect to its first argument, that is, there exists a class $\mathscr{K}_\infty$ function $\hat{\alpha}$ such that

$$|\kappa(t,\xi)| \leq \hat{\alpha}(|\xi|), \quad \text{for all } t \geq 0. \tag{5}$$

We design a predictor-based controller for the plant (1) as

$$U(t) = \kappa\left(\sigma(t), P(t)\right), \tag{6}$$

where for all $t - D(X(t)) \leq \theta \leq t$

$$\begin{aligned} P(\theta) &= X(t) + \int_{t-D(X(t))}^{\theta} \frac{f(P(s),U(s))\,ds}{1 - \nabla D(P(s))\, f(P(s),U(s))} & (7)\\ \sigma(\theta) &= \theta + D(P(\theta)). & (8) \end{aligned}$$

The initial predictor $P(\theta)$, $\theta \in [-D(X(0)), 0]$, is given by (7) for $t = 0$.

The quantity $P(t)$ given in (7) is the $\sigma(t) - t$ time units predictor of $X(t)$, which can be seen as follows. Differentiating relation (7) with respect to θ, setting $\theta = t$ and performing a change of variables $\tau = \sigma(t)$ in the ODE for $X(\tau)$ given in (1) (where t is replaced by τ), one observes that $P(t)$ satisfies the same ODE in t as $X(\sigma(t))$. Since from (7) for $t = 0$ it follows that $P(-D(X(0))) = X(0)$, by defining

$$\begin{aligned} \phi(t) &= t - D(X(t)), \quad t \geq 0, & (9)\\ \sigma(\theta) &= \phi^{-1}(\theta), \quad t - D(X(t)) \leq \theta \leq t, & (10) \end{aligned}$$

we get $P(0) = X(\sigma(0))$. Hence, indeed $P(t) = X(\sigma(t)), t \geq 0$. For implementing the control law (6)–(8) one has to compute the integral in (7) at each step for all $t - D(X(t)) \leq \theta \leq t$.

Noting from (9) that $D\left(X\left(\phi^{-1}(t)\right)\right) = \phi^{-1}(t) - t$, differentiating this relation, we get for all $t - D(X(t)) \leq \theta \leq t$

$$\nabla D\left(X(\sigma(\theta))\right) f\left(X(\sigma(\theta)), U(\theta)\right) \dot{\sigma}(\theta) = \dot{\sigma}(\theta) - 1. \tag{11}$$

Solving for $\dot{\sigma}(\theta)$ and observing that $X\left(\phi^{-1}(\theta)\right)$ is the predictor signal $P(\theta)$, we get for all $t - D(X(t)) \leq \theta \leq t$

$$\dot{\sigma}(\theta) = \frac{1}{1 - \nabla D(P(\theta))\, f(P(\theta), U(\theta))}. \tag{12}$$

Motivated by the need to keep the denominator in (7) and (12) positive, throughout the paper we consider the condition on the solutions which is given for all $\theta \geq -D(X(0))$ by

$$\mathscr{F}_c: \quad \nabla D(P(\theta))\, f(P(\theta), U(\theta)) < c, \tag{13}$$

for $c \in (0,1]$. We refer to $\mathscr{F}_1$ as the *feasibility condition* of the controller (6)–(8).

III. Stability Analysis

Theorem 1: Consider the plant (1) together with the control law (6)–(8). Under Assumptions 1 and 2 there exist a class $\mathscr{KC}$ function ψ_{RoA}, class $\mathscr{KC}_\infty$ functions $\bar{\rho}_c$, ρ and a class $\mathscr{KL}$ function β such that for all initial conditions of the plant (1) that satisfy

$$|X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)| < \psi_{RoA}\left(\bar{\rho}_c(c,c), c\right), \tag{14}$$

for some $0 < c < 1$, the following holds

$$\Omega(t) \leq \beta\left(\rho\left(\Omega(0), c\right), t\right), \tag{15}$$

for all $t \geq 0$, where

$$\Omega(t) = |X(t)| + \sup_{t-D(X(t)) \leq \theta \leq t} |U(\theta)|. \tag{16}$$

Furthermore, there exists a class $\mathscr{K}_\infty$ function δ_1 such that, for all $t \geq 0$,

$$\begin{aligned} D(X(t)) &\leq D(0) + \delta_1\left(\bar{\rho}_c(c,c)\right) & (17)\\ \left|\dot{D}(X(t))\right| &\leq c. & (18) \end{aligned}$$

