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Robustness Analysis and Controller Synthesis with Non-Normalized Coprime Factor Uncertainty Characterisation

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Abstract—**A new robust controller synthesis method is proposed, based on a non-normalized coprime factorization of an uncertain plant. It achieves robust stabilization for very large uncertainties as measured by the ν-gap metric, as well as robust performance. It is shown that a non-normalized coprime factorization enables less conservative robust stability and performance guarantees than a normalized coprime factorization. In a numerical example, a controller is synthesized for a benchmark robust control problem, demonstrating the good robustness properties obtainable with the method of this paper.**

I. INTRODUCTION

The impact of model perturbations on the stability and performance of closed-loop systems with linear plants can be quantified using distance measures. Given a nominal and a perturbed plant model, the distance between the two corresponds to the size of an admissible uncertainty block in a certain topology or uncertainty structure. The gap metric [1], graph metric [2] and ν-gap metric [3], [4] all assume a normalized coprime factor (or four-block) uncertainty structure. This structure also forms the basis for a widely used robust controller synthesis method [5]. With the general framework of [6], distance measures and the associated robust stability margins can also be defined for other uncertainty structures. An engineer can choose the most suitable structure from among additive, multiplicative [7], normalized coprime factor/four-block [8] and not necessarily normalized coprime factor uncertainty [6]. Sections III-V of this paper show that non-normalized coprime factor uncertainty provides less conservative robust stability and performance guarantees than *normalized* coprime factor uncertainty for a given combination of nominal plant, controller and perturbed plant, when a particular factorization is chosen. This factorization minimizes the ratio of coprime factor distance to robust stability margin, which is an important quantity for bounding robust stability and robust performance degradation.

In Section VI, a controller synthesis method is described that exploits the advantages of non-normalized coprime factor uncertainty. The proposed objective function is an $\mathscr{H}_\infty$ norm comprising the non-normalized coprime factor distance and robust stability margin. An optimal coprime factorization reduces these two objects into one single quantity. A controller is therefore synthesized for a given nominal and worst-case perturbed plant. An additional part of the objective function ensures a minimal level of normalized coprime factor robust stability margin. Robust stability and performance guarantees for this synthesis method are described and it is shown that the guarantees are less conservative than those that can be obtained using four-block methods.

A benchmark motion control example illustrates the proposed controller synthesis method. It is shown how a robustly stabilizing controller can be obtained for a lightly damped uncertain plant, for which normalized coprime factor methods do not guarantee the existence of a stabilizing controller.

A. Notation

Notation is standard. $\mathscr{R}$ denotes the set of proper real-rational transfer functions, $\mathscr{RL}_\infty$ the space of proper real-rational functions bounded on $j\mathbb{R}$ including ∞, and $\mathscr{RH}_\infty$ denotes the space of proper real-rational functions bounded and analytic in the open right half complex plane. Denote the space of functions that are units in $\mathscr{RH}_\infty$ by $\mathscr{GH}_\infty$ ($f \in \mathscr{GH}_\infty \Leftrightarrow f, f^{-1} \in \mathscr{RH}_\infty$). Let $P^*(s)$ denote the adjoint of $P(s) \in \mathscr{R}$ defined by $P^*(s) = P(-s)^T$. The ordered pair $\{N,M\}$, $M,N \in \mathscr{RH}_\infty$, is a right-coprime factorization (*rcf*) of $P \in \mathscr{R}$ if M is invertible in $\mathscr{R}$, $P = NM^{-1}$, and N and M are right-coprime over $\mathscr{RH}_\infty$. Furthermore, the ordered pair $\{N,M\}$ is a normalized *rcf* of P if $\{N,M\}$ is a *rcf* of P and $M^*M + N^*N = I$. The ordered pair $\{\tilde{N},\tilde{M}\}$, with $\tilde{M},\tilde{N} \in \mathscr{RH}_\infty$, is a left-coprime factorization (*lcf*) of $P \in \mathscr{R}$ if $\tilde{M}$ is invertible in $\mathscr{R}$, $P = \tilde{M}^{-1}\tilde{N}$, and $\tilde{N}$ and $\tilde{M}$ are left-coprime over $\mathscr{RH}_\infty$. Furthermore, the ordered pair $\{\tilde{N},\tilde{M}\}$ is a normalized *lcf* of P if $\{\tilde{N},\tilde{M}\}$ is a *lcf* and $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$. Let $\{N,M\}$ be a *rcf* and $\{\tilde{N},\tilde{M}\}$ be a *lcf* of a plant P. Also, let $\{U,V\}$ be a *rcf* and $\{\tilde{U},\tilde{V}\}$ be a *lcf* of a controller C. Define

$$G := \begin{bmatrix} N \\ M \end{bmatrix}, \tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix}, K := \begin{bmatrix} V \\ U \end{bmatrix}, \tilde{K} := \begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix},$$

where G and $\tilde{G}$ will be referred to as the right and left graph symbols of P, and K and $\tilde{K}$ will be referred to as the right and left inverse graph symbols of C, respectively. Let $\mathscr{F}_l(\cdot,\cdot)$ (resp. $\mathscr{F}_u(\cdot,\cdot)$) denote a lower (resp. upper) linear fractional transformation (LFT). For a scalar $p(s) \in \mathscr{R}$,

