Compact Sets in the Graph Topology and Applications to Approximation of System Design

Yutaka Yamamoto*   Mathukumalli Vidyasagar**

Abstract— The graph topology plays a central role in characterizing the robustness of feedback systems. In particular, it provides necessary and sufficient conditions for the continuity properties of the transfer matrices of stabilized closed-loop systems. It is possible to derive stronger conclusions by confining our attention to a *compact* set of controllers. Specifically, if a family of plants is stabilized by each controller belonging to a compact set of controllers, then the closed-loop transfer matrix is *uniformly continuous*. However, at present a precise characterization of compactness in the graph topology is not available. That is the topic of the present paper. In general it appears difficult to give a necessary and sufficient conditions for a set to be compact. Hence we give a necessary condition and a sufficient condition, and discuss the gap between the two. The necessary condition is standard, while the proof of the sufficient condition is based on three major theorems in analysis: the Baire category theorem, Montel's theorem on normal families of analytic functions, and the corona theorem for H^∞. Finally, it is shown how the notion of a compact set of controllers can be applied to the problem of approximate design and performance estimation for sampled-data control systems.

I. INTRODUCTION

Many control system design problems inherently or inevitably involve system approximation. If nothing else, the system model itself is an approximation since no model can be exact. One then encounters the following question: Let a model M be an approximation of system Σ in some sense. Let K be a controller that is designed for M. How do we guarantee the performance of the closed-loop system when we connect K to Σ?

This is of course a question of robustness, typically robust stability. But normally, it is placed in the context of an analysis problem, to guarantee that the designed controller satisfies certain performance criteria. Or else, one can formulate a synthesis problem in such a way that a certain performance estimate be satisfied. But what if we have only a rough estimate on the set of controllers and still wish to guarantee overall performance and "convergence" of the closed-loop system? We need to guarantee some uniform estimate for a closed-loop performance for a certain prescribed class of controllers out of which a desired controller is to be chosen.

To deal with such a problem, the notion of the graph topology (or the gap metric) has been introduced in [16],

* yy@i.kyoto-u.ac.jp, Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, JAPAN. Author to whom all correspondence should be addressed.
** m.vidyasagar@utdallas.edu, Erik Jonsson School of Engineering & Computer Science, Room 3.908A, ECSS, The University of Texas at Dallas, Richardson, TX 75080-3021, U.S.A.

[17], [22]. Roughly speaking, the graph topology is the weakest topology on controllers that makes closed-loop stability a robust property. Suppose $\{P_\lambda\}$ is a family of plant models parametrized by some parameter λ assuming values in some topological space Λ, and let P_{λ_0} denote the 'nominal' plant model. Then there exists a controller that stabilizes P_λ for all λ in some neighborhood of λ_0, and in addition the closed-loop transfer function is continuous at λ_0, if and only if the open-loop transfer function P_λ is continuous in the graph topology at λ_0. The graph topology is metrizable and there are several metrics that generate the same topology, one of which is the gap metric.

To be more specific, consider the following rather specialized, but quite realistic, problem: Suppose that we are given a sequence of plant approximants P_n, $n = 1, 2, \ldots$, that converges to the "true" plant P, and we have a prescribed set $\mathscr{K}$ from which a controller K may be chosen. Can we guarantee the performance of the closed-loop in the limit? The essence of the problem here is that we do not know a priori the controller K until it is chosen; but on the other hand, we do wish to guarantee the closed-loop performance for the whole class of problems where K is chosen from $\mathscr{K}$. In other words, we aim at guaranteeing the convergence of the *design problem*. Precisely stated, the question is the following: Can we ensure that the closed-loop transfer matrix which is denoted by $H(P_n, K)$ converges *uniformly* to $H(P, K)$ for *every* controller $K \in \mathscr{K}$? If the set of controllers $\mathscr{K}$ is 'too large' then uniform convergence will not hold. Thus the set $\mathscr{K}$ has to be 'sufficiently small' in order for uniform convergence to hold. It is easy to see that a simple *sufficient condition* for such uniform convergence is that the set $\mathscr{K}$ should be *compact* in the graph topology. That is the motivation for studying the question of compactness in the graph topology.