Lemma 1: The infinite-dimensional backstepping transformation of the actuator state for $t - D(X(t)) \leq \theta \leq t$

$$W(\theta) = U(\theta) - \kappa\left(\sigma(\theta), P(\theta)\right), \tag{19}$$

together with the predictor-based control law given in relations (6)–(8) transform system (1) to the "target system":

$$\begin{aligned} \dot{X}(t) &= f\left(X(t), \kappa(t, X(t)) + W(t - D(X(t)))\right) & (20)\\ W(t) &= 0, \quad \forall t \geq 0. & (21) \end{aligned}$$

Proof: Using (6) and the facts that $P(t - D(X(t))) = X(t)$, $\sigma(t - D(X(t))) = t$, which are immediate consequences of (7)–(8), we get the statement of the lemma. ■

Lemma 2: The inverse of the infinite-dimensional backstepping transformation (19) for all $t - D(X(t)) \leq \theta \leq t$ is

$$U(\theta) = W(\theta) + \kappa\left(\sigma(\theta), \Pi(\theta)\right), \tag{22}$$

where for all $t - D(X(t)) \leq \theta \leq t$

$$\Pi(\theta) = \int_{t-D(X(t))}^{\theta} \frac{f\left(\Pi(s), \kappa(\sigma(s), \Pi(s)) + W(s)\right) ds}{1 - \nabla D(\Pi(s))\, f\left(\Pi(s), \kappa(\sigma(s), \Pi(s)) + W(s)\right)} + X(t). \tag{23}$$

Proof: We first point out that $\Pi(\theta)$ is for the closed-loop system (20) what $P(\theta)$ is for system (1). Although $\Pi(\theta) = P(\theta)$ for all $t - D(X(t)) \leq \theta \leq t$, $\Pi(\theta)$ is driven by the transformed input $W(\theta)$, whereas $P(\theta)$ is driven by the input $U(\theta)$ for $t - D(X(t)) \leq \theta \leq t$. In other words, the direct backstepping transformation is defined as $(X(t), U(\theta)) \mapsto (X(t), W(\theta))$ and is given in (19), where $P(\theta)$ is given as a function of $X(t)$ and $U(\theta)$ through relations (7)–(8). Analogously, the inverse transformation is defined as $(X(t), W(\theta)) \mapsto (X(t), U(\theta))$ and is given in (22) where $\Pi(\theta)$ is given in terms of $X(t)$ and $W(\theta)$ through (23). ∎

Lemma 3: There exist a positive constant g, a function $\delta_1 \in \mathscr{K}_\infty$ and a function $\beta_2 \in \mathscr{KL}$ such that for all solutions of the system satisfying (13) for $0 < c < 1$, the following holds

$$\Xi(t) \quad \leq \quad \beta_2 \left(\rho_* \left(\Xi(0), c\right), t\right), \tag{24}$$

for all $t \geq 0$, where

$$\Xi(t) \quad = \quad |X(t)| + \sup_{t - D(X(t)) \leq \theta \leq t} |W(\theta)| \tag{25}$$

$$\rho_*(s, c) \quad = \quad \frac{e^{\frac{g}{1-c}}}{1-c} e^{\frac{g}{1-c}\left(D(0) + \delta_1(s)\right)} s. \tag{26}$$

Proof: Based on Assumption 2 and [22] we can conclude that there exist a smooth positive definite function $S(X(t))$ and class $\mathscr{K}_\infty$ functions α_5, α_6, α_7 and α_8 such that

$$\alpha_5 (|X(t)|) \leq S(X(t)) \leq \alpha_6 (|X(t)|) \tag{27}$$

$$\frac{\partial S(X(t))}{\partial X(t)} f\left(X(t), \kappa(X(t)) + W(t - D(X(t)))\right) \leq -\alpha_7 (|X(t)|) + \alpha_8 (|W(t - D(X(t)))|). \tag{28}$$