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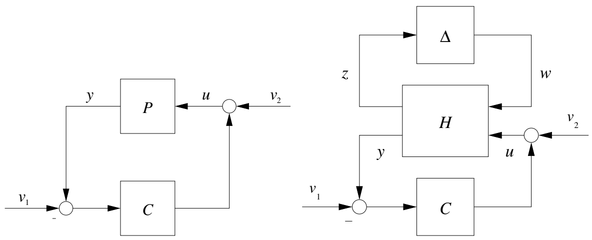

Fig. 1. The nominal closed-loop system $[P,C]$ (left) and the linear fractional interconnection $\langle H,C\rangle$ with the perturbation block Δ (right).

its winding number $\operatorname{wno} p(s)$ is defined as the number of encirclements of the origin made by $p(s)$ as s follows the standard Nyquist D-contour, indented into the right half plane around any imaginary axis poles or zeros of $p(s)$. Let $\eta(P)$ denote the number of open right half plane poles of $P\in\mathscr{R}$. Also, let $\operatorname{Ric}(H)$ denote the stabilizing solution to an algebraic Riccati equation. For a plant $P\in\mathscr{R}$ and a controller $C\in\mathscr{R}$, let $[P,C]$ denote the nominal feedback interconnection displayed in Fig. 1, and let $\langle H,C\rangle$ denote the linear fractional interconnection of H and C with input w, exogenous inputs v_1, v_2 and measured output z as displayed in Fig. 1.

II. Generalized Distance Measures and Robust Stability Margins

Given a nominal and a perturbed plant model $P\in\mathscr{R}^{p\times q}$ and $P_\Delta\in\mathscr{R}^{p\times q}$, how will robust stability and performance of the closed-loop system with nominally stabilizing controller $C\in\mathscr{R}^{q\times p}$ be affected if P is replaced by P_Δ? One way to answer this question is by measuring the distance between P and P_Δ in terms of the smallest infinity norm of an uncertainty block $\Delta\in\mathscr{RL}_\infty$ that produces the perturbed plant P_Δ in a linear fractional interconnection as shown in Fig. 1.

With a generalized plant $H=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\in\mathscr{R}$ with $H_{22}=P$, and if $(I-H_{11}\Delta)^{-1}\in\mathscr{R}$, we can formulate

$$P_\Delta=\mathscr{F}_u(H,\Delta)=P+H_{21}\Delta(I-H_{11}\Delta)^{-1}H_{12}. \qquad (1)$$

Since $H_{22}=P$, the nominal plant is recovered when $\Delta=0$.

Given P, H and P_Δ, $\Delta\in\mathscr{RL}_\infty$ is called an admissible uncertainty if it fulfills the well-posedness condition $(I-H_{11}\Delta)^{-1}\in\mathscr{R}$ and the consistency eqn. (1). A smallest admissible uncertainty in terms of the infinity norm (there are possibly several admissible Δ, or none) is called the distance between P and P_Δ.

Definition 1 ([6])**.** *Given a plant $P\in\mathscr{R}^{p\times q}$, a generalized plant $H\in\mathscr{R}$ with $H_{22}=P$, and a perturbed plant $P_\Delta\in\mathscr{R}^{p\times q}$. Let the set of all admissible perturbations be given by*

$$\boldsymbol{\Delta}=\left\{\Delta\in\mathscr{RL}_\infty:(I-H_{11}\Delta)^{-1}\in\mathscr{R},\ P_\Delta=\mathscr{F}_u(H,\Delta)\right\}.$$

Define the distance measure $d_{\mathrm{H}}(P,P_\Delta)$ between plants P and P_Δ for the uncertainty structure implied by H as:

$$d_{\mathrm{H}}(P,P_\Delta):=\begin{cases}\inf_{\Delta\in\boldsymbol{\Delta}}\|\Delta\|_\infty & \text{if } \boldsymbol{\Delta}\neq\emptyset\\ \infty & \text{otherwise.}\end{cases}$$

The counterpart to the distance measure is the robust stability margin of the feedback interconnection $\langle H,C\rangle$. The robust stability margin (in conjunction with the distance measure) captures the loop-gain part of the stability conditions for $\Delta\in\mathscr{RL}_\infty$:[1]

Definition 2 ([6])**.** *Given a plant $P\in\mathscr{R}^{p\times q}$, a generalized plant $H\in\mathscr{R}$ with $H_{22}=P$, and a controller $C\in\mathscr{R}^{q\times p}$. Define the stability margin $b_{\mathrm{H}}(P,C)$ of the feedback interconnection $\langle H,C\rangle$ as:*

$$b_{\mathrm{H}}(P,C):=\begin{cases}\|\mathscr{F}_l(H,C)\|_\infty^{-1} & \text{if } 0\neq\mathscr{F}_l(H,C)\in\mathscr{RL}_\infty \text{ and}\\ & [P,C] \text{ is internally stable,}\\ 0 & \text{otherwise.}\end{cases}$$

By specifying the precise structure of the generalized plant H, $d_{\mathrm{H}}(P,C)$ and $b_{\mathrm{H}}(P,C)$ can be adapted for various standard uncertainty structures.[2] In the case of normalized coprime factor uncertainty (which will be referred to as four-block uncertainty), they reduce to the well known ν-gap distance measure and robust stability margin (see e.g. [4]). The definitions for normalized coprime factors are given here for convenience.