Such a problem arises naturally in varied situations where we have approximate system models. One example is the case of fast-sampling/fast-hold approximations for sampled-data systems (see Section IV-A below); yet another can arise often in discrete approximation of distributed parameter systems. For such systems, we often execute system design based on approximate models, and wish to guarantee the performance of the obtained controller in the limit.

This problem in itself presents an interesting mathematical challenge. While it seems difficult to give a complete necessary and sufficient condition, we can give a necessary condition, and an interesting sufficient condition, based on the notion of boundedness in the space H^∞. Using such a

notion of compactness, we will prove some results on the estimate of the limiting behavior of approximate design.

II. PRELIMINARIES

In this subsection we give a very brief introduction to the graph topology. Complete details can be found in [18, Chapter 7]. We begin with the extremely important notion of coprimeness.

A. Coprimeness

Let $\mathbb{B}$ denote a commutative Banach algebra with identity over the complex field $\mathbb{C}$. Actually the graph topology can be defined in far broader settings, but this is sufficient for our purposes. Examples of commutative Banach algebras include H^∞, the set of functions on $\mathbb{C}$ that are analytic on $\mathbb{D}$ and bounded on $\overline{\mathbb{D}}$, and the disc algebra, consisting of the functions in H^∞ that are not merely bounded on $\overline{\mathbb{D}}$ but also continuous on $\overline{\mathbb{D}}$. Suppose $n, d \in \mathbb{B}$. Then we say that n, d are *coprime* or *Bézout* if there exist other elements $x, y \in \mathbb{B}$ such that

$$xn + yd = 1,$$

where $1 = 1_{\mathbb{B}}$, the identity element of $\mathbb{B}$. More generally, suppose $N \in \mathbb{B}^{p \times m}, D \in \mathbb{B}^{m \times m}$. Then we say that N, D are *right coprime* or *right Bézout* if there exist matrices X, Y over $\mathbb{B}$ of compatible dimensions such that

$$XN + YD = I,$$

where I denotes the identity matrix over $\mathbb{B}$.

It is possible to give an abstract necessary and sufficient condition for two matrices to be coprime, using the notion of the Gel'fand transform. Let $\mathcal{M}$ denote the space of maximal ideals of $\mathbb{B}$. For every $a \in \mathbb{B}$, and $I \in \mathcal{M}$, $[a]_I$ denotes the coset $a + I$ in the quotient algebra $\mathbb{B}/I$. By the well-known Gel'fand-Mazur theorem [3], $\mathbb{B}/I \cong \mathbb{C}$. Now suppose $a \in \mathbb{B}$. As a result, for each $I \in \mathcal{M}$, the coset $a + I$ is (or more accurately, can be uniquely identified with) a complex number, which is denoted by $\hat{a}_I$. Note that $\hat{a}_I$ and $[a]_I$ are two different ways of denoting the same object. The association $a \mapsto \hat{a}_I$ as I varies over $\mathcal{M}$ maps the element a into a complex-valued function on $\mathcal{M}$ and is denoted by $\hat{a}$. This mapping is called the **Gel'fand transform** of a. The so-called 'carrier space topology' on $\mathcal{M}$ is the weakest topology on $\mathcal{M}$ in which the mapping $\hat{a} : \mathcal{M} \to \mathbb{C}$ is continuous for every $a \in \mathbb{B}$. It is customary to denote the set $\mathcal{M}$ with the carrier space topology by Ω, and individual elements of Ω (which are actually maximal ideals of $\mathbb{B}$) by ω. One of the most useful results of Gel'fand theory is that, in the carrier-space topology, the set Ω is compact [3]. Moreover, if the Banach algebra $\mathbb{B}$ is 'semi-simple', meaning that the intersection of all maximal ideals of $\mathbb{B}$ consists of the singleton set $\{0\}$, then the Gel'fand transform is a one-to-one mapping from $\mathbb{B}$ into $C(\Omega)$, the set of continuous functions on Ω.