Consider now the following Lyapunov function for (20)–(21)

$$V(t) = S(X(t)) + k \int_0^{L(t)} \frac{\alpha_8(r)}{r} dr, \tag{29}$$

where $L(t) = \sup_{t-D(X(t)) \leq \theta \leq t} \left| e^{g(1+\sigma(\theta)-t)} W(\theta) \right| = \lim_{n \to \infty} \left(\int_{t-D(X(t))}^t e^{2ng(1+\sigma(\theta)-t)} W(\theta)^{2n} d\theta \right)^{\frac{1}{2n}}$, and $g > 0$. We now upper- and lower-bound $L(t)$ in terms of $\sup_{t-D(X(t)) \leq \theta \leq t} |W(\theta)|$. From (13) for $0 < c < 1$ we get that $\dot{\sigma}(\theta) \leq \frac{1}{1-c}$. Integrating the relation $\dot{\sigma}(\theta) \leq \frac{1}{1-c}$ from $t - D(X(t))$ to θ and, since $\sigma(t - D(X(t))) = t$, we have for all $t - D(X(t)) \leq \theta \leq t$ that $1 + \sigma(\theta) - t \leq \frac{1-c+D(X(t))}{1-c}$. Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ there exists $\delta_1 \in \mathscr{K}_\infty$ such that

$$D(X) \quad \leq \quad D(0) + \delta_1(|X|). \tag{30}$$

Therefore, $L(t) \leq \frac{e^{\frac{g}{1-c}\left(1 + D(0) + \delta_1(|X(t)|)\right)}}{1-c} \sup_{t-D(X(t)) \leq \theta \leq t} |W(\theta)|$. With relation $\sigma(t - D(X(t))) = t$ and since σ is increasing we get $1 \leq 1 + \sigma(\theta) - t$ for all $t - D(X(t)) \leq \theta \leq t$. Hence, $L(t) \geq e^g \sup_{t-D(X(t)) \leq \theta \leq t} |W(\theta)|$. Taking the time derivative of $L(t)$, with (21) we get $\dot{L}(t) = \lim_{n \to \infty} \frac{1}{2n} \left(\int_{t-D(X(t))}^t e^{2ng(1+\sigma(\theta)-t)} W(\theta)^{2n} d\theta \right)^{\frac{1}{2n}-1} \times \left(-\left(1 - \frac{dD(X(t))}{dt}\right) e^{2ng} W(t - D(X(t)))^{2n} - 2ng \int_{t-D(X(t))}^t e^{2ng(1+\sigma(\theta)-t)} W(\theta)^{2n} d\theta \right)$. Using (13) we get $\frac{dD(X(t))}{dt} < 1$. Hence $\dot{L}(t) \leq -gL(t)$. Taking the derivative of $V(t)$, and using (28), $\dot{L}(t) \leq -gL(t)$ we get $\dot{V}(t) \leq -\alpha_7(|X(t)|) + \alpha_8(|W(t - D(X(t)))|) - kg\alpha_8(L(t))$. Setting $k = 2g^{-1}$, the lower bound of $L(t)$ gives $\dot{V}(t) \leq -\alpha_7(|X(t)|) - \alpha_8(L(t))$. With (27), the definition of $L(t)$ and (29) there exists $\gamma_1 \in \mathscr{K}$ such that $\dot{V}(t) \leq -\gamma_1(V(t))$. Using Lemma 4.4 in [10] there exists $\beta_1 \in \mathscr{KL}$ such that $V(t) \leq \beta_1(V(0), t)$. Using (27), (29) and the properties of class $\mathscr{K}$ functions we get $|X(t)| + L(t) \leq \beta_2(|X(0)| + L(0), t)$, for a $\beta_2 \in \mathscr{KL}$. Using the upper and lower bounds of $L(t)$ the lemma is proved. ∎

Lemma 4: There exists a class $\mathscr{KC}_\infty$ function ρ_1 such that for all solutions of the system satisfying (13) for $0 < c < 1$, the following holds for all $t - D(X(t)) \leq \theta \leq t$

$$|P(\theta)| \quad \leq \quad \rho_1(\Omega(t), c). \tag{31}$$

Proof: Consider the following ODE in θ for all $t - D(X(t)) \leq \theta \leq t$ which follows by differentiating (7)

$$\dot{P}(\theta) = \frac{f(P(\theta), U(\theta))}{1 - \nabla D(P(\theta)) f(P(\theta), U(\theta))}. \tag{32}$$