Definition 3 ([3])**.** *Given two plants $P,P_\Delta\in\mathscr{R}^{p\times q}$ with graph symbols $\tilde{G}$ and G_Δ as defined in the Notation subsection. Define the ν-gap or the distance measure for four-block uncertainty as*

$$\delta_\nu(P,P_\Delta):=\begin{cases}\left\|\tilde{G}G_\Delta\right\|_\infty & \text{if } \det\left(G_\Delta^*G\right)(j\omega)\neq 0\,\forall\omega\\ & \text{and } \operatorname{wno}\det\left(G_\Delta^*G\right)=0;\\ 1 & \text{otherwise.}\end{cases} \qquad (2)$$

Definition 4 ([11])**.** *Given a positive feedback interconnection $[P,C]$ of a plant $P\in\mathscr{R}^{p\times q}$ and a controller $C\in\mathscr{R}^{q\times p}$. Define the robust stability margin in the left four block uncertainty structure as*

$$b(P,C):=\begin{cases}\left\|\begin{bmatrix}I\\ C\end{bmatrix}(I-PC)^{-1}\begin{bmatrix}I & -P\end{bmatrix}\right\|_\infty^{-1} & \text{if } [P,C] \text{ is internally stable;}\\ 0 & \text{otherwise.}\end{cases} \qquad (3)$$

Robust stability and robust performance theorems using the ν-gap are given in [3], [4]. Standard $\mathscr{H}_\infty$ controller synthesis optimizes the robust stability margin $b(P,C)$. In the following sections, we will give improved analysis and synthesis results for an uncertainty structure using coprime factors that are not normalized.

[1] In classical robust stability analysis for stable uncertainties $\Delta\in\mathscr{RH}_\infty$, the small gain theorem [9] provides sufficient conditions for internal stability of the feedback loop under perturbation. The theorem can be extended for $\Delta\in\mathscr{RL}_\infty$ [10], [4], [6] by introducing an additional winding number condition (cf. Theorem 1).

[2] This includes additive, multiplicative [7], coprime factors [6] and normalized coprime factors [8]

III. Coprime factor uncertainty and robust stability

Less conservative robust stability and performance analysis results can be obtained when the coprime factors of the plant and uncertainty description are not necessarily normalized. The nominal closed-loop system is shown in Fig. 1. The interconnection of the nominal plant P with an uncertainty block Δ is represented by a generalized plant H of appropriate dimensions, as shown in Fig. 1. For left coprime factor uncertainty H has the form

$$H = \left[\begin{array}{c|c} \tilde{M}_0^{-1} & P \\ 0 & I \\ \hline \tilde{M}_0^{-1} & P \end{array}\right], \tag{4}$$

where the left coprime factors $\{\tilde{M}_0, \tilde{N}_0\}$ over $\mathscr{RH}_\infty$ of the plant $P = \tilde{M}_0^{-1}\tilde{N}_0$ are not necessarily normalized, and can be related to normalized left coprime factors $\{\tilde{M}, \tilde{N}\}$ via a denormalization factor $R \in \mathscr{GH}_\infty$ s.t. $\{\tilde{M}_0 = R\tilde{M}, \tilde{N}_0 = R\tilde{N}\}$. The distance measure and robust stability margin for coprime factor uncertainty are defined below.

Definition 5 ([6])**.** *Given two plants $P, P_\Delta \in \mathscr{R}^{p\times q}$ with graph symbols $\tilde{G}$ and G_Δ as defined in the Notation subsection, and a denormalization factor $R \in \mathscr{GH}_\infty$. Define the distance measure for a left coprime factor uncertainty structure as*

$$d_{\mathrm{cf}}^R(P, P_\Delta) := \left\| R\tilde{G}G_\Delta \right\|_\infty. \tag{5}$$

Definition 6 ([6])**.** *Given a positive feedback interconnection $[P,C]$ of a plant $P \in \mathscr{R}^{p\times q}$ with lcf $\{\tilde{M}_0, \tilde{N}_0\}$ and a controller $C \in \mathscr{R}^{q\times p}$. Define the robust stability margin in the left coprime factor uncertainty structure as*

$$b_{\mathrm{cf}}^R(P,C) := \begin{cases} \left\| \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1}\tilde{M}_0^{-1} \right\|_\infty^{-1} & \text{if } [P,C] \text{ is internally stable;} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note that the coprime factor robust stability $b_{\mathrm{cf}}^R(P,C)$ carries the superscript R to denote its dependence on the denormalization factor $R \in \mathscr{GH}_\infty$ of the lcf $\{\tilde{M}_0 = R\tilde{M}, \tilde{N}_0 = R\tilde{N}\}$. It will be shown that a suitable choice of R is important for obtaining optimal analysis and synthesis results. Also note that when R is unitary, the factorization is normalized.[3]

Using the concepts of distance measure and robust stability margin, a small-gain type condition can be formulated for systems with coprime factor uncertainty: $d_{\mathrm{cf}}^R(P,P_\Delta) < b_{\mathrm{cf}}^R(P,C)$ or equivalently $\frac{d_{\mathrm{cf}}^R(P,P_\Delta)}{b_{\mathrm{cf}}^R(P,C)} < 1$ if $[P,C]$ is stable. For uncertainties $\Delta \in \mathscr{RH}_\infty$, this is a sufficient condition for stability; for $\Delta \in \mathscr{RL}_\infty$, an additional winding numer condition is required (see Theorem 1). Throughout this paper, we will use the fraction of $d_{\mathrm{cf}}^R(P,P_\Delta)$ and $b_{\mathrm{cf}}^R(P,C)$, and will show that its infimum over $R \in \mathscr{GH}_\infty$ is important in bounding robust stability and robust performance degradation in both coprime factor and four-block measures.