Yet another very important result in Gel'fand theory is that the spectrum of an element $a \in \mathbb{B}$ (i.e., the set of λ for which $\lambda 1 - a$ is not invertible) consists precisely of the set $\{\hat{a}(\omega), \omega \in \Omega\}$. Hence $\lambda 1 - a$ has an inverse in $\mathbb{B}$ if and only if $\hat{a}(\omega) \neq \lambda$ for every $\omega \in \Omega$. In particular a has an inverse in $\mathbb{B}$ if and only if $\hat{a}(\omega) \neq 0 \, \forall \omega$. This leads to the following easy consequence [18, Lemma 8.1.9]:

Lemma 2.1: Let $a_1, \ldots, a_n \in \mathbb{B}$. Then there exist $x_1, \ldots, x_n \in \mathbb{B}$ such that

$$\sum_{i=1}^{n} x_i a_i = 1 \tag{1}$$

if and only if

$$\text{rank}[\hat{a}_1(\omega), \ldots, \hat{a}_n(\omega)] = 1 \tag{2}$$

for every $\omega \in \Omega$.

Observe now that a pair (N, D) over $\mathbb{B}$ is right coprime if and only if it is left-invertible over $\mathbb{B}$. Hence we now give the following general characterization of left invertibility [18, Theorem 8.1.12]:

Theorem 2.2: Suppose $A \in \mathbb{B}^{n \times m}$ with $n \geq m$. Then A admits a left inverse in $\mathbb{B}^{m \times n}$ if and only if

$$\text{rank}[\hat{A}(\omega)] = m \tag{3}$$

for every $\omega \in \Omega$.

A brief indication of its proof will be given in Appendix.

B. The Graph Topology

Given $\mathbb{B}$, let $\mathbb{F}$ denote the associated field of fractions. So we can think of $\mathbb{B}$ as the set of 'stable' systems, while $\mathbb{F}$ is the set of 'unstable' systems. Since $\mathbb{B}$ is a Banach algebra, there is a natural topology on $\mathbb{B}$. The graph topology extends the topology on $\mathbb{B}$ to a topology on the associated field $\mathbb{F}$ of fractions. Specifically, suppose $n, d \in \mathbb{B}$ are coprime with $d \neq 0$, and let $p = n/d$. Then the set of all fractions n'/d', where n', d' belong to some open balls around n, d respectively, and in addition the ball around d' does not contain zero, is defined as a neighborhood of p. In the case of multi-input, multi-output systems, suppose N, D are matrices over $\mathbb{B}$, with D being square and nonsingular (meaning that its determinant is not the zero element of $\mathbb{B}$). Then $P = ND^{-1}$ is well-defined as a matrix over $\mathbb{F}$. Now suppose that in addition N, D are right Bézout. Then the set of all ratios $N'(D')^{-1}$, where N', D' belong to open balls around N, D respectively, and in addition, all matrices in the ball around D are nonsingular, constitutes a neighborhood of P.

As mentioned in the Introduction, the significance of the graph topology is that it is the weakest topology in which feedback stability is a continuous property. Specifically we have the following following [17]:

Theorem 2.3: Suppose $\{P_\lambda\}$ is a family of plant models parametrized by some parameter λ assuming values in some topological space Λ, and let P_{λ_0} denote the 'nominal' plant model. Then there exists a controller that stabilizes P_λ for all λ in some neighborhood of λ_0, and in addition the closed-loop transfer function is continuous at λ_0, if and only if the open-loop transfer function P_λ is continuous in the graph topology at λ_0.

III. Compact sets in the graph topology

As stated in the Introduction, the topic of study in the present paper is the compactness of sets in the graph topology. The topic can be motivated quite simply. As we have seen in Theorem 2.3, feedback stability is a continuous property in the graph topology. Thus if P_n is a sequence of plants converging to some P_0 in the graph topology, and if K is a controller that stabilizes P_0, then K also stabilizes P_n for all sufficiently large n, and in addition, the closed-loop transfer function $H(P_n, K)$ converges to $H(P_0, K)$ as n approaches infinity. As noted in the Introduction, however, while this convergence guarantees the convergence of $H(P_n, K)$ for a chosen K, the problem is that we do not know a priori which K will be chosen until the synthesis is done. Instead, suppose we would like to guarantee the *convergence of the design problem before choosing a controller*. That is to say, we wish to guarantee the convergence of the closed-loop performance for a prefixed class $\mathcal{K}$ of controllers. In particular, this entails us to guarantee $H(P_n, K)$ to converge *uniformly* to $H(P_0, K)$ for $K \in \mathcal{K}$. Since a continuous function on a compact set is uniformly continuous, it readily follows that the desired uniform convergence behavior holds true if the set of controllers $\mathcal{K}$ is compact in the graph topology.

Next we will present conditions for compact sets in the graph topology.