Define the change of variables $y = \sigma(\theta)$ and rewrite (32) as

$$\frac{dP(\phi(y))}{dy} = f\left(P(\phi(y)), U(y - D(P(\phi(y))))\right), \tag{33}$$

for all $t \leq y \leq \sigma(t)$. Using (4) we get $\frac{dR(P(\phi(y)))}{d\theta} \frac{d\theta}{dy} \leq R(P(\phi(y))) + \alpha_4(|U(y - D(P(\phi(y))))|)$. With (13) we have $\frac{dR(P(\theta))}{d\theta} \leq \frac{1}{1-c}\left(R(P(\theta)) + \alpha_4(|U(\theta)|)\right)$. Using the comparison principle and (30) we get $R(P(\theta)) \leq e^{\frac{(D(0)+\delta_1(|X(t)|))}{1-c}} \left(R(X(t)) + \sup_{t-D(X(t)) \leq s \leq t} \alpha_4(|U(s)|) \right)$. With the properties of class $\mathscr{K}_\infty$ functions the lemma is proved with $\rho_1(s, c) = \alpha_2^{-1} \circ (\alpha_3(s) + \alpha_4(s)) e^{(D(0)+\delta_1(s))(1-c)^{-1}}$. ∎

Lemma 5: There exists a class $\mathscr{K}$ function γ_5 such that for all solutions of the system satisfying (13) for $0 < c < 1$, the following holds for all $t - D(X(t)) \leq \theta \leq t$

$$|\Pi(\theta)| \leq \gamma_5 \left(|X(t)| + \sup_{t-D(X(t)) \leq s \leq t} |W(s)| \right). \tag{34}$$

Proof: Under Assumption 2 and [22], there exists class $\mathscr{KL}$ function $\beta_3(\cdot, \tau)$ and class $\mathscr{K}$ function γ_3 such that

$$|Y(\tau)| \leq \beta_3(|Y(0)|, \tau) + \gamma_3 \left(\sup_{s \geq 0} |\omega(s)| \right), \quad \tau \geq 0, \tag{35}$$

where $Y(\tau)$ satisfies $\dot{Y}(\tau) = f(Y(\tau), \kappa(\tau, Y(\tau)) + \omega(\tau))$. Defining $y = \sigma(\theta)$. Equations (23), (33) and the properties of class $\mathscr{KL}$ functions, give for all $t \leq y \leq \sigma(t)$, $\frac{d\Pi(\phi(y))}{dy} = f(\Pi(\phi(y)), \kappa(y, \Pi(\phi(y))) + W(\phi(y)))$. From (35) and the properties of class $\mathscr{K}$ functions we get (34) with $\gamma_5 \in \mathscr{K}$ as $\gamma_5(s) = \gamma_3(s) + \beta_3(s, 0)$. ∎

Lemma 6: There exist a class $\mathscr{KC}_\infty$ function ρ_2 and a class $\mathscr{K}_\infty$ function α_9 such that for all solutions of the system satisfying (13) for $0 < c < 1$, the following holds

$$\Omega(t) \quad \leq \quad \alpha_9^{-1}(\Xi(t)), \quad t \geq 0 \tag{36}$$

$$\Xi(t) \quad \leq \quad \rho_2(\Omega(t), c), \quad t \geq 0. \tag{37}$$

Proof: Using (22) and (34) we get (36) with $\alpha_9^{-1}(s) = s + \hat{\alpha} \circ \gamma_5(s)$. With (19) and (31) we get (37) with $\rho_2(s, c) = s + \hat{\alpha} \circ \rho_1(s, c)$. ∎

Lemma 7: There exists a class $\mathscr{KC}_\infty$ function $\bar{\rho}_c$ such that all the solutions that satisfy

$$|X(t)| + \sup_{t-D(X(t))\leq\theta\leq t} |U(\theta)| \quad < \quad \bar{\rho}_c(c,c), \quad t \geq 0, \tag{38}$$

for $0 < c < 1$ also satisfy (13).