Definition 7. *Given plants $P, P_\Delta \in \mathscr{R}^{p\times q}$ and a controller $C \in \mathscr{R}^{q\times p}$ such that $[P,C]$ is internally stable, with graph symbols $\tilde{G}$, G_Δ and K as in the Notation subsection. Define the robustness ratio*

$$r(P,P_\Delta,C) := \inf_{R\in\mathscr{GH}_\infty} \frac{d_{\mathrm{cf}}^R(P,P_\Delta)}{b_{\mathrm{cf}}^R(P,C)} = \left\| \left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \right\|_\infty.$$

One possible (non-unique) choice for R achieving this infimum is $R = \left(\tilde{G}K\right)^{-1}$. To see that the infimum is indeed given by Definition 7 note that $b_{\mathrm{cf}}^R(P,C) = \left\|(R\tilde{G}K)^{-1}\right\|_\infty^{-1}$ and

$$\begin{aligned} \inf_{R\in\mathscr{GH}_\infty} \frac{d_{\mathrm{cf}}^R(P,P_\Delta)}{b_{\mathrm{cf}}^R(P,C)} &= \inf_{R\in\mathscr{GH}_\infty} \left\|R\tilde{G}G_\Delta\right\|_\infty \left\| \left(R\tilde{G}K\right)^{-1} \right\|_\infty, \\ &\geq \inf_{R\in\mathscr{GH}_\infty} \left\| \left(\tilde{G}K\right)^{-1} R^{-1} R\tilde{G}G_\Delta \right\|_\infty. \end{aligned}$$

This robustness ratio can be used to formulate the small gain condition for robust stability given in the theorem below, and is used in the following two sections to provide new and improved bounds on robust performance degradation.

Theorem 1 ([6])**.** *Given a plant $P \in \mathscr{R}^{p\times q}$, a perturbed plant $P_\Delta \in \mathscr{R}^{p\times q}$, a controller $C \in \mathscr{R}^{q\times p}$, and left coprime factors $\tilde{N}_0, \tilde{M}_0$ over $\mathscr{RH}_\infty$ of P. Define normalized graph symbols G, $\tilde{G}$, G_Δ, $\tilde{G}_\Delta$ as in the notation subsection and let $R \in \mathscr{GH}_\infty$ satisfy $\begin{bmatrix} -\tilde{M}_0 & \tilde{N}_0 \end{bmatrix} = R\tilde{G}$. Define a stability margin $b_{\mathrm{cf}}^R(P,C)$ as in* (6) *and a distance measure $d_{\mathrm{cf}}^R(P,P_\Delta)$ as in* (5)*. Furthermore, suppose $d_{\mathrm{cf}}^R(P,P_\Delta) < b_{\mathrm{cf}}^R(P,C)$ and $\overline{\sigma}\left(\tilde{G}G_\Delta\right)(\infty) < 1$.*

Then,

$$[P_\Delta, C] \text{ is internally stable} \Leftrightarrow \mathrm{wno}\det(G_\Delta^* G) = 0.$$

The condition $\frac{d_{\mathrm{cf}}^R(P,P_\Delta)}{b_{\mathrm{cf}}^R(P,C)} < 1$ (valid whenever $[P,C]$ is internally stable) can be made least conservative by choosing an infimizing $R \in \mathscr{GH}_\infty$, in which case it reduces to $r(P,P_\Delta,C) < 1$. While the results of Theorem 1 look very similar to results obtained using four-block theory, e.g. [4, Theorem 3.8], the set of plants guaranteed to be robustly stable (for a given controller C) by using a non-normalized coprime factorization

$$\begin{aligned} \mathscr{P}_{\mathrm{lcf}} := \{ P_\Delta \in \mathscr{R} :\ & r(P,P_\Delta,C) < 1,\ \overline{\sigma}\left(\tilde{G}G_\Delta\right)(\infty) < 1 \\ & \text{and } \mathrm{wno}\det\left(G_\Delta^* G\right) = 0 \}, \end{aligned}$$

will always include as a subset the set of plants guaranteed to be robustly stable by using a normalized coprime factorization and the ν-gap (for the same given controller C), i.e.

$$\begin{aligned} \mathscr{P}_\nu := \{ P_\Delta \in \mathscr{R} :\ & \left\|(\tilde{G}K)^{-1}\right\|_\infty \left\|\tilde{G}G_\Delta\right\|_\infty < 1, \\ & \overline{\sigma}\left(\tilde{G}G_\Delta\right)(\infty) < 1 \text{ and } \mathrm{wno}\det\left(G_\Delta^* G\right) = 0 \}. \end{aligned}$$

It is clear that $\mathscr{P}_\nu \subseteq \mathscr{P}_{\mathrm{lcf}}$, and the difference will be especially marked when the minimum value of the robust stability margin $\sup_{\omega\in\mathbb{R}} \left(\overline{\sigma}\left(\tilde{G}K\right)^{-1}\right)^{-1}$ occurs in a different channel and/or frequency region than the maximum distance $\sup_{\omega\in\mathbb{R}} \overline{\sigma}\left(\tilde{G}G_\Delta\right)$. For an example, see [6].