We start with the following easy necessary condition:

Theorem 3.1: Let (N_Λ, D_Λ) be a family of functions in $(H^\infty)^{p \times m} \times (H^\infty)^{m \times m}$. Suppose (N_Λ, D_Λ) is compact. Then it is a closed bounded subset of $(H^\infty)^{p \times m} \times (H^\infty)^{m \times m}$.

This follows readily from the standard fact that in *any* normed linear space, an unbounded set cannot be compact.

The sufficiency part is more subtle:

Theorem 3.2: Let (N_Λ, D_Λ) be a family of functions in $(H^\infty)^{p \times m} \times (H^\infty)^{m \times m}$. Suppose that there exists an open subset $U \subset \mathbb{C}$ that contains the closure $\overline{\mathbb{D}}$ such that (N_Λ, D_Λ) is a subset of $(H^\infty)^{p \times m}(U) \times (H^\infty)^{m \times m}(U)$ and is a closed bounded subset there. Then (N_Λ, D_Λ) is a compact subset of $(H^\infty)^{p \times m}(\mathbb{D}) \times (H^\infty)^{m \times m}(\mathbb{D})$.

In other words, the above conditions require that (N_Λ, D_Λ) constitutes a closed bounded subset in the topology of $H^\infty(U)$ (which is a smaller space than $H^\infty(\mathbb{D})$), and then it is compact. The gap between boundedness over $\mathbb{D}$ and over U yields the gap between necessity (which requires boundedness only on $\mathbb{D}$) and sufficiency that requires boundedness on $U \supset \overline{\mathbb{D}}$.

We give a proof of this theorem in two steps: first for the SISO case, i.e., $m = p = 1$, for simplicity, and then generalize it to the MIMO case.

A. SISO case

Let us start with the following lemma:

Lemma 3.3: The subset of pairs (n, d) that are coprime is an open subset of $H^\infty \times H^\infty$.

Proof Take any (n, d) that is coprime. Then by the well-known corona theorem [4], [8], there exists a positive con-

stant c such that

$$|n(z)| + |d(z)| \geq c > 0, \quad \forall z \in \mathbb{D}.$$

Take ε neighborhoods of n and d, and let $\tilde{n}$ and $\tilde{d}$ belong to these neighborhoods, respectively. It follows that $|\tilde{n}(z)| \geq |n(z)| - \varepsilon$, and similarly for $|\tilde{d}(z)|$. They by taking $\varepsilon = c/4$, we see that

$$|\tilde{n}(z)| + |\tilde{d}(z)| \geq c/2 > 0$$

for every $z \in \mathbb{D}$. Then $(\tilde{n}, \tilde{d})$ is coprime again by the corona theorem. That is, the set of all coprime pairs constitute an open subset of $H^\infty \times H^\infty$. □

To prove the compactness of (N_Λ, D_Λ), we assume, without loss of generality, that the cardinality of the set $(N_\Lambda, D_\Lambda) = \infty$; for otherwise, (N_Λ, D_Λ) is clearly compact. Then (N_Λ, D_Λ) is a complete metric space as a closed subset of $H^\infty \times H^\infty$.

Now let $\{(n_\lambda, d_\lambda)\}$ be any infinite sequence in (N_Λ, D_Λ). Since (N_Λ, D_Λ) is a bounded subset of $(H^\infty)^{p \times m}(U) \times (H^\infty)^{m \times m}(U)$, $\{(n_\lambda, d_\lambda)\}$ constitutes a normal family in $(H^\infty)^{p \times m}(U) \times (H^\infty)^{m \times m}(U)$ by Montel's theorem [6]. That is, there exists a subsequence $\{(n_{\lambda_p}, d_{\lambda_p})\}_{p=1}^\infty$ that is convergent to $(n_\infty, d_\infty) \in (H^\infty)^{p \times m}(U) \times (H^\infty)^{m \times m}(U)$ uniformly on every compact subset of U. But $\overline{\mathbb{D}} \subset U$ is a compact subset of U in itself, so this convergence is uniform on $\overline{\mathbb{D}}$. This means that $(n_{\lambda_p}, d_{\lambda_p})$ converges to (n_∞, d_∞) in the H^∞ norm of $H^\infty(\mathbb{D})$. However, this by itself is not sufficient to establish the desired result, because (n_∞, d_∞) might not be a coprime pair. For that purpose, further reasoning is required.