Proof: Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ there exists a class $\mathscr{K}_\infty$ function δ_2 such that

$$|\nabla D(X)| \quad \leq \quad |\nabla D(0)| + \delta_2(|X|). \tag{39}$$

If a solution satisfies for all $t - D(X(t)) \leq \theta \leq t$ the inequality $(|\nabla D(0)| + \delta_2(|P(\theta)|))\ \alpha_1\left(|P(\theta)| + \sup_{t-D(X(t))\leq s\leq t}|U(s)|\right) < c$, for $0<c<1$, then it also satisfies (13). Using Lemma 4 this inequality holds for $0<c<1$, if (38) holds, where $\rho_c \in \mathscr{KC}_\infty$ is defined as $\rho_c(s,c) = (|\nabla D(0)| + \delta_2(\rho_1(s,c)))\,\alpha_1(\rho_1(s,c)+s)$, and $\bar{\rho}_c$ denotes the inverse function of ρ_c with respect to its first argument. ■

Lemma 8: There exist function $\gamma_2 \in \mathscr{K}$ such that for all initial conditions of the closed-loop system (1), (6)–(8) that satisfy relation (14) for the function $\psi_{RoA} \in \mathscr{KC}$ defined as

$$\bar{\psi}_{RoA}(s,c) = \gamma_2(\rho_*(\rho_2(s,c),c)), \tag{40}$$

where $\bar{\psi}_{RoA}$ stands for the inverse function of ψ_{RoA} with respect to ψ_{RoA}'s first argument, the solutions of the system satisfy (38) for $0<c<1$ and hence satisfy (13).

Proof: Using Lemma 6, with (24) we have that

$$\Omega(t) \leq \alpha_9^{-1} \circ \beta_2(\rho_*(\rho_2(\Omega(0),c),c),t). \tag{41}$$

Defining the function $\gamma_2 \in \mathscr{K}$, $\gamma_2(s) = \alpha_9^{-1} \circ (\beta_2(s,0))$, we get $\Omega(t) \leq \gamma_2(\rho_*(\rho_2(\Omega(0)),c),c)$. Hence, for all initial conditions that satisfy (14), with $\psi_{RoA(s,c)}$ as in (40), the solutions satisfy (38). Furthermore, for all those initial conditions, the solutions verify (13) for all $\theta \geq -D(X(0))$. ■

Proof of Theorem 1: Using (41) we get (15) of Theorem 1 with $\beta(s,t) = \alpha_9^{-1}(\beta_2(s,t))$ and $\rho(s,c) = \rho_*(\rho_2(s,c),c)$. Using Lemma 8, and (30), (39) we get (17)–(18). □

IV. Application to Linear Systems

In this section we apply the general theory of Sections II and III to linear systems of the form

$$\dot{X}(t) = AX(t) + BU(t - D(X(t))), \tag{42}$$

where $X \in \mathbb{R}^n$, $U \in \mathbb{R}$, $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$. Assumption 1 is satisfied for any A by means of the fact $\frac{d|Y(\tau)|^2}{d\tau} \leq (2|A|+1)|Y(\tau)|^2 + |B|^2\omega^2(\tau)$. Assumption 2 is satisfied under the controllability condition of the pair (A,B) with

$$U(t) = KP(t) \tag{43}$$

$$P(\theta) = X(t) + \int_{t-D(X(t))}^{\theta} \frac{(AP(s)+BU(s))\,ds}{1 - \nabla D(P(s))(AP(s)+BU(s))},$$
$$t - D(X(t)) \leq \theta \leq t, \tag{44}$$

where K is chosen such that the matrix $A+BK$ is Hurwitz. The initial conditions for (44) are derived by setting $t=0$ in (44). One can observe that the predictor signal $P(\theta)$, even for the linear case, is not given explicitly. However, we establish explicit estimates, that highlight the nonlinear role of the delay function $D(X)$, and exponential decay in time.

Corollary 1: Consider the plant (42) together with the control law (43)–(44) and K chosen such that $A+BK$ is Hurwitz. Then there exist class $\mathscr{KC}_\infty$ functions $\bar{\zeta}_c$, ζ_4, ζ_{RoA} and a positive constant λ such that for all initial conditions of the plant that satisfy

$$|X(0)| + \sup_{-D(X(0))\leq\theta\leq 0} |U(\theta)| < \zeta_{RoA}\left(\bar{\zeta}_c(c,c),c\right), \tag{45}$$

for some $0<c<1$, the following holds

$$\Omega(t) \quad \leq \quad \zeta_4(\Omega(0),c)\,e^{-\lambda t}, \tag{46}$$

for all $t \geq 0$. Furthermore, inequalities (17) and (18) hold with $\bar{\rho}_c(c,c)$ replaced by $\bar{\zeta}_c(c,c)$.