[3] As a consequence of the denormalization, both $d_{\mathrm{cf}}^R(P,P_\Delta)$ and $b_{\mathrm{cf}}^R(P,C)$ will have values on the closed and open positive real line, respectively. This is in contrast to their normalized counterparts, which always take values less or equal to unity.

IV. Coprime factor uncertainty and coprime factor performance

This section describes bounds on the robust performance degradation under perturbation, with the coprime factor robust stability margin $b^R_{\mathrm{cf}}(P_\Delta, C)$ serving as one of the performance measures. The following reformulation of [6, Theorem 9] gives bounds on the ratio of residual to nominal robust stability margins and on performance degradation when P is replaced by P_Δ.

Theorem 2. *Given the suppositions of Theorem 1 and assuming* $\mathrm{wno}\det(G^*_\Delta G) = 0$. *Let* H, H_Δ *as in* (4). *Then*

$$1 - \frac{d^R_{\mathrm{cf}}(P,P_\Delta)}{b^R_{\mathrm{cf}}(P,C)} \le \frac{b^R_{\mathrm{cf}}(P_\Delta,C)}{b^R_{\mathrm{cf}}(P,C)} \le 1 + \frac{d^R_{\mathrm{cf}}(P,P_\Delta)}{b^R_{\mathrm{cf}}(P,C)} \tag{7}$$

and

$$\frac{\|\mathscr{F}_l(H_\Delta,C) - \mathscr{F}_l(H,C)\|_\infty}{\|\mathscr{F}_l(H,C)\|_\infty} \le \frac{\frac{d^R_{\mathrm{cf}}(P,P_\Delta)}{b^R_{\mathrm{cf}}(P,C)}}{1 - \frac{d^R_{\mathrm{cf}}(P,P_\Delta)}{b^R_{\mathrm{cf}}(P,C)}}. \tag{8}$$

Proof: The two bounds in inequality (7) follow from [6, eqn. (23)]:

$$\left| b^R_{\mathrm{cf}}(P_\Delta,C) - b^R_{\mathrm{cf}}(P,C) \right| \le d^R_{\mathrm{cf}}(P,P_\Delta)$$

by considering the cases $b^R_{\mathrm{cf}}(P,C) \le b^R_{\mathrm{cf}}(P_\Delta,C)$ and $b^R_{\mathrm{cf}}(P_\Delta,C) \le b^R_{\mathrm{cf}}(P,C)$ and noting that $b^R_{\mathrm{cf}}(P,C) > 0$ by assumption. To obtain inequality (8), note that $b^R_{\mathrm{cf}}(P,C) = \left\| \left(R\tilde{G}K\right)^{-1} \right\|^{-1}_\infty = \|\mathscr{F}_l(H,C)\|^{-1}_\infty$ if $[P,C]$ is internally stable. It then follows from [6, eqn. (24)] that

$$\frac{\|\mathscr{F}_l(H_\Delta,C) - \mathscr{F}_l(H,C)\|_\infty}{\|\mathscr{F}_l(H,C)\|_\infty} \le \frac{d^R_{\mathrm{cf}}(P,P_\Delta)}{b^R_{\mathrm{cf}}(P,C) - d^R_{\mathrm{cf}}(P,P_\Delta)}$$

via eqn. (23) in the same paper, which entails eqn. (8) since $b^R_{\mathrm{cf}}(P,C) > 0$ by assumption. ■

Remark 1. *The bounds given in eqn.* (7) *and* (8) *can be tightened via the choice of R. E.g. for* $R = \left(\tilde{G}K\right)^{-1}$,

$$1 - r(P,P_\Delta,C) \le b^R_{\mathrm{cf}}(P_\Delta,C)\Big|_{R=\left(\tilde{G}K\right)^{-1}} \le 1 + r(P,P_\Delta,C),$$

$$\|\mathscr{F}_l(H_\Delta,C) - \mathscr{F}_l(H,C)\|_\infty\Big|_{R=\left(\tilde{G}K\right)^{-1}} \le \frac{r(P,P_\Delta,C)}{1 - r(P,P_\Delta,C)}.$$

It can be seen from the results of this and the previous section that the robustness ratio $r(P,P_\Delta,C)$ *bounds both the change in robust stability margin when* P *is replaced by* P_Δ *and the normalized robust performance degradation. It is in the interval* $[0,1)$ *by assumption (this being one of two sufficient conditions for robust stability of* $[P_\Delta,C]$*).*

V. Coprime factor uncertainty and four-block performance

The same uncertainty structure with a left coprime factor plant description can also be analyzed using a four-block performance measure. This corresponds to the setting shown in Fig. 2: The nominal plant is factorized as $P = \tilde{M}_0^{-1}\tilde{N}_0$, where $\{\tilde{M}_0, \tilde{N}_0\}$ are (left) coprime factors of P over $\mathscr{RH}_\infty$. The

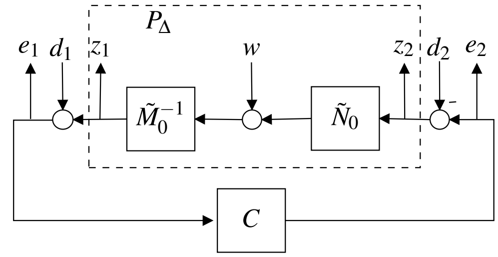

Fig. 2. Coprime factor uncertainty with four-block performance measure.