Let M be the closure of $\{(n_{\lambda_p}, d_{\lambda_p})\}_{p=1}^\infty$, i.e., $M = \{(n_{\lambda_p}, d_{\lambda_p})\}_{p=1}^\infty \cup \{(n_\infty, d_\infty)\}$. Note that M could in principle be strictly smaller than the closure of the *original* collection $\{(n_\lambda, d_\lambda)\}$. Then M is a complete metric space as a closed subset of (N_Λ, D_Λ). Moreover, since $(n_{\lambda_p}, d_{\lambda_p})$ converges to (n_∞, d_∞), it follows that $(n_{\lambda_p}, d_{\lambda_p})$ is the only possible limit point of the set M.

Now let

$$\mathscr{S} := \{(n, d) \in M : (n, d)\text{: coprime}\}.$$

By Lemma 3.3, $\mathscr{S}$ is an open subset of M with respect to its relative topology. As an open subset of a complete metric space, $\mathscr{S}$ is in itself a space of second category by the Baire category theorem [13].

For each positive integer n, define a subset C_n of $\mathscr{S}$ as

$$C_n := \{(n, d) \in \mathscr{S} \mid \inf_{z \in \mathbb{D}} |n(z)| + |d(z)| \geq 1/n\}.$$

C_n is clearly a closed subset of M.

Now observe that

$$\mathscr{S} = \cup_{n=1}^\infty C_n. \tag{4}$$

Since $\mathscr{S}$ is of second category, at least one, say C_m, contains an open subset. Hence it contains an infinitely many element. Moreover, such infinitely many elements should converge to (n_∞, d_∞) because they constitute a subset of M which is convergent to (n_∞, d_∞). This means (N_Λ, D_Λ) is compact, and completes the proof. □

B. MIMO case

We first quote the following result:

Theorem 3.4 (Fuhrmann [7], Treil[14], Vidyasagar[18]):
Let $N \in (H^\infty)^{p \times m}$ and $D \in (H^\infty)^{m \times m}$. Then there exist $X \in (H^\infty)^{m \times p}$ and $Y \in (H^\infty)^{m \times m}$ such that the Bézout identity

$$XN + YD = I \tag{5}$$

is satisfied if and only if

$$\inf_{z \in \mathbb{D}} \sigma_{\min} \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} > 0 \tag{6}$$

where $\sigma_{\min}$ denotes the minimum singular value.

We give a brief indication of the proof in the Appendix.

Let us now give a proof for Theorem 3.2 for the MIMO case.

We start with the following generalization of Lemma 3.3:

Lemma 3.5: The subset of pairs (N, D) that are coprime in the sense of (5) is an open subset of $(H^\infty)^{p \times m} \times (H^\infty)^{m \times m}$.

Sketch of Proof

By Theorem 3.4, there exists $c > 0$ such that

$$\sigma_{\min} \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} \geq c > 0$$

for all $z \in \mathbb{D}$. Take sufficiently small neighborhoods of N and D, and let $\tilde{N}$ and $\tilde{D}$ belong to these neighborhoods, respectively. Then similarly as in the proof of Lemma 3.3, we readily see that

$$\inf_{z \in \mathbb{D}} \sigma_{\min} \begin{bmatrix} \tilde{N}(z) \\ \tilde{D}(z) \end{bmatrix} > 0$$

for all $\tilde{N}$ and $\tilde{D}$ belonging to these neighborhoods. It follows that $(\tilde{N}, \tilde{D})$ is coprime again by Theorem 3.4, and this completes the proof. □

Let us again assume that $\#(N_\Lambda, D_\Lambda) = \infty$ without loss of generality. Then (N_Λ, D_Λ) is a complete metric space as a closed subset of $(H^\infty)^{p \times m} \times (H^\infty)^{m \times m}$.

The rest of the proof can be almost a verbatim repetition of that for the SISO case. That is,

1) We take any infinite sequence in (N_Λ, D_Λ), take its convergent subsequence (convergent uniformly on $\mathbb{D}$ due to the boundedness in $H^\infty(U)$) according to Montel's theorem.

2) Complete the subsequence as M by adding the limit point to it, making the closure again a complete metric space, and observing that the set M can have only one limit point.