Proof: The proof of Corollary 1 follows the lines of the proof of Theorem 1, and hence, it is omitted due to space limitation. ■

V. Global Stabilization

The key challenge for the stabilization of systems with state-dependent input delay is to maintain the feasibility condition (13), i.e., keep the delay rate below one. This condition can be satisfied a priori by assuming the following.

Assumption 3: $\nabla D(X) f(X,\omega) < c$, for some $0<c<1$ and all $(X,\omega) \in \mathbb{R}^{n+1}$.

Corollary 2: Consider the plant (1) together with the controller (6)–(8). Under Assumptions 1, 2 and 3 there exist a function $\beta_{GL} \in \mathscr{KL}$ and a function $\rho_{GL} \in \mathscr{KC}_\infty$ such that

$$\Omega(t) \leq \beta_{GL}(\rho_{GL}(\Omega(0),c),t), \tag{47}$$

for all $t \geq 0$ and some $0<c<1$.

Proof: Based on Assumption 3 the condition (13) for $0<c<1$ is satisfied and hence Lemmas 1–6 apply. Using Lemmas 3 and 6 we get (41) which completes the proof. ■

VI. Examples

Example 1: We consider the following prototype scalar system with a Lyapunov-like delay

$$\dot{X}(t) = X(t) + U\left(t - X(t)^2\right). \tag{48}$$

The delay-compensating controller is $U(t) = -2P(t)$, $P(\theta) = X(t) + \int_{t-X(t)^2}^{\theta} \frac{(P(s)+U(s))ds}{1-2P(s)(P(s)+U(s))}$, for all $\theta \geq -X(0)^2$. In Fig. 1 we show the response of the system and the function $\phi(t) = t - X(t)^2$ for four different initial conditions of the state, $X(0) = 0.15, 0.25, 0.35, X^*$. With X^* we denote the critical value of $X(0)$ for the given initial condition of the input, such that, for any $X(0) \geq X^*$, the control inputs produced by the delay-compensating feedback law for positive t never reach the plant. The function $\phi(t) = t - X(0)^2e^{2t}$ has a maximum at t^* if $-\log\left(\sqrt{2X(0)^2}\right) = t^* > 0$. Since $\phi(t^*) = -\log\left(\sqrt{2X(0)^2}\right) - \frac{1}{2}$ has to be positive for the control to reach the plant, it follows $X^* = \frac{1}{\sqrt{2e}} = 0.43$.

Example 2: We consider the problem of stabilizing a mobile robot (non-holonomic unicycle) subject to an input delay that grows with the distance relative to the reference position.

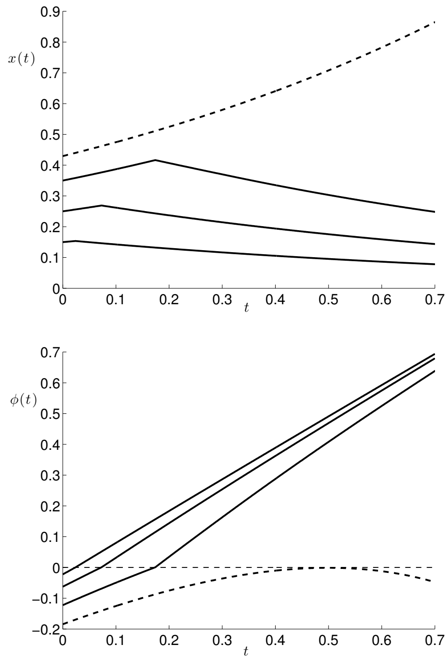

Fig. 1. Response of system (48) with initial conditions as $U(\theta)=0$ for all $-X(0)^2 \le \theta \le 0$ and four different initial conditions for the state $X(0)$.