perturbed plant is obtained by $P_\Delta = \left(\tilde{M}_0 + \Delta_{\tilde{M}}\right)^{-1}\left(\tilde{N}_0 + \Delta_{\tilde{N}}\right)$. In contrast to the previous section, performance is here measured using a four-block performance measure, between the additional outputs $e = [e_1,\ e_2]^T$ and inputs $d = [d_1,\ d_2]^T$. The resulting theorem describes how coprime factor uncertainty affects the degradation of the four-block robust stability margin $b(P,C)$, which in turn can be related to classical measures of robustness and nominal performance [4]. It will be shown that the bounds that can be obtained using this uncertainty structure are in fact tighter than those that can be obtained using only the ν-gap and four-block uncertainty.

Theorem 3. *Given the suppositions of Theorem 2. Define a robust stability margin for four-block uncertainty* $b(P,C)$ *as in* (3). *Then,*

$$1 - r(P,P_\Delta,C) \le \frac{b(P_\Delta,C)}{b(P,C)} \le 1 + r(P,P_\Delta,C). \tag{9}$$

Proof: A sketch of the proof follows. The full proof will be published elsewhere. The inequalities (9) can be obtained by manipulations on the relationship between d and e as in Fig. 2, using trigonometric identities and [4, Lemma 2.2]. ■

As this theorem and Theorem 2 show, deviation in both the four-block and the coprime factor robust stability margin is bounded by the robustness ratio $r(P,P_\Delta,C)$. Eqns. (7) and (9) are similar in their structure, but the crucial difference is that the former bounds the deviation in non-normalised coprime factor robust stability margin, and the latter the deviation in four-block or normalized coprime factor robust stability margin.

The bounds in Theorem 3 are in fact tighter than the equivalent bounds using the ν-gap and $b(P,C)$, e.g. from [4, Theorem 3.8], which can be formulated as:

$$1 - \frac{\delta_\nu(P,P_\Delta)}{b(P,C)} \le \frac{b(P_\Delta,C)}{b(P,C)} \le 1 + \frac{\delta_\nu(P,P_\Delta)}{b(P,C)}$$

$$\Leftrightarrow\ 1 - \left\|\tilde{G}G_\Delta\right\|_\infty \left\|\left(\tilde{G}K\right)^{-1}\right\|_\infty \le \frac{b(P_\Delta,C)}{b(P,C)}$$

$$\le 1 + \left\|\tilde{G}G_\Delta\right\|_\infty \left\|\left(\tilde{G}K\right)^{-1}\right\|_\infty.$$

Clearly, $r(P,P_\Delta,C) \le \left\|\tilde{G}G_\Delta\right\|_\infty \left\|\left(\tilde{G}K\right)^{-1}\right\|_\infty$. Furthermore, these bounds can be computed even for cases in which $\delta_\nu(P,P_\Delta) > b(P,C)$, for which four-block bounds do not exist.

VI. Controller synthesis using a coprime factor uncertainty structure

In the previous sections, the importance of the robustness ratio $r(P,P_\Delta,C) = \left\| \left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \right\|_\infty$ for bounding deviations in the robust stability margin and the performance given a nominal plant P, a perturbed plant P_Δ and a nominally stabilizing controller C was demonstrated. The robustness ratio would be an obvious candidate for an objective function for controller synthesis. However, minimizing $\left\| \left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \right\|_\infty$ does not imply any bounds on $b(P,C)$, since the distance is possibly zero at some frequencies ($\tilde{G}G_\Delta(j\omega) = 0$), allowing $\overline{\sigma}\left(\tilde{G}K\right)^{-1}(j\omega)$ to take any value for such ω. The importance of $b(P,C) = \left\| \left(\tilde{G}K\right)^{-1} \right\|_\infty^{-1}$ as an indicator of good robustness has already been emphasised. To ensure a minimal level of $b(P,C)$, an augmented objective function is chosen, combining both $r(P,P_\Delta,C)$ and $b(P,C)$. A design parameter ε is introduced as a multiplier for $\left(\tilde{G}K\right)^{-1}$, allowing a trade off between the two quantities. The interval for this parameter is $\varepsilon \in (0, b_{\text{opt}}(P))$, ensuring that $\left\| \varepsilon \left(\tilde{G}K\right)^{-1} \right\|_\infty < 1$. This in turn allows the combined objective to be minimized below one if $r(P,P_\Delta,C) < 1$ for some C.

Controller synthesis using both coprime factor and four-block robustness measures. *Given plants $P, P_\Delta \in \mathscr{R}^{p\times q}$ with graph symbols $\tilde{G}$ and G_Δ satisfying the assumptions of Theorem 2. Given $\varepsilon \in (0, b_{\text{opt}}(P))$. Find a controller $C \in \mathscr{R}^{q\times p}$ such that $[P,C]$ is internally stable which satisfies $\left\| \left[\left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \quad \varepsilon \left(\tilde{G}K\right)^{-1} \right] \right\|_\infty < \gamma$, where $\gamma > \gamma_0$ and*

$$\gamma_0 = \inf_{C \text{ stab.}} \left\| \left[\left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \qquad \varepsilon \left(\tilde{G}K\right)^{-1} \right] \right\|_\infty . \quad (10)$$

The robust stability and performance guarantees provided by such a controller are summarised in the theorem below.

Theorem 4. *Given $P, P_\Delta \in \mathscr{R}^{p\times q}$ along with graph symbols G, $\tilde{G}$ and G_Δ as defined in the Notation subsection. Also given a generalized plant H for P and H_Δ for P_Δ as in eqn.* (4)*. Assume that $\overline{\sigma}\left(\tilde{G}G_\Delta\right)(\infty) < 1$ and $\operatorname{wno}\det\left(G_\Delta^* G\right) = 0$. Choose $\varepsilon \in (0, b_{\text{opt}}(P))$. If $\exists C \in \mathscr{R}^{q\times p}$ such that $[P,C]$ is internally stable and*

$$\left\| \left[\left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \quad \varepsilon \left(\tilde{G}K\right)^{-1} \right] \right\|_\infty < \gamma \le 1 \quad (11)$$

then

a) $b(P,C) > \frac{\varepsilon}{\gamma}$;
b) $b_{\text{cf}}^{\left(\tilde{G}K\right)^{-1}}(P,C) = 1$;
c) $[P_\Delta, C]$ *is internally stable;*
d) $b(P_\Delta, C) > b(P,C)(1-\gamma)$.
e) $b_{\text{cf}}^{\left(\tilde{G}K\right)^{-1}}(P_\Delta, C) > 1-\gamma$;

Furthermore, if $\gamma < 1$, then

e)

$$\left\| \mathscr{F}_l(H_\Delta, C) - \mathscr{F}_l(H, C) \right\|_\infty \Big|_{R=\left(\tilde{G}K\right)^{-1}} < \frac{\gamma}{1-\gamma}. \quad (12)$$

Proof:

a) This follows from noting that $b(P,C) = \left\| \left(\tilde{G}K\right)^{-1} \right\|_\infty^{-1}$, since $[P,C]$ is internally stable. From eqn. (11) it follows that $\left\| \left(\tilde{G}K\right)^{-1} \right\|_\infty < \frac{\gamma}{\varepsilon}$, from which a) is easily derived.
b) Simple substitution of $R = \left(\tilde{G}K\right)^{-1}$ into Definition 6.
c) Under the assumptions of this theorem, robust stability of $[P_\Delta, C]$ requires $d_{\text{cf}}^R(P,P_\Delta) < b_{\text{cf}}^R(P,C)$ via Theorem 1. Choosing $R = \left(\tilde{G}K\right)^{-1}$, we find that
$$d_{\text{lcf}}^{\left(\tilde{G}K\right)^{-1}}(P,P_\Delta) < b_{\text{lcf}}^{\left(\tilde{G}K\right)^{-1}}(P,C) \;\Leftrightarrow\; r(P,P_\Delta,C) < 1$$
which is guaranteed to hold since $\gamma \le 1$.
d) This is a consequence of a) together with Theorem 3.
e) Follows from Theorem 2 upon noting that $r(P,P_\Delta,C) < \gamma$ via eqn. (11).
f) This follows from Theorem 2 upon noting that for $R = \left(\tilde{G}K\right)^{-1}$, $\left.\frac{d_{\text{cf}}^R(P,P_\Delta)}{b_{\text{cf}}^R(P,C)}\right|_{R=\left(\tilde{G}K\right)^{-1}} = \left\| \left(\tilde{G}K\right)^{-1} \tilde{G}G_\Delta \right\|_\infty < \gamma$ via eqn. (11).

■

By incorporating the robustness ratio $r(P,P_\Delta,C)$—which is based on a non-normalized coprime factorization—into the objective function, this synthesis method allows controller synthesis even for plants for which the ν-gap exceeds the normalized coprime factor robust stability margin. This can easily occur in uncertain lightly damped systems. The example in the following section illustrates such a case.

While the robust stabilization part of Theorem 4 is formulated for $[P_\Delta, C]$ only, the controller is actually guaranteed to stabilize the set of plants $\{\hat{P} \in \mathscr{R}^{p\times q} : \operatorname{wno}\det\left(\hat{G}^* G\right) = 0,\ \overline{\sigma}\left(\tilde{G}\hat{G}\right)(\infty) < 1,\ \left\| \left(\tilde{G}K\right)^{-1} \tilde{G}\hat{G} \right\|_\infty < 1\}$, via an argument as in part a) of the proof.

The $\mathscr{H}_\infty$ norm minimization problem following from Theorem 4 can be solved using the linear matrix inequality (LMI) approach [12] via a suitable generalized state-space representation for the problem. Given state-space realizations $P = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right]$, $P_\Delta = \left[\begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array}\right]$, with $(A_0,B_0)/(A_\Delta,B_\Delta)$ stabilizable and (C_0,A_0) detectable, and assuming w.l.o.g. that $D_0 = 0$, a generalized state-space model is given by

$$H_{\text{e}}(s) := \left[\begin{array}{cc|ccc:c} A_0 & -B_0F_\Delta & -B_0R_\Delta^{-1/2} & 0 & -\varepsilon B_0 & B_0 \\ 0 & A_\Delta + B_\Delta F_\Delta & B_\Delta R_\Delta^{-1/2} & 0 & 0 & 0 \\ \hline C_0 & C_\Delta + D_\Delta F_\Delta & D_\Delta R_\Delta^{-1/2} & \varepsilon I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \hdashline C_0 & C_\Delta + D_\Delta F_\Delta & D_\Delta R_\Delta^{-1/2} & \varepsilon I & 0 & 0 \end{array}\right], \quad (13)$$

where $R_\Delta := I + D_\Delta^* D_\Delta$; $\tilde{R}_\Delta := I + D_\Delta D_\Delta^*$;

$$X_\Delta := \operatorname{Ric}\begin{bmatrix} A_\Delta - B_\Delta R_\Delta^{-1} D_\Delta^* C_\Delta & -B_\Delta R_\Delta^{-1} B_\Delta^* \\ -C_\Delta^* \tilde{R}_\Delta^{-1} C_\Delta & -\left(A_\Delta - B_\Delta R_\Delta^{-1} D_\Delta^* C_\Delta\right)^* \end{bmatrix};$$

$$F_\Delta := -R_\Delta^{-1}\left(B_\Delta^* X_\Delta + D_\Delta^* C_\Delta\right).$$

Precise LMI solvability conditions and controller reconstruction formulae can be obtained for the above problem.

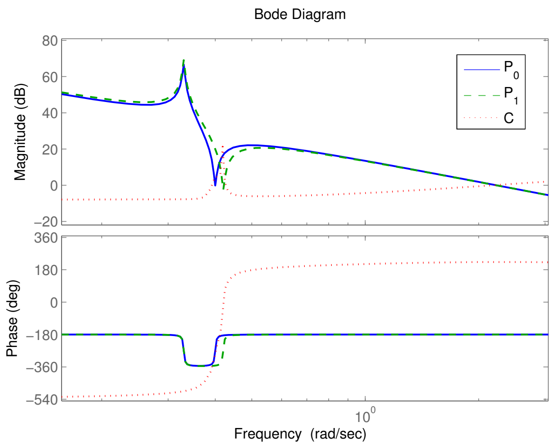

Fig. 3. Bode plots of P_0, P_1 and of the optimal controller C.

These will be reported elsewhere. However, given a suitable state-space representation, software routines like Matlab's 'hinflmi' can carry out the optimization and reconstruct a controller.

VII. Numerical Example

In this section, the synthesis methodology described in the previous section is applied to a physically motivated robust motion control problem which has been proposed as a benchmark example in [13] (c.f. [4]). The system consists of two masses M_1, M_2, coupled by a spring of stiffness k, with the assembly sliding on a frictionless table. The input u is a force applied to one mass, the output x is the displacement of that mass. The nominal transfer function is:

$$P_0(s) = \frac{x(s)}{u(s)} = \frac{M_2 s^2 + k}{s^2(M_1 M_2 s^2 + (M_1 + M_2)K)}.$$

Slight changes in the location of the poles and zeros of this system can cause very large distances as measured by the ν-gap [4]. Consider the nominal system $P_0(s)$ ($M_1 = M_2 = 1$kg, $k = 1\frac{\mathrm{N}}{\mathrm{m}}$, sensor gain of 10) and the perturbed system $P_1(s)$, respectively defined as

$$P_0(s) = \frac{10(s^2+1)}{s^2(s^2+2)}, \; P_1(s) = \frac{10(s^2+1.1)}{s^2(s^2+2)}.$$

The Bode plots of these transfer functions are displayed in Fig. 3. The ν-gap between both systems is $\delta_\nu(P_0, P_1) = 0.8012$, i.e. very large compared to the optimal four-block robust stability margin $b_{\text{opt}}(P_0) = 0.3919$. Robust stability of $[P_\Delta, C]$ can not be guaranteed using four-block methods since $\delta_\nu(P_0, P_1) > b_{\text{opt}}(P_0)$. And indeed, the controller achieving $b(P_0, C) = b_{\text{opt}}(P_0)$ does not stabilize $[P_1, C]$ [4]. Applying the synthesis method of Section VI, we choose $\varepsilon = 0.3$, which we assume will give us an adequate level of four-block robust stability margin for the nominal system if $\gamma \approx 1$. We use the generalized state-space model (13) to solve the combined $\mathscr{H}_\infty$-optimization problem (11) with the Matlab LMI solver 'hinflmi', approximating the infimum as $\gamma \approx 0.8884$. The controller C that achieves this (approximate) infimum is guaranteed to be robustly stabilizing for $P_1(s)$ via Theorem 4. It achieves a four-block robust stability margin $b(P, C) = 0.3385 > \frac{\varepsilon}{\gamma}$. The four-block stability margins of the perturbed system is remarkeably similar: $b(P_1, C) = 0.3360$. The Bode plot of the optimal controller C is shown in Fig. 3. C approximates a simple lead-lag controller, which is widely known to give very good robustness properties for systems as in this example [4].

VIII. Conclusion

A new robust controller synthesis method has been proposed, based on a coprime factorization of an uncertain plant in which the coprime factors are not necessarily normalized. It has been shown that by choosing a particular factorization, robust stability and robust performance guarantees can be obtained that are less conservative than those using *normalized* coprime factors. In the example, a robustly stabilizing controller has been obtained for an uncertain plant for which normalized coprime factor methods cannot guarantee stabilization. The proposed method will be especially useful in the controller synthesis for lightly damped systems.

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