3) Consider a subset $\mathscr{S}$ consisting of coprime pairs.

Then by Lemma 3.5, $\mathscr{S}$ becomes an open subset, which is in in itself a space of second category by the Baire category theorem.

Now define a subset C_n of $\mathscr{S}$ as

$$C_n := \{(N, D) \in \mathscr{S} \mid \inf_{z \in \mathbb{D}} \sigma_{\min} \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} \geq 1/n\},$$

and then C_n is a closed subset of M.
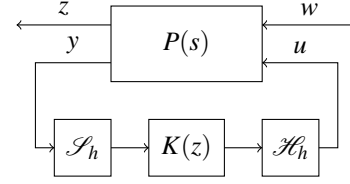

Fig. 1. Sampled-Data Control System

The rest is exactly the same. We cover $\mathscr{S}$ with C_n as

$$\mathscr{S} = \cup_{n=1}^\infty C_n, \tag{7}$$

and extract a convergent subsequence there. This completes the proof. □

Remark 3.6: It may appear odd or possibly unnatural that we require boundedness in $H^\infty(U)$ rather than $H^\infty(\mathbb{D})$. Let us first note that boundedness in $H^\infty(\mathbb{D})$ cannot be a sufficient condition. For otherwise, it would imply that a closed unit ball of $H^\infty(\mathbb{D})$ (which is a Banach space) would become a compact set. However, this contradicts a well-known result in functional analysis, i.e., every normed linear space whose closed unit ball is compact must be necessarily a finite-dimensional space; see, e.g., [15]. On the other hand, there are topological vector spaces in which every closed bounded set is compact. Montel's theorem [6] guarantees that the space of analytic functions $H(\mathbb{D})$ is such a space. In fact, every infinite sequence of $H(\mathbb{D})$ contains a subsequence that is convergent on every compact subset of $\mathbb{D}$, but this does not yield uniform convergence on the whole of $\mathbb{D}$ (otherwise it will yield a contradiction as above). The boundedness in $H^\infty(U)$ guarantees such uniformity because $\overline{\mathbb{D}}$ is a compact subset of U. The gap between the boundedness on $\mathbb{D}$ (i.e., in $H^\infty(\mathbb{D})$) and the boundedness on U (i.e., in $H^\infty(U)$) thus yield the gap between necessity and sufficiency here.

As an aside we point out that the set $\mathscr{A}(\mathbb{D})$ of all functions that are analytic over $\overline{\mathbb{D}}$, with no additional assumptions of being bounded over $\overline{\mathbb{D}}$, is an example of a 'Bezout domain'. Thus $\mathscr{A}(\mathbb{D})$ a commutative ring with identity in which every finitely generated ideal is principal. As a result, every finite collection of elements in $\mathscr{A}(\mathbb{D})$ has a greatest common divisor. However, $\mathscr{A}(\mathbb{D})$ is *not* a principal ideal domain (PID). Consequently, an *infinite* collection of elements in $\mathscr{A}(\mathbb{D})$ need not have a greatest common divisor.

IV. APPLICATION TO APPROXIMATE SYSTEM DESIGN

A. Approximation in sampled-data systems

Consider the sampled-data control system in Fig. IV-A consisting of a continuous-time generalized plant $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and a discrete-time controller K with sampler $\mathscr{S}_h$ and zero-order hold $\mathscr{H}_h$ with sampling period h. The closed-loop transfer operator from w to z will be denoted by $\mathscr{T}_{zw}(K)(z)$.

Assume that P_{21} is strictly proper, which is necessary to assure that sampling is well-defined. Likewise, we assume that P_{11} and P_{12} are also strictly proper.
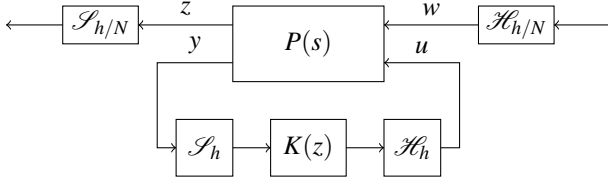
Fig. 2.   Sampled-Data Control System

Whereas the sampled-data H^∞ controller design problem of Fig. IV-A can be reduced to a norm-equivalent discrete-time H^∞ design problem [1], [5], [9], discretization via fast-sampling and fast-hold as studied in [10], [11], [21] is practically effective and simple in obtaining a solution for this problem.

Let us thus consider the fast-sampling approximation shown in Fig. IV-A for this design problem. The N-step closed-loop transfer operator from w to z will be denoted by $\mathscr{T}_{zw}^N(K)(z)$. The fast hold device $\mathscr{H}_{h/N}$ approximates inputs with step functions with step size h/N, and the fast-sampling component $\mathscr{S}_{h/N}$ approximates the output by taking samples with the faster rate h/N. For detailed formulae, see, e.g., [10], [21].

It is proven in [21] that the frequency response gain of such approximants converges to that of the limit uniformly for $0 \le \omega \le 2\pi/h$, *for a fixed* controller K.

Finally, let us note the following lemma on the continuity of the closed-loop operator with respect to K, which is nothing but a consequence of the graph topology [17]:

Lemma 4.1: Consider the closed-loop operator $\mathscr{T}_{zw}(K)$ where K is assumed to be stabilizing. Then $\mathscr{T}_{zw}(K)$ is continuous in K with respect to the graph topology.

Although this is sufficient for analysis purposes, we still need to go one step further. In synthesis, we do not know in advance which controller will enter in the closed-loop. The convergence result of [21] requires that the controller be fixed in advance, and this assumption is (in effect) not satisfied for synthesis problems. To this end, we must guarantee the *uniform* convergence, with respect to controller K, of the norms of the fast-sampling approximants if we are going to use such approximants in designing the controller. This motivates the following theorem.

Theorem 4.2: Let $\mathscr{K}$ be a set of controllers K such that i) every $K \in \mathscr{K}$ is stabilizing for all approximating plants, and ii) $\mathscr{K}$ is compact with respect to the graph topology. Then the frequency response gain $\left\| \mathscr{T}_{zw}^n(K)(e^{j\omega h}) \right\|$ of the n-step fast-sampling approximant $\mathscr{T}_{zw}^n(K)$ converges to $\left\| \mathscr{T}_{zw}(K)(e^{j\omega h}) \right\|$ uniformly in $K \in \mathscr{K}$. This convergence is also uniform in $\omega \in [0, 2\pi/h)$.

Proof   Fix $\varepsilon > 0$, and take any $K \in \mathscr{K}$. By the convergence result of [21], there exists N such that

$$\left| \left\| \mathscr{T}_{zw}^n(K)(e^{j\omega h}) \right\| - \left\| \mathscr{T}_{zw}(K)(e^{j\omega h}) \right\| \right| < \varepsilon \text{ for all } n \ge N,$$

and this is uniform in ω. We will thus omit the dependence on ω below. Take the least such N and name it $N(K, \varepsilon)$.

Let $d(\cdot, \cdot)$ denote a metric that defines the graph topology. Since $\left\| \mathscr{T}_{zw}^n(K) \right\| - \left\| \mathscr{T}_{zw}(K) \right\|$ is continuous with respect to K as Lemma 4.1 below shows, there exists a neighborhood $B(K, \delta) := \{ K' : d(K', K) < \delta \}$ of K such that

$$\left| \left\| \mathscr{T}_{zw}^n(K') \right\| - \left\| \mathscr{T}_{zw}(K') \right\| \right| < \varepsilon$$

for all $n \ge N(K, \varepsilon)$ and $K' \in B(K, \delta)$. This yields a covering of the controller set:

$$\mathscr{K} = \cup_{K \in \mathscr{K}} B(K, \delta).$$

By the compactness of $\mathscr{K}$, there exists a subcovering

$$\mathscr{K} = B(K_1, \delta_1) \cup \cdots \cup B(K_m, \delta_m).$$

Taking $N_{\max} := \{ N(K_1, \varepsilon), \ldots, N(K_m, \varepsilon) \}$, we readily have that $n \ge N_{\max}$ implies

$$\left| \left\| \mathscr{T}_{zw}^n(K) \right\| - \left\| \mathscr{T}_{zw}(K) \right\| \right| < \varepsilon \tag{8}$$

for all $K \in \mathscr{K}$. $\square$

Remark 4.3: A preliminary version of this theorem has been obtained in [20]. However, a crucial assumption there was that the controller set was constrained to be *stable*. As a result, the topology for the controller set was taken in the H^∞-norm. To remedy this problem, we have now modified the proof to be compatible with the graph topology. The question, however, still remains: what are the compact sets in the graph topology. This was indeed the basic motivation for the present study.