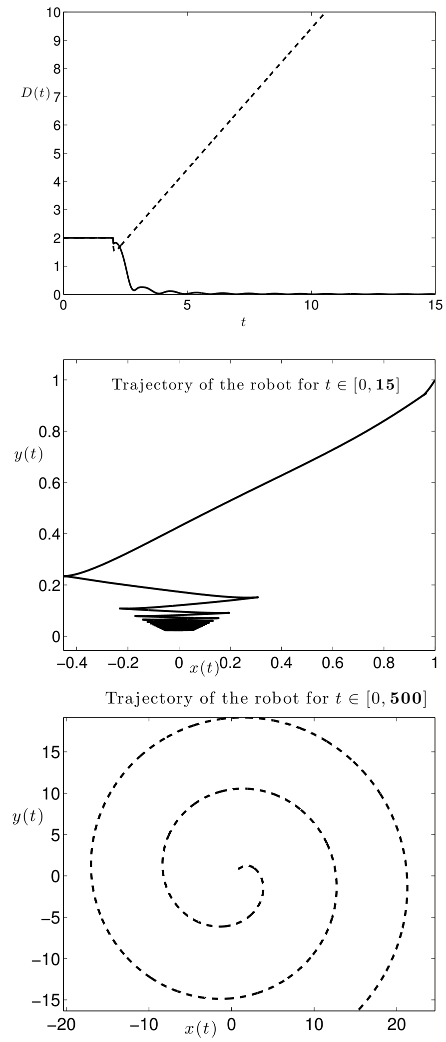

Fig. 2. The delay and the trajectory of the robot with the delay-compensating controller (top two plots) and the trajectory of the robot with the un-compensating controller (bottom plot).

The model is $\dot{x}(t) = v(t-D(x(t),y(t)))\cos(\theta(t))$, $\dot{y}(t) = v(t-D(x(t),y(t)))\sin(\theta(t))$, $\dot{\theta}(t) = \omega(t-D(x(t),y(t)))$, where $D(x(t),y(t)) = x(t)^2+y(t)^2$ and $(x(t),y(t))$ is the position of the robot, $\theta(t)$ is the heading, $v(t)$ is the speed and $\omega(t)$ is the turning rate. When $D=0$ a stabilizing controller is ([20]), $\omega(t) = -5p(t)^2\cos(3t) - p(t)q(t)\left(1+25\cos(3t)^2\right)$ and $v(t) = -p(t) + 5q(t)(\sin(3t)-\cos(3t)) + q(t)\omega(t)$, where $p(t) = x(t)\cos(\theta(t)) + y(t)\sin(\theta(t))$ and $q(t) = x(t)\sin(\theta(t)) - y(t)\cos(\theta(t))$. The predictor-based version of this controller is $\omega(t) = -5P(t)^2\cos(3\sigma(t)) - P(t)Q(t)\left(1+25\cos(3\sigma(t))^2\right) - \Theta(t)$ and $v(t) = -P(t) + 5Q(t)(\sin(3\sigma(t)) - \cos(3\sigma(t))) + Q(t)\omega(t)$, where respectively $P(t) = X(t)\cos(\Theta(t)) + Y(t)\sin(\Theta(t))$ and $Q(t) = X(t)\sin(\Theta(t)) - Y(t)\cos(\Theta(t))$, and the predictors are $X(t) = x(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)v(s)\cos(\Theta(s))ds$, $Y(t) = y(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)v(s)\sin(\Theta(s))ds$, $\Theta(t) = \theta(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)\omega(s)ds$, $\sigma(t) = t + D(X(t),Y(t))$, $\dot{\sigma}(s) = \frac{1}{1-2v(s)(X(s)\cos(\Theta(s))+Y(s)\sin(\Theta(s)))}$. The initial conditions are $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \le s \le 0$, and hence, $X(s) = Y(s) = \Theta(s) = 1$ for all $-2 \le s \le 0$. The controller "kicks in" at t_0, where $t_0 = x(t_0)^2 + y(t_0)^2$. Since $v(s) = \omega(s) = 0$ for $s<0$ we get $x(t) = y(t) = \theta(t) = 1$ for all $0 \le t \le t_0$. Hence, $t_0 = 2$. In Fig. 2 we show the delay and the trajectory of the robot for the uncompensated and the delay-compensating controller. In Fig. 3 we show the control efforts $v(t)$ and $\omega(t)$. From Fig. 2 one can observe that in the case of the uncompensated controller the delay grows aproximately linearly in time and the vehicle's trajectory is a divergent Archimedean spiral.

