2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Enforcing a System model to be Negative Imaginary via Perturbation of Hamiltonian Matrices

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Abstract— **Flexible structure dynamics with collocated force actuators and position sensors lead to negative imaginary (NI) systems. However, in some cases, the models obtained for these systems may not satisfy the NI property. This paper provides a new method for enforcing such models to be NI. The results are based on a study of the spectral properties of related Hamiltonian matrices. A test for the negativity of the imaginary part of a corresponding transfer function matrix is first performed by checking for the existence of imaginary eigenvalues of the associated Hamiltonian matrix. In the presence of imaginary eigenvalues, the system is not NI. In such cases, a first-order perturbation is presented for the precise characterization of frequency bands where violations of the NI property occur. This characterization is then used for the design of an iterative perturbation scheme for state matrices aimed at displacing the imaginary eigenvalues of the Hamiltonian matrix away from the imaginary axis.**

Index Terms— **Negative imaginary systems, Positive real systems, Hamiltonian matrices, and Passivity.**

I. Introduction

Negative imaginary (NI) systems theory was introduced by Lanzon and Petersen in [1], [2]. NI systems are defined by considering the property of the imaginary part of a certain frequency dependent transfer function matrix $G(j\omega) = D + C(j\omega I - A)^{-1}B$, where $G(j\omega)$ belongs to the set of real-rational stable transfer function matrices.

Flexible structure dynamics can be found in many systems, such as in aircrafts, bridges, buildings, robots, and optical systems. The resonant dynamics resulting from the flexibility of these systems can affect their performance. Also, these dynamics can lead to structural modes that can limit the ability of control systems in achieving the desired performance.

Precise modeling of structural dynamics is often difficult as it is sensitive to boundary conditions and environmental effects. Therefore, using force actuators combined with collocated measurements of velocity, position, or acceleration can improve the performance of control systems by increasing active damping [3]. Collocated here means that sensors and actuators have the same location and direction. Since flexible structures with collocated force actuators and position sensors are typically strictly negative imaginary (SNI), the NI theory can be effectively applied to these systems [3]. Also, it has been shown in [1], [2] that the necessary and sufficient condition for the internal stability of the positive-feedback interconnection of a NI and a SNI system is for the corresponding closed-loop DC gain to be less than unity.

For applications containing flexible structure dynamics, it is hard to get an exact system model by constructing differential equations for such systems. An alternative method for obtaining the system model is by means of system identification. However, the resulting mathematical model may not completely describe the true dynamics of the underlying system.

Identified system models can sometimes be misleading in the sense that they might not reflect the actual characteristics of the underlying system. For example, linear time-invariant (LTI) systems which are known to be NI might be identified as non-NI systems. In such cases, the system model can be enforced to satisfy NI system characteristics. In this paper, we achieve such an enforcement by using results from the theory of passivation for LTI systems; see e.g., [4]–[9].

NI systems can be transformed into positive real (PR) systems and vice versa under some technical assumptions. However, this equivalence is not complete. For instance, such a transformation applied to a strictly negative imaginary (SNI) system always leads to a non-strict PR system. Hence, the passivity theorem [10], [11] cannot capture the stability of the closed-loop interconnection of a NI and a SNI system. Also, any approach based on strictly PR synthesis cannot be used for the control of a NI system irrespective of whether it is strict or non-strict. Also, transformations of NI systems to bounded-real systems for application of the small-gain theorem also suffer from the exact same difficulty of giving a non-strict bounded real system despite the original system being SNI, see [12] for details.

This paper is further organized as follows: Section II introduces the concept of PR and NI systems and presents a relationship between them. In Section III, an algebraic procedure is presented that allows us to pinpoint the frequency bands where the NI property for a given transfer function is violated. Section IV shows how we can enforce such a system model to be NI and Section V concludes the paper with a note on future work.

II. Preliminaries

A. Positive Real Systems

The definition of PR systems is motivated by the study of linear electric circuits composed of resistors, capacitors,

This work was supported by the Australian Research Council

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and inductors. The same definition applies for analogous mechanical and hydraulic systems. This idea can be extended to study electric circuits with nonlinear passive components and magnetic couplings. Here, we present definitions and a lemma describing PR systems in terms of their transfer function matrices. For a detailed discussion on PR systems, see [10], [11] and references therein.

Definition 1: A transfer function $f(s)$ is said to be positive real if:

1. $f(s)$ is analytic in $\mathfrak{Re}[s] > 0$.
2. $\mathfrak{Re}(f(s)) \geq 0$ for all $\mathfrak{Re}[s] > 0$.
3. $f(s)$ is real for positive real s.

Definition 2: A square transfer function matrix $F(s)$ is positive real if:

1. $F(s)$ has no pole in $\mathfrak{Re}[s] > 0$.
2. $F(s)$ is real for all positive real s.
3. $F(s) + F(s)^* \geq 0$ for all $\mathfrak{Re}\ [s] > 0$.

Here $F(s)^*$ denotes the complex conjugate transpose of $F(s)$.

B. Negative Imaginary Systems

Definition 3: [1], [2], [13], [14] A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1. $G(s)$ has no pole at the origin and in $\mathfrak{Re}[s] > 0$.
2. For all $\omega > 0$, such that $j\omega$ is not a pole of $G(s)$, and $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$.
3. If $j\omega_0$ is a pole of $G(j\omega)$, it is at most a simple pole and the residual matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)sG(s)$ is positive semidefinite Hermitian.

C. Relationship between Negative Imaginary and Positive Real Systems

Since the theory of PR systems is well-researched, it is useful to establish a relationship between PR and NI systems to further develop the theory for NI systems. The following lemma provides a relationship between NI an PR systems [13], [14].

Lemma 1: [13], [14] Given a real rational proper transfer function matrix $G(s)$ with minimal state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and the transfer function matrix $\tilde{G}(s) = G(s) - D$, the transfer function matrix $G(s)$ is negative imaginary if and only if,

1) $G(s)$ has no poles at the origin.
2) The transfer function matrix $F(s) = s\tilde{G}(s)$ is positive real.

III. Characterization of Frequency Bands where Negative Imaginary Property is violated

In this section, we describe an algebraic procedure that allows us to pinpoint frequency bands where the NI property is violated, i.e., frequency bands where, $j\left(G(j\omega) - G(j\omega)^*\right) < 0$.

Theorem 1: Given a transfer function matrix with minimal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Assume A has no imaginary eigenvalues and $\delta \geq 0$ is not an eigenvalue of $\frac{CB + B^TC^T}{2}$. Then, $\delta \in \lambda(H(j\omega_0))$ if and only if $j\omega_0 \in \lambda(N_\delta)$. Here, $\lambda(\cdot)$ denotes the set of eigenvalues of a matrix, $H(j\omega) = \frac{j\omega}{2}(G(j\omega) - G^*(j\omega))$ and the Hamiltonian matrix N_δ is given by

$$N_\delta = \begin{bmatrix} A + BQ^{-1}CA & BQ^{-1}B^T \\ -A^TC^TQ^{-1}CA & -A^T - A^TC^TQ^{-1}B^T \end{bmatrix} \quad (1)$$

with $Q = 2\delta I - CB - B^TC^T$.

Proof: The proof of this theorem follows from the proof of Theorem 3 in [15] for the PR case and then using Lemma 1.

■

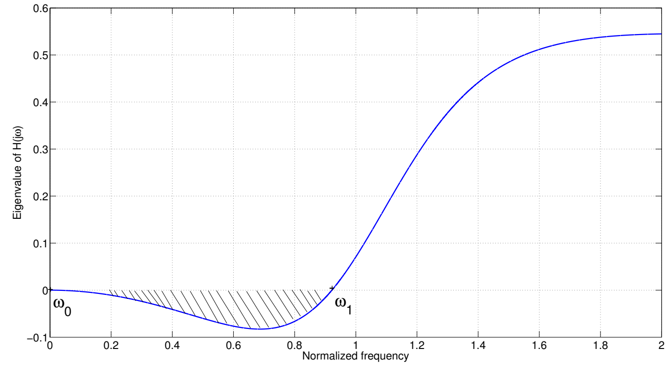

Fig. 1. Eigenvalue distribution versus normalized frequency for the Hermitian part of $F(s) = sG(s)$, where $G(s) = \frac{0.5s+0.2}{s^2+s+1.25}$ is NI.

Theorem 1 allows us to compute the frequencies at which the eigenvalues of the Hermitian part of a transfer function matrix cross or touch any given threshold (or a critical level), $\delta = \delta_0$. A NI property test can be readily designed by using the critical level $\delta_0 = 0$.

To illustrate this theorem, as explained in [9] for the PR case, consider the situation depicted in Fig. 1 which describes the eigenvalues of the Hermitian part of the transfer function $F(s) = sG(s)$, where $G(s) = \frac{0.5s+0.2}{s^2+s+1.25}$. $G(s)$ is NI, except in the shaded frequency band from ω_0 to ω_1. The number of imaginary eigenvalues of the associated Hamiltonian matrix are two, which corresponded to frequencies ω_0 and ω_1. The frequency axis is therefore subdivided into two frequency bands (ω_0, ω_1) and (ω_1, ∞).

From this example, we can conclude that using only the number of imaginary eigenvalues of $H(j\omega)$ does not allow for the local characterization of the NI property in each frequency band. In order to obtain such a characterization, we need to consider the slope of the eigenvalue curve at points crossing the critical level $\delta = 0$. Since the eigenvalues are continuous functions of frequency, the number of successive crossings with positive (negative) slopes can be precisely related to the number of eigenvalues crossing the threshold in each frequency band.

Let us consider the set of all imaginary eigenvalues (with positive imaginary part) of the Hamiltonian matrix at the critical level $\delta_0 = 0$, defined as

$$\Omega = \{\omega_i > 0 : j\omega_i \in \lambda(N_0)\}. \quad (2)$$

We will assume that the multiplicity of each eigenvalue $j\omega_i$ is unity. Now, the perturbation of the original eigenvalue $j\omega_i$ for $\delta_0 \simeq 0$, denoted by $\omega_{p_i}^{\delta}$, can be computed as a convergent power series,

$$\omega_{p_i}^{\delta} = j\omega_i + k_i'\delta + h.o.t. \quad (3)$$

Here, the first-order coefficient k' is related to the slope of the eigenvalue curve at the crossing and is given by

$$k_i' = \frac{\partial \omega_{p_i}^{\delta}}{\partial \delta}|_{\delta=0}. \quad (4)$$

Also, we need to express the Hamiltonian matrix N_δ as a first-order expansion about the critical level $\delta = 0$. A straightforward calculation leads to the following expression [9]:

$$N_\delta = N_0 + \delta N_0' + h.o.t., \quad (5)$$

where,

$$N_0' = \begin{bmatrix} -2BQ_0^{-2}CA & -2BQ_0^{-2}B^T \\ 2A^TC^TQ_0^{-2}CA & 2A^TC^TQ_0^{-2}B^T \end{bmatrix}$$

with $Q_0 = -(CB + B^TC^T)$.

According to [9] k_i' in Eq. (4) can be given as,

$$k_i' = \frac{\upsilon_i^* J N_0' \upsilon_i}{\upsilon_i^* J \upsilon_i}, \quad (6)$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $(\cdot)^*$ denotes the complex conjugate transpose, and υ_i is the corresponding eigenvector of $j\omega_i$.

Remark 1: determination of the set Ω requires structure-preserving Hamiltonian eigensolvers. Perturbation theory results can be used to prove that only if the structure is preserved, simple purely imaginary eigenvalues might remain on the imaginary axis while unstructured solvers are used see; [16]–[18]. Thus, it is recommended to use structured solvers in determination of the set Ω.

Now, consider the following result:

Theorem 2: Let Ω (defined in (2)), be a set of all simple positive imaginary parts of the imaginary eigenvalues of the Hamiltonian matrix N_δ at the critical level $\delta = 0$, sorted in ascending order. Also, let ζ_i be defined as

$$\zeta_i = \frac{j\upsilon_i^* J \upsilon_i}{\upsilon_i^* J N_0' \upsilon_i}. \quad (7)$$

Then, $G(jw)$ locally satisfies the NI property for $\omega \in (\omega_{i-1}, \omega_i)$, if and only if

$$\Lambda_i = \sum_{k \geq i} \text{sgn}(\zeta_k) = 0, \quad (8)$$

where $\text{sgn}(\cdot)$ extracts the sign of its argument and $\omega_0 = 0$.

Proof: The proof of this theorem follows from the proof of Theorem 4 in [9] and then applying Lemma 1 which transforms a NI transfer function matrix to a corresponding PR transfer function matrix. ■

Using the above theorem, Algorithm 1 in [9] can be used to determine the frequency bands where the NI property is not satisfied. Also, in terms of considering the general case, where multiplicity of the eigenvalues is more than one, the generalization in [9] for PR systems holds in the case of NI systems as well.

IV. Enforcing a System Model to be Negative Imaginary

In this section, we address the problem of finding an approximate NI system model for a given stable but non-negative imaginary system model for some frequency bands.

Let us consider a state-space representation for Σ as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (9)$$
$$y(t) = Cx(t) + Du(t), \quad (10)$$

where, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. The aim is to find a perturbed state-space realization Σ_p, which satisfies the NI property.

Since Σ is assumed to be stable, the A matrix will stay the same in the new perturbed model Σ_p. Also, as we transfer the model Σ to an equivalent PR system $\bar{\Sigma}$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (11)$$
$$\bar{y}(t) = \bar{C}x(t) + \bar{D}u(t) \quad (12)$$

via Lemma 1. In order to use the result in [9], we need to perturb the matrix $\bar{C}$ in $\bar{\Sigma}$ which is achieved by perturbing the matrix C in Σ. The state-space realization for the perturbed model Σ_p is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (13)$$
$$\bar{y}(t) = \bar{C}_p x(t) + \bar{D}u(t), \quad (14)$$

where only the matrix C is perturbed in order to satisfy the NI property.

Let

$$d\bar{C} = \bar{C}_p - \bar{C} = (C_p - C)A \quad (15)$$

which is the perturbation of the state matrix $\bar{C}$. The difference in the impulse response of the two systems induced by this perturbation is expressed as

$$dh(t) = L^{-1}\{dF(s)\} = d\bar{C}e^{At}B = (\bar{C}_p - \bar{C})e^{At}B, \quad (16)$$

where $L^{-1}(\dot{})$ is inverse laplace transform. The matrix $\bar{C}_p$ is selected to minimize the cumulative energy of the impulse response perturbations which can be computed as [9]

$$E = tr(d\bar{C}Wd\bar{C}^T), \quad (17)$$

where, W is the controllability Gramian [19].

Then, substituting (15) into (17) gives

$$E = tr(dCAWA^TdC^T) \quad (18)$$

and the matrix AWA^T can be factorized using Cholesky factorization as $AWA^T = K^TK$. Now, we can express the perturbation of dC in a new coordinate system as $dC_k = dCK^T$. This leads to,

$$E = tr(dC_k dC_k^T) = \| dC_k \|_F^2 = \| vec(dC_k) \|_2^2, \tag{19}$$

where, $vec(X)$ denotes a vector storing stacked columns of the matrix X, $\| \cdot \|_F^2$ denotes the Frobenius norm, and $\| \cdot \|_2^2$ denotes the Euclidean norm.

Now, we apply a first-order perturbation to the Hamiltonian matrix N_0 in order to move the imaginary eigenvalues off the imaginary axis. The perturbation on the Hamiltonian matrix in N_0 induced by small perturbations in the state matrix C is given by

$$N_0|_p = N_0 + dN_0, \tag{20}$$

where,

$$dN_0 = \begin{bmatrix} BQ_0^{-1}dCA & 0 \\ A^T C^T Q_0^{-1} dCA & A^T dC^T Q_0^{-1} B^T \end{bmatrix}.$$

The aim now is to find the matrix dC in order to displace each imaginary eigenvalue $j\omega_i$ to a new location $j\omega_{i,p}$, where

$$j\omega_{i,p} - j\omega_i \simeq \frac{\upsilon_i^* J dN_0 \upsilon_i}{\upsilon_i^* J \upsilon_i}. \tag{21}$$

As in [9], (21) can be expressed as

$$2\mathfrak{Re}((\upsilon_{i1}^T K^{-1}) \otimes z_i^*) vec(dC_k) = \mathfrak{Im}(\upsilon_i^* J \upsilon_i)(j\omega_{i,p} - j\omega_i), \tag{22}$$

where

$$z_i = -Q_0^{-1} B^T \upsilon_{i2} - Q_0^{-1} C \upsilon_{i1}. \tag{23}$$

Here, $(\upsilon_{i1}, \upsilon_{i2})$ is a partition of the eigenvector υ_i using the induced block partition of the Hamiltonian matrix, $\otimes$ is the Kronecker product, and $\mathfrak{Re}(\cdot), \mathfrak{Im}(\cdot)$ are the real and imaginary parts. The state matrix perturbation dC_k that is required to move the imaginary eigenvalue $j\omega_i$ to $j\omega_{i,p}$ must satisfy the linear constraint (22). In summary, these constraints can be described as a standard least squares problem

$$\begin{aligned} Zvec(dC_k) &= r, \\ \min \|vec(dC_k)\|_2, \end{aligned} \tag{24}$$

where each row in matrix Z stores the left-hand side of (22), satisfying $CB + B^T C^T > 0$ to ensure that the transfer function $G(j\omega)$ is asymptotically NI.

The determination of the location of new eigenvalues $j\omega_{i,p}$ is explained in [9].

The method outlined in this paper can be executed stepwise as

1) Set $m = 0$ and $C_0 = C$;
2) Apply Algorithm 1 in [9] to form the set Ω;
3) Increase the iteration count $m := m + 1$;
4) Apply the bisection algorithm in [15] to each violation bandwidth;
5) Determine the new eigenvalues locations $j\omega_{i,p}$ as in [9];
6) Solve the linear least squares problem (24) and compute dC_m using (22);
7) Update the state matrix $C_m = C_{m-1} + dC_m$;
8) Apply Algorithm 1 in [9] using C_m and form the set Ω;
9) Repeat from step 2) until Ω is empty.

V. Conclusion

In this paper, we have developed a method that allows us to drive a system model to satisfy the NI property. This was achieved by considering spectral perturbations of certain Hamiltonian matrices associated with the system. The main results of this paper were based on the assumption that the imaginary eigenvalues of the system Hamiltonian matrix were simple or characterized by complete sets of eigenvectors.

For some applications with multiple eigenvalues, the Hamiltonian matrix results presented here may not be valid. In such cases, further investigations are needed for the precise characterization of the NI property violations.

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