B. General approximation results

As briefly mentioned in the Introduction, distributed parameter systems are often discretized in the spatial domain and this yields a sequence of finite-dimensional systems (with increasing dimensions) that is shown to "converge" to the original system. For example, the so-called averaging approximation for delay-differential systems is such an example [2]; see also [19] for a more general setting on finite-dimensional approximations. Both these results are in some sense for open-loop plants. For closed-loop plants, a relevant approximation result is found in [12]. It is shown in this paper that if a controller is designed on the basis of a sequence of finite-dimensional Galerkin-type approximations to an infinite-dimensional system, then the resulting closed-loop responses converge to that of the infinite-dimensional system with that particular controller.

Let us now state the following theorem:

Theorem 4.4: Let P_n be a sequence of plants that converges to a plant P_0 in the graph topology, and let $\mathscr{K}$ be a set of controllers K such that i) every $K \in \mathscr{K}$ is stabilizing for all approximating plants, and ii) $\mathscr{K}$ is compact with respect to the graph topology. Then $H(P_n, K)$ converges uniformly to $H(P_0, K)$ for $K \in \mathscr{K}$.

Sketch of Proof

The proof is entirely similar to that of Theorem 4.2. Then take $\varepsilon > 0$ and take any $K \in \mathscr{K}$. There exists N such that

$$d(H(P_n, K), H(P_0, K)) < \varepsilon$$

for all $n \geq N$. By Lemma 4.1, there exists a neighborhood $B(K, \delta) := \{K' : d(K', K) < \delta\}$ of K such that

$$d(H(P_n, K'), H(P_0, K)) < \varepsilon$$

for all $n \geq N(K, \varepsilon)$ and $K' \in B(K, \delta)$. Using compactness, find a finite subcovering, and this readily yields uniformity of convergence as the proof of Theorem 4.2. $\square$

APPENDIX: PROOF OF THEOREM 3.4

For the sake of completeness, we here give a proof of the generalization Theorem 3.4 based on the result of the scalar case. The proof here goes along the line of the treatment in [18, Chapter 8].

Let R be a Gel'fand algebra, i.e., complex commutative Banach algebra with identity. H^∞ is a well-known example. We defer the proof until the very end of Appendix, and proceed to the proof of Theorem 3.4.

An easy consequence of Theorem 2.2 is the following:

Corollary 4.5: Suppose N and D have the same number of columns. Then N and D are right coprime if and only if

$$\begin{bmatrix} N \\ D \end{bmatrix}$$

has full column rank for every $I \in \mathcal{M}$.

Theorem 3.4 then easily follows from this and the corona theorem:

Proof of Theorem 3.4

For every $a \in \mathbb{D}$, let I_a denote $\{f \in H^\infty(\mathbb{D}) : f(a) = 0\}$. I_a is easily seen to be a maximal ideal of $H^\infty(\mathbb{D})$. The corona theorem [8] asserts that $\{I_a : a \in \mathbb{D}\}$ constitute a dense subset of $\mathcal{M}$. Hence (3) is satisfied for H^∞ if and only if

$$\inf_{z \in \mathbb{D}} \sigma_{\min} \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} > 0,$$

that is, if and only if (6) holds. $\square$

It remains only to prove Theorem 2.2.

Sketch and Indication of Proof of Theorem 2.2

The necessity is obvious.

For sufficiency, first note that the case $m = n$ is obvious, since in this case condition (3) means that $\det A$ does not belong to any maximal ideal, and hence is invertible.

For the case $m < n$, we only give an indication and the flavor. The detailed proof can be found in [18, Theorem 8.12]. Condition (3) implies that for every maximal ideal I there exists a full-sized minor (size m) that does not vanish over R/I. Since there are only finitely many full-sized minors $f_1, \ldots, f_L$, they together satisfy

$$\mathrm{rank}[[f_1]_I, \ldots, [f_L]_I] = 1$$

and by Lemma 2.1 they generate R, i.e.,

$$\sum x_i f_i = 1.$$

For each f_j, define B_j as a suitable cofactor matrix corresponding to the minor f_j such that

$$B_j A = f_j I.$$

Define $B := \sum x_j B_j$. Then BA is easily seen to be equal to I, and this completes the proof. $\square$

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