Example 3: We consider the following scalar system

$$\dot{X}(t) = \frac{X(t) + U(t-D(X(t)))}{U(t-D(X(t)))^2+1}, \tag{49}$$

with $D(X(t)) = \frac{1}{4}\log\left(X(t)^2+1\right)$. Young's inequality gives $\dot{D}(X(t)) \le \frac{6}{7}$. System (49) satisfies Assumptions 1–3. Hence, Corollary 2 applies. The control law is $U(t) = -2P(t)$, where $P(t) = X(t) + \int_{t-D(X(t))}^{t} \frac{2\left(P(\theta)^2+1\right)(P(\theta)+U(\theta))d\theta}{\Gamma(\theta)}$ and $\Gamma(\theta) = 2\left(U(\theta)^2+1\right)\left(P(\theta)^2+1\right) - P(\theta)\left(P(\theta)+U(\theta)\right)$. As one can observe from Fig. 4, initially $X(t)$ runs in open loop and grows exponentially. During this time, $D(t)$ grows roughly linearly since $D(X(t)) = \frac{1}{4}\log\left(X(t)^2+1\right)$. This goes on until the predictor control "kicks in", which is at the time when $t^* = \frac{1}{4}\log\left(1+X(0)^2e^{2t^*}\right) = 0.4835$. For $t > t^*$, $X(t)$ no longer grows exponentially and the controller starts bringing it back to zero. As $X(t)$ decays according to the target system $\dot{X}(t) = -\frac{X(t)}{1+4X(t)^2}$, the delay $D(t)$ also decays. Starting from a large $X(t^*)$, $X(t)^2$ first decays roughly linearly in t, making

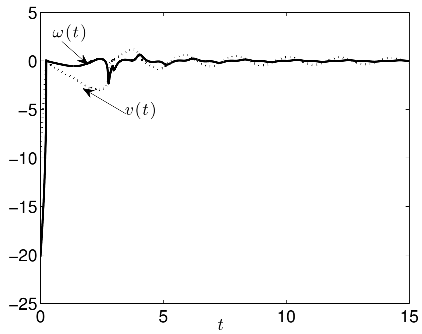

Fig. 3. The control efforts $v(t)$ (dot line) and $\omega(t)$ (solid line) for the non-holonomic unicycle with the delay-compensating controller.

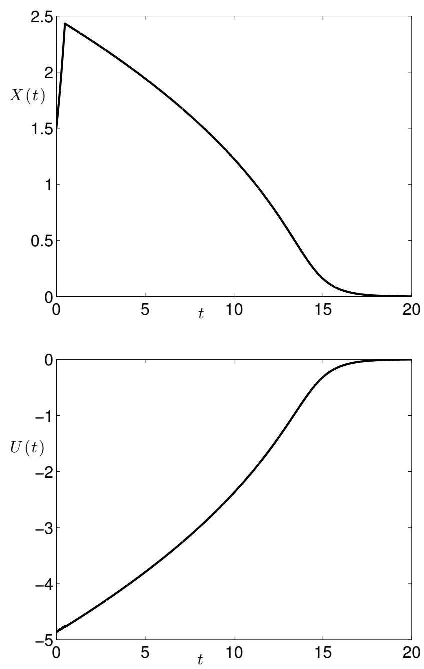

Fig. 4. Response of system (49) with the predictor controller and initial conditions as $X(0) = 1.5$ and $U(\theta) = 0$ for all $-\frac{1}{4}\log\left(X(0)^2+1\right) \leq \theta \leq 0$.

the decay of $D(t)$ logarithmic. When $X(t)$ becomes small, its decay becomes exponential and the decay of $D(t)$ is also exponential. The behavior of $U(t)$ is governed by $-2P(t)$ for $t > t^*$. Since $P(t)^2$ initially decays linearly in t and later $P(t)$ decays exponentially, $U(t)$ decays with the same pattern as $P(t)$.

VII. Conclusions

The paper's key design idea is how to define the predictor state (5). The gradient-of-delay term in the denominator of (5) is the result of a change in the time variable, which allows the predictor to be defined using an integral from the known delayed time $\phi(t) = t - D(X(t))$ until present time t, rather than an integral from the present time t until the unknown prediction time $\sigma(t) = t + D(P(t))$.

Though the stability results in the paper are not global, the size of the delay is not limited. By examining the estimates in detail, the reader can observer that, when $D(X)|_{X=0}$ is large, namely when the system is regulated to an equilibrium where the delay is necessarily large, the stability estimates dictate that the initial conditions of the state and the input be small. However, no restrictions on $D(X)|_{X=0}$ are imposed. A tradeoff exists between the size of the state-dependent delay and the achievable region of attraction in closed loop.

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