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Consensus on nonlinear spaces and graph coloring

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Abstract— **This paper comments on the complexity of equilibria reached by agents that evolve on a nonlinear space by interacting according to a fixed undirected graph. In particular, it considers agents on the projective space of $\mathbb{R}^k$, which links to the algorithmic problem of graph k-coloring. It is thereby shown that characterizing stable equilibria of repulsive agents on the projective space can be as difficult as graph coloring, that is NP-hard for $k > 2$.**

Disclaimer: The author must apologize for failing to pay a fair tribute to the broad combinatorial literature on graph coloring and to the large body of work on analog computation. Both these communities have accumulated an impressive body of work, in comparison to which the present comment more than pales. It therefore seemed preferable to let the interested reader pursue his/her own selection of the literature on these subjects.

I. INTRODUCTION

There has recently been a surge of interest in dynamical systems where agents, moving on a continuous state space, interact with each other according to a graph structure. For instance, interconnected agents attract each other in the "consensus" or "synchronization" problem: the goal is to reach agreement, i.e. agents finally all have the same state; see e.g. [14], [15], [20] among the literature on consensus in vector spaces. Repulsive agents can also be considered, for compact configuration spaces: the goal is to "maximally distribute" the agents, see e.g. [8], [18], [19]. The dynamics of interacting agents on nonlinear spaces can be complicated, both for attractive and for repulsive agents; some points in their characterization remain open even for fixed undirected graphs [16], [17]. In contrast, mutually attracting agents on a *linear* or convex space always converge to state agreement under sufficient connectivity assumptions, see e.g. [20]. The present paper comments on the complