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Model Falsification using SVOs for a Class of Discrete-Time Dynamic Systems: A Coprime Factorization Approach

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Abstract— **This paper introduces a new method for model falsification using Set-Valued Observers (SVOs), which can be applied to a class of discrete Linear Time-Invariant (LTI) dynamic systems with time-varying model uncertainties. In comparison with previous studies, the main advantages of this approach are as follows: the computation of the convex hull of the set-valued estimate of the state can be avoided under certain circumstances; in order to guarantee convergence of the set-valued estimate of the state, the required number of previous set-valued estimates is at most as large as the number of states of the nominal plant; it provides a straightforward *non-conservative* method to falsify uncertain models of dynamic systems, including open-loop unstable plants. The results obtained are illustrated in simulation.**

I. Introduction

The problem of *model falsification* or *model invalidation* is relevant in a wide range of applications, such as Fault Detection and Isolation (FDI), Multiple-Model Adaptive Control (MMAC) and model identification methodologies. In any of these situations, the key aspect to take into account is the fact that a model can never be *validated* in practice, as stressed in [1].

Unmodeled and/or unknown dynamics (present in virtually every physical system) and adverse exogenous disturbances, can result in erroneous model falsification. Hence, it is imperative that the formulation of the problem takes into account these uncertainty terms, in order to avoid undue invalidation of models. As an example, the solution proposed in [1] for uncertain Linear Time-Invariant (LTI) systems, and later on extended to Linear Parameter-Varying (LPV) systems [2], assumes that the system is described by an LTI nominal model interconnected with an LTI or LTV unknown system which can be used, for instance, to describe unmodeled dynamics and parametric uncertainty. However, the methods provided in [1], [2] are not recursive, which means that, after a given amount of input/output data is obtained, we verify whether or not the data sequence is compatible with the model of the system. Hence, the complexity of the algorithms grows with the number of measurements. Other alternatives in the literature include frequency-based model validation and probabilistic estimates – see [3], [4], [5] and references therein.

A different approach to model falsification can be found in the FDI literature – see, for instance, [6]. The main idea in such architectures stems from the designing of filters that are more reactive to faults than to disturbances and model uncertainty. This can be achieved, for instance, by using geometric considerations regarding the plant (see [7], [8], [9], [10]), or by optimizing a particular norm minimization objective, such as the $\mathcal{H}_\infty$- or ℓ_1-norm (see [11], [12], [13], [14], [15]). The latter approach provides, in general, important robustness properties, as stressed in [11], [16], [17], [18], by explicitly accounting for model uncertainty. After synthesizing the filters, a set of residuals is then generated by comparing the actual output of the plant with the ones estimated by each filter. A model is thereafter invalidated if the corresponding residual is greater than a given threshold, which may be time-varying and that, in general, depends on the model uncertainty and on the amplitude of the disturbances. As a caveat, these methodologies are typically conservative, or can only be applied to a particular class of systems.

A novel model falsification strategy was proposed in [19], [20], [21], which relies on Set-Valued Observers (SVOs) to invalidate discrete-time Linear Parameter-Varying (LPV) models of dynamic systems. The reasoning behind this approach is similar to that of [1], [2], but a recursive algorithm is proposed instead, allowing it to run in real-time. Due to the properties inherited from the SVOs, this model falsification method guarantees that valid models of the plant are never invalidated. Moreover, under certain distinguishability conditions briefly discussed in the sequel, it can also be shown that the *correct* model of the plant is selected.

In [19], [20], an extension of the SVOs introduced in [22] to LPV uncertain systems is presented. The proposed solution is able to cope with descriptions of the plant that can be time-varying and partially unknown. In order to constrain the number of faces of the set-valued estimate of the state of the system, an overbound was proposed which is guaranteed not to grow unbounded, under certain assumptions on the plant. Nevertheless, a few questions regarding the implementation of these SVOs were left unresolved. In particular, in order to guarantee a bounded set-valued estimate of the state, an arbitrarily large number of previous state estimates was required, possibly leading to excessive computational requirements. Furthermore, it was assumed that the plant (at least in closed-loop) was asymptotically stable.

Thus, this paper describes a new SVO-based method to invalidate a class of discrete-time dynamic systems, guaranteeing that the set-valued estimates of the state remain bounded under mild assumptions, and requiring at most the n previous state estimates, where n is the number of states of the model of the system. Moreover, the proposed solution can be applied to a class of dynamic systems described by LTI models with time-varying uncertainties.

The remainder of this paper is organized as follows. We start by introducing the notation used in this work and describing some of the techniques available in the literature

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for the design of SVOs in Section II. In Section III, the main results of this paper regarding the convergence of the SVOs and its applicability to a class of dynamic systems is presented. The theory is illustrated by means of a simulation example, in Section IV. Finally, some conclusions regarding this work are discussed in Section V.

II. Preliminaries and Notation

A. Set-Valued Observers

The subspace of all proper and real rational stable transfer matrices is denoted by $\mathcal{RH}_\infty$. We represent the elements of $v(k) \in \mathbb{R}^m$, for some $m, k \in \mathbb{Z}, m > 0$, as $v_i(k)$, so that $v(k) = [v_1(k),\, v_2(k),\, \cdots,\, v_m(k)]^{\mathrm{T}}$. The concatenation of vectors $v(k)$, $v(k-1)$, $\cdots$, $v(k-N+1)$, for $N \in \mathbb{Z}^+$ is denoted as

$$v_N = \begin{bmatrix} v(k) \\ \vdots \\ v(k-N+1) \end{bmatrix}.$$

For the sake of simplicity, v is used instead of v_N whenever N can be inferred from the context. For $a, b \in \mathbb{R}^n$, we say that $a \leq b$ if $a_i \leq b_i$ for all $i \in \{1, \cdots, n\}$.

At this point, we assume that the input/output data available for model falsification can be obtained through an LTI system, described by

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Ld(k), \\ y(k) &= Cx(k) + n(k), \end{cases} \tag{1}$$

with bounded exogenous disturbances, $d(\cdot)$, uncertain initial state, $x(0) \in X(0)$, control input, $u(\cdot)$, and measurement output, $y(\cdot)$, corrupted by additive noise, $n(\cdot)$. It is also assumed that $|d(k)| := \max_i |d_i(k)| \leq 1$, and $|n(k)| := \max_i |n_i(k)| \leq \bar{n}$. At each sampling time, k, the vector of states is denoted by $x(k)$, and we define $X(0) := \mathrm{Set}(M_0, m_0)$, where

$$\mathrm{Set}(M, m) := \{q : Mq \leq m\} \tag{2}$$

represents a convex polytope. Moreover, let $x(k) \in \mathbb{R}^n$, $d(k) \in \mathbb{R}^{n_d}$, $u(k) \in \mathbb{R}^{n_u}$, and $y(k) \in \mathbb{R}^{n_y}$.

Let $X(k+1)$ represent the set of possible states at time-instant $k+1$, *i.e.*, the state $x(k+1)$ verifies (1) with $x(k) \in X(k)$ if and only if $x(k+1) \in X(k+1)$. An SVO aims to find $X(k+1)$, based upon (1) and with the additional knowledge that $x(k) \in X(k), x(k-1) \in X(k-1), \cdots, x(k-N) \in X(k-N)$ for some finite N. We further require that for all $x \in X(k+1)$, there exists $x^\star \in X(k)$ such that, for $x(k) = x^\star$, the observations are compatible with (1). In other words, we want $X(k+1)$ to be the smallest set containing all the solutions to (1). A procedure for time-varying discrete-time linear systems was introduced in [22], and a preliminary extension to uncertain plants is presented in [19], [20].

The computation of $X(k+1)$ based upon $X(k)$ for systems with no model uncertainty can be performed using the technique described in [22]. Indeed, let the system be described by (1). Then, as shown in [22], $x(k+1) \in X(k+1)$ if and only if there exist $x(k)$, $n(k)$ and $d(k)$, such that, for the current measurement, $y(k+1)$, we have

$$P(k) \begin{bmatrix} x(k+1) \\ x(k) \\ d(k) \end{bmatrix} \leq \begin{bmatrix} Bu(k) \\ -Bu(k) \\ \mathbf{1} \\ \mathbf{1} \\ \tilde{m}(k) \\ m(k-1) \end{bmatrix} =: p(k) \tag{3}$$

where

$$P(k) := \begin{bmatrix} I & -A & -L \\ -I & A & L \\ 0 & 0 & I \\ 0 & 0 & -I \\ \tilde{M}(k) & 0 & 0 \\ 0 & M(k\text{-}1) & 0 \end{bmatrix}, \tilde{M}(k) = \begin{bmatrix} C \\ -C \end{bmatrix}, \tilde{m}(k) = \begin{bmatrix} \bar{n} + y(k+1) \\ \bar{n} - y(k+1) \end{bmatrix},$$

and where $M(k-1)$ and $m(k-1)$ are defined such that $X(k) = \mathrm{Set}\,(M(k-1), m(k-1))$. Notice that the disturbances and measurement noise, $d(\cdot)$ and $n(\cdot)$, respectively, are treated differently by the SVOs, as the former typically impact the dynamics in more than a single direction.

The inequality in (3) provides a description of a set in $\mathbb{R}^{2n+n_d}$, denoted by $\Gamma(k+1) = \mathrm{Set}\,(P(k), p(k))$. Therefore, it is straightforward to conclude that

$$\hat{x} \in X(k+1) \Leftrightarrow \underset{x \in \mathbb{R}^n, d \in \mathbb{R}^{n_d}}{\exists} : \begin{bmatrix} \hat{x} \\ x \\ d \end{bmatrix} \in \Gamma(k+1)$$

Hence, the set $X(k+1)$ can be obtained by projecting $\Gamma(k+1)$ onto the subspace of the first n coordinates.

The projection of $\Gamma(k+1)$ onto $\mathbb{R}^n$ can be done resorting to the *Fourier-Motzkin elimination method* (see [22], [23]). Therefore, we obtain a description of all the admissible $x(k+1)$, which does not depend upon specific $x(k)$ nor $d(k)$.

The formulation in (3) can be easily extended, in case it is convenient to compute $X(k+1)$ not only based upon $X(k)$, but also upon $X(k-1)$, $\cdots$, $X(k-N)$. Indeed, $x(k+1) \in X(k+1)$ if and only if there exist $x(k+1), \cdots$, $x(k-N+1)$, $y(k)$, and $d(k), \cdots, d(k-N+1)$, such that,

$$P_N(k) \begin{bmatrix} x(k+1)^{\mathrm{T}} & x_N^{\mathrm{T}} & d_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \leq p_N(k) \tag{4}$$

where

$$P_N(k) := \begin{bmatrix} I & \text{-}A & \cdots & 0 & \text{-}L & 0 & \cdots & 0 \\ \text{-}I & A & \cdots & 0 & L & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 & \text{-}L & \text{-}AL & \cdots & 0 \\ \text{-}I & 0 & \cdots & 0 & L & AL & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ I & 0 & \cdots & \text{-}A^N & \text{-}L^k & \cdots & \cdots & \text{-}A^{N\text{-}1}L \\ \text{-}I & 0 & \cdots & A^N & L & \cdots & \cdots & A^{N\text{-}1}L \\ 0 & \cdots & \cdots & 0 & I & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \text{-}I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & I \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \text{-}I \\ \tilde{M}(k) & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & M(k\text{-}1) & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & M(k\text{-}N) & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and $p_N(k)$ can be inferred from (1).

For plants with uncertainties, the set $X(k+1)$ is, in general, non-convex, even if $X(k)$ is convex. Thus, it cannot be represented by a linear inequality as in (2). The approach suggested in [19] is to overbound this set by a convex polytope, $\hat{X}(k+1)$, therefore adding some conservatism to the solution. A generalization of this result is presented in [21]. An alternative in the literature to the design of SVOs uses Luenberger observers to provide bounded errors for the estimates of the states – see [24] and references therein.

B. Coprime Factorization of LTI Systems

The so-called *left-coprime factorization of discrete-time LTI systems* will be used in the following section:

Definition 1: Let $M, N \in \mathcal{RH}_\infty$. Then, M and N are *left-coprime over* $\mathcal{RH}_\infty$ if there exist $X, Y \in \mathcal{RH}_\infty$ such that

$$[M \quad N]\begin{bmatrix} X \\ Y \end{bmatrix} = MX + NY = I.$$

Moreover, if P is a proper real-rational matrix, then a *left-coprime factorization* of P is a factorization $P = M^{-1}N$, where N and M are left-coprime over $\mathcal{RH}_\infty$. $\diamond$

We also recover the following result (see, for instance, [25, p. 554]):

Proposition 1: Let

$$P(z) := D + C(zI - A)^{-1}B := \left[\begin{array}{c|c} zI - A & B \\ \hline C & D \end{array}\right]$$

be observable and define

$$[N \quad M] = \left[\begin{array}{c|cc} zI - A + KC & -K & B - KD \\ \hline RC & R & RD \end{array}\right],$$

where R is any non-singular matrix. Then, $P = N^{-1}M$.

III. Main Results

We are now in conditions of stating the main results in this paper. We will start by extending the applicability of the SVOs in the context of model falsification. Thereafter, guarantees of convergence of the SVOs are provided and, finally, we illustrate how to use these methods to falsify models with uncertain dynamics.

A. Model Falsification using SVOs

In [19], [21], the main idea was to invalidate dynamic models associated with SVOs whose state estimate, at a given time, is the empty set. Thus, as long as a given SVO provides non-empty set-valued estimates for the state of the plant, the corresponding dynamic model cannot be discarded. Hence, the first part of this subsection is devoted to the development of a method to invalidate dynamic models, while in the second part we show how to use this method on a model selection architecture.

We consider the class of discrete-time dynamic systems described by LTI models driven by unknown but bounded disturbances, connected to time-varying uncertainties, as depicted in Fig. 1.

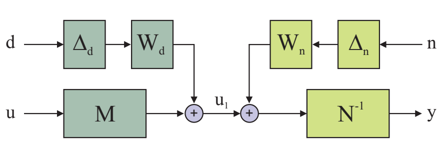


Fig. 1. Block-diagram of a class of discrete-time LTI dynamic systems with time-varying uncertainties and unknown disturbances.

We assume that M, N, W_d and W_n are LTI dynamic systems, and that $\Delta_d(k) \in \mathbb{R}^p$, $\Delta_n(k) \in \mathbb{R}^q$, $|\Delta_d(k)| \leq 1$, $|\Delta_n(k)| \leq 1$, for all $k \geq 0$. Moreover, the control input, $u(\cdot)$, and noisy measurements, $y(\cdot)$, are assumed known, and $|d| \leq 1$, $|n| \leq 1$.

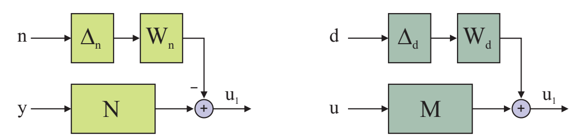
Notice that

$$\begin{aligned} y &= N^{-1}\big(u_1 + W_n\Delta_n n\big) \\ \Leftrightarrow u_1 &= Ny - W_n\Delta_n n, \end{aligned} \tag{5}$$

and

$$u_1 = Mu + W_d\Delta_d d. \tag{6}$$

Therefore, u_1 can either be estimated using (5) or (6). Due to the uncertainties and to the exogenous disturbances, the values of $u_1(k)$ for each k is also, in general, uncertain. Thus, an SVO as in [22], [19] and as described in Section II, referred to as SVO$_A$ and depicted in Fig. 2(a), can be designed to generate the set-valued estimates of u_1 based upon (5), while an SVO, designated by SVO$_B$ and illustrated in Fig. 2(b), can be synthesized to obtain the set-valued estimates of u_1 based upon (6).



(a) Block-diagram used by SVO$_A$ to obtain u_1 based on u and on the bounds on m.

(b) Block-diagram used by SVO$_B$ to obtain u_1 based on y and on the bounds on n.

Fig. 2. Block-diagrams used to compute the set-valued estimates of u_1.

Using this line-of-thought, the architecture depicted in Fig. 3 is proposed for single-model falsification using SVOs, as described in the sequel.

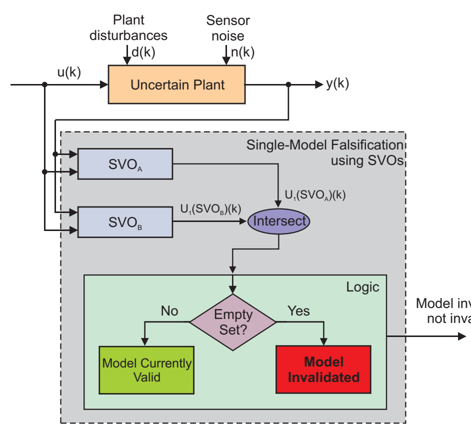


Fig. 3. Architecture for single-model falsification using SVOs, including the interconnection between the *true* plant, SVO$_A$ and SVO$_B$.

The set-valued estimates of $u_1(k)$, generated by SVO$_A$ and SVO$_B$, denoted by $U_1\big(\text{SVO}_A\big)(k)$ and $U_1\big(\text{SVO}_B\big)(k)$, respectively, are obtained by driving the SVOs with the (noisy) measurements output, $y(\cdot)$, and with the control inputs, $u(\cdot)$, respectively. If, at a given time k, the set-valued estimate of $u_1(k)$, obtained with SVO$_A$, *i.e.*, $U_1\big(\text{SVO}_A\big)(k)$, does not intersect with the set-valued estimate of $u_1(k)$, obtained with SVO$_B$, *i.e.*, $U_1\big(\text{SVO}_B\big)(k)$, the model of the system is not compatible with the true dynamics. Hence, such a model is falsified or invalidated.

In summary, we have that

- if $\underset{k_o \geq 0}{\exists} : U_1\big(\text{SVO}_A\big)(k_o) \bigcap U_1\big(\text{SVO}_B\big)(k_o) = \emptyset$, then the model of the plant is *not* compatible with the observations and input commands, for $k \geq k_o$;
- if $\underset{k \leq k_o}{\forall} : U_1\big(\text{SVO}_A\big)(k) \bigcap U_1\big(\text{SVO}_B\big)(k) \neq \emptyset$, then the model of the plant *is* compatible with the observations and input commands, at least up to $k \leq k_o$.

Therefore, if a set of *plausible* dynamic models of a given plant is available, then a couple of SVOs can be designed for each of these models, in order to invalidate them or not. Thus, the architecture in Fig. 4 is proposed as a model selection approach, where N_S denotes the number of considered dynamic models.

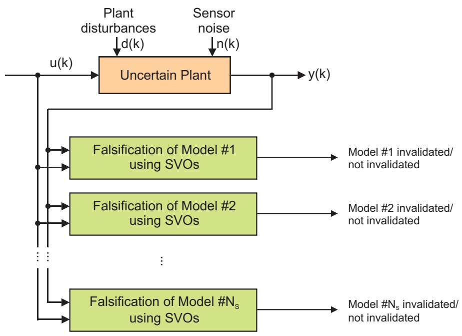


Fig. 4. Multiple-model falsification architecture using SVOs.

Remark 1: The architecture in Fig. 4 does not guarantee that only a single model is not going to be invalidated. Indeed, as shown in [20], this approach only guarantees that the "correct" model of the plant is not falsified. It does not, however, provide any guarantees in terms of invalidating all the other *plausible* models of the plant. These topics are still under research and some preliminary results are presented in [26]. The interested reader is further referred to [27], [28], [29], [30], [31]. $\diamond$

B. Guarantees of Convergence

As stated in [19], one possible shortcoming of the SVOs is related to the numerical approximations used during the computation of the set-valued estimates. In other words, since we do not have infinite precision in the computations that have to be carried out every sampling time to obtain the set-valued estimate $\hat{X}(k)$, the actual set where the state can take value, $X(k)$, need not be entirely contained inside $\hat{X}(k)$. Therefore, it may happen that the true state does not belong to $\hat{X}(k)$, and hence we may end up by discarding the "correct" model of the plant.

The solution proposed in [19] is to "robustify" the algorithm by slightly enlarging the set $\hat{X}(k)$. As long as the maximum error in the computation of the set $X(k)$ is known, we have, for every time k, a vector $\epsilon^*(k)$ such that $X(k) \subseteq \text{Set}\left(M(k), m(k) + \epsilon^*(k)\right)$.

Moreover, it may happen that, from time-step to time-step, the number of faces of the polytope containing the set-valued estimate of the state of the system increases exponentially. Hence, it is useful to overbound, in such circumstances, that polytope by another one, with a constrained number of faces.

Nonetheless, using an overbound to guarantee that we do not discard valid states of the plant also has its shortcomings. Besides adding conservatism to the solution, it may be responsible for the unbounded increase with time of the area of the polytope of the set-valued estimate.

Remark 2: One of the first algorithms developed to compute (ellipsoidal) set-valued estimates of the state of a system was presented in [32] and [33]. Using ellipsoids to describe the set-valued estimates of the state is an alternative method to the one discussed in this paper, with the advantage of having less computationally demanding calculations. However, unlike the convex polytope-based approach presented herein, the ellipsoid-based approach does not guarantee convergence of the set of state estimates, even if the system at hand is stable. $\diamond$

Consider a time-invariant plant described by (1), and let $x(0) \in X(0)$, with $X(0)$ bounded.

Let $\Psi(k)$ denote the smallest hyper-cube centered at the origin, such that $\Psi(0)$ is the smallest hyper-cube containing $X(0)$ and $\Psi(k)$ for $k > 0$ is obtained through (1). Define $\epsilon(k)$ as the maximum distance between a face of $\Psi(k)$ and the corresponding face of the estimate $\hat{\Psi}(k)$. The following proposition provides sufficient conditions to guarantee that $\hat{\Psi}(k)$ is bounded. It should be noticed that $\hat{\Psi}(k)$ can be interpreted as a rough approximation of $\hat{X}(k)$, in the sense that $\hat{X}(k) \subseteq \hat{\Psi}(k)$, which means that if $\hat{\Psi}(k)$ is bounded, so does $\hat{X}(k)$.

***Proposition** 2 ([19]):* Consider an *asymptotically stable* plant described by (1) with the aforementioned constraints. Suppose that the maximum numerical error (previously defined) at every sampling time is $\epsilon(k)$, with $\epsilon(k) \leq \epsilon^*|x_i(k)|$, for some $0 \leq \epsilon^* < \infty$ and for every $x(k) \in X(k)$. Further suppose that $\epsilon^* < 1 - \rho(A)$, where $\rho(A)$ denotes the spectral radius of A. Then, $\hat{\Psi}(k)$ is bounded.

The main drawback associated with Proposition 2 is that the required number of previous set-valued estimates of the state – *i.e.*, the size of N, in (4) – can be arbitrarily large – see [19]. This can obviously jeopardize the implementability of the SVOs, due to a pronounced computational burden. Moreover, this result can only be applied to asymptotically stable systems, which constrains the applicability of the SVOs for model falsification of unstable plants.

The solution proposed in this paper to both problems is to use the left-coprime factorization (see Definition 1) of the dynamic model of the system, together with the model falsification architecture depicted in Fig. 3.

***Theorem** 1:* Consider a system described by the observable realization

$$P(z) := D + C(zI - A)^{-1}B := \left[\begin{array}{c|c} zI - A & B \\ \hline C & D \end{array}\right],$$

with state $x(\cdot) \in \mathbb{R}^{n_x}$, actuated by control input $u(\cdot)$, $|u(\cdot)| < \infty$, with exogenous disturbances $d(\cdot)$, and with measurements $y(\cdot)$, $|y(\cdot)| < \infty$, corrupted by additive noise $n(\cdot)$, such that $|d(\cdot)| \leq 1$ and $|n(\cdot)| \leq 1$. Moreover, suppose that the (previously defined) maximal numeric error at each sampling time is $\epsilon(k)$, with $\epsilon(k) \leq \epsilon^\star|x_i(k)|$, for some $0 \leq \epsilon^\star < 1$ and for every $x(k) \in X(k)$. Then, there exist $M(z)$ and $N(z)$ such that

i) $P(z) = N(z)^{-1}M(z)$,
ii) the set-valued estimates of the states of N and M, respectively $\hat{\Psi}(\text{SVO}_A)(k)$ and $\hat{\Psi}(\text{SVO}_B)(k)$, obtained from the overbounding of (4), are bounded, for $N \geq n_x$.

Proof: The first part of the proof comes directly from Proposition 1. In particular, let K in Proposition 1 satisfy $(A - KC)^{n_x} = 0$. It should be noticed that the existence of such K is guaranteed, from the existence of a *deadbeat* observer for any observable LTI system – see, for instance, Theorem 5.3 in [34]. We now show that the set-valued estimates of the state of system $N(z)$ are bounded.

Consider the smallest hypercubes, denoted by $\hat{\Psi}(k)$, $\hat{\Psi}(k+1)$, $\cdots$, $\hat{\Psi}(k+N)$, that contain the sets $\hat{X}(k)$, $\hat{X}(k+1)$, $\cdots$, $\hat{X}(k+N)$, respectively, plus the maximal numeric error at each sampling time, $\epsilon(k)$, with $\epsilon(k) \leq \epsilon^\star |x_i(k)|$, for some $0 \leq \epsilon^\star < 1$ and for every $x(k) \in X(k)$. Then, for $N \geq n_x$, an overly conservative SVO can be synthesized to generate the sets $\hat{\Psi}(k)$, $\hat{\Psi}(k+1)$, $\cdots$, $\hat{\Psi}(k+N)$, using the following inequality:

$$|x(k+N)| \leq |(A-KC)^N x(k)| + \epsilon^\star |x(k)| + \delta_N = \epsilon^\star |x(k)| + \delta_N,$$

since $(A-KC)^N = 0$, for $N \geq n_x$, and where

$$\delta_N = \max_{d(k),\cdots,d(k+N-1)} |(A-KC)^{N-1} Ld(k) + \cdots + Ld(k+N-1)|.$$

Notice that is suffices to show that the sequence $\hat{\Psi}(k)$, $\hat{\Psi}(k+1)$, $\cdots$, $\hat{\Psi}(k+N)$ does not grow without bound, since it contains the sequence of set-valued estimates provided by an SVO as described in Section II. Given that $\epsilon^\star < 1$ by assumption, and that $|\delta_N| < \infty$ since $|y| < \infty$, the sets defined by (4) for system $N(z)$, with maximal numeric error at each sampling time, $\epsilon(k)$, are bounded. A similar result can be obtained for the set-valued estimates of the state of $M(z)$. ■

C. Model Uncertainty in the Dynamics

Notice that the model in Fig. 1 can be used not only to represent dynamic systems with exogenous disturbances and measurement noise, but also models with uncertainty in the input and in the outputs. Moreover, this technique can also be used to model uncertainty in matrix A, if the linear combination of states multiplying this uncertainty can be obtained by the measured outputs.

To see this, assume that $A = A_o + \Delta A_1$, where $\operatorname{rank}(A_1) = 1$ and with $|\Delta| \leq 1$. Hence, there exist vectors e_1 and f_1 such that $A_1 = e_1 f_1^{\mathrm{T}}$. Now, suppose that the signal $f_1^{\mathrm{T}} x(\cdot)$ can be obtained from the outputs of the plant, assuming no measurement noise[1]. Then, as described in [25], we can obtain a feedback description of the uncertain plant, where Δ is interconnected with the nominal plant, *i.e.*, the plant for $\Delta = 0$ – see Fig. 5.

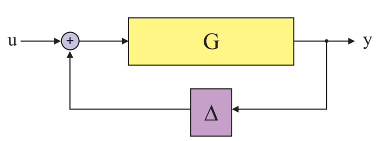


Fig. 5. Feedback interconnection between the nominal plant and the uncertainty in A.

Using the left-coprime decomposition, this interconnection can be transformed into the one depicted in Fig. 6. Hence, we can use the results in the previous subsection to assess whether or not the model with uncertain A matrix can describe a given input/output sequence.

In comparison to our previous results [19], we are now able of handling uncertainties in matrix A, without using the convex hull of the set-valued state estimates. Nevertheless, this is only true if the linear combination of the states that multiplies the uncertainty can be recovered from the outputs of the plant.

[1]The measurement noise will impact the estimate of the state by the SVO as a bounded disturbance. Therefore, it can be considered as such during the design of the SVO.

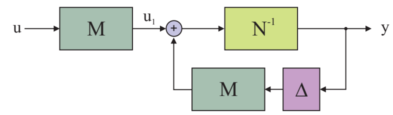


Fig. 6. Feedback interconnection between the nominal plant and the uncertainty in A, using the left-coprime decomposition.

IV. Simulations

In this section, some advantages of the methods described in Section III are illustrated by means of an example. We consider a plant with a continuous-time realization

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + Ld(t), \\ y(t) &= Cx(t) + n(t), \end{cases} \tag{7}$$

where $x(\cdot) \in \mathbb{R}^5$ denotes the state of the system, $y(\cdot) \in \mathbb{R}$ is the measured output, corrupted by noise $n(\cdot)$, $u(\cdot)$ is the control input and $d(\cdot)$ is an exogenous disturbance. Moreover, we have that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 2 & -0.2 & 0.2 & 0 \\ 2 & -2.15 & 0.2 & -0.3 & 1 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{bmatrix},$$

$$C = [0 \quad 1 \quad 0 \quad 0 \quad 0].$$

There are several real life applications that share the dynamics of the aforementioned described. In particular, these dynamics have been used in our previous studies, in order to describe a double mass-spring-dashpot plant, where the control input is non-collocated with the measured output – see, for example, [35], [36]. In this case, the output of the plant is the position of mass m_2, while the control input is the force applied to mass m_1.

The system in (7) was discretized using a sampling period of $T = 300$ ms. Moreover, the disturbances and measurement noise are assumed to follow a uniform distribution, with zero mean, and maximum absolute values of $\bar{d} := 1$ N and $\bar{n} := 0.001$ m, respectively. For the sake of simplicity, the control signal is defined as $u(k) := u(kT) := A_s \sin(\omega kT)$, where $A_s = 2$ N and $\omega = 2$ rad/s. Hence, the state of the system at time kT can be described by

$$\begin{cases} x(k+1) &= A_d x(k) + B_d u(k) + L_d d(k), \\ y(k) &= C_d x(k) + n(k), \end{cases} \tag{8}$$

where the matrices A_d, B_d, L_d and C_d are straightforwardly obtained from the discretization of (7).

The model falsification architecture depicted in Fig. 4 was adopted, using a set of 3 plausible models of the plant, described by (8), but with different C_d matrices:

- Model $M\#1$: $C_d = [0 \quad 1 \quad 0 \quad 0 \quad 0]$;
- Model $M\#2$: $C_d = [0 \quad 2 \quad 0 \quad 0 \quad 0]$;
- Model $M\#3$: $C_d = [0 \quad 0.5 \quad 0 \quad 0 \quad 0]$.

As a physical interpretation, these models represent different gains in the sensor that measures the position of mass m_2. For each of these models, a pair of SVOs for the corresponding coprime factorization was designed.

Notice that, if an SVO is designed for (8) as in [20], then the convergence of the set-valued estimate of the state would only be guaranteed if $N > 177$ in (4), since there exist $k \in \mathbb{N}$ with $k \leq 177$ such that $\|A_d^k\| > 1$. However, by using the coprime factorization-based approach introduced in this paper, we guarantee the convergence of the set-valued estimates of the state for any $N \geq 5$ in (4). Hence, the computational burden associated with the implementation of the SVOs is significantly decreased.

The results obtained for a typical Monte-Carlo run of the aforementioned scenario are depicted in Fig. 7. In this case, model $M\#3$ was falsified in 1.8 s, while 5.4 s were required to invalidate model $M\#2$. Hence, the only remaining model is the one compatible with (8) and with the observations.

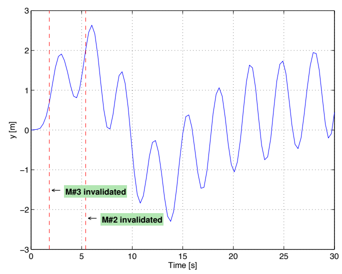


Fig. 7. Output of the plant for a typical Monte-Carlo run.

V. Conclusions

A coprime factorization-based approach was proposed in this paper to address the problem of model falsification of dynamic systems, using Set-Valued Observers (SVOs). The results presented indicate that using SVOs as a means of model invalidation is possible, not only for stable, but also for unstable systems. Moreover, in terms of implementability, using the coprime factors of a transfer function matrix, rather than the "original" transfer function matrix, may also have its own advantages. In particular, this method allows us to bound the number of required previous estimates of the state not to be larger than the number of states of the system. This particular benefit of the proposed methodology was also illustrated in simulation.

As a caveat, SVO-based model falsification is a *worst case* approach, in the sense that a model can only be invalidated if none of the allowable sequences of disturbances and measurement noise explains the measured output sequence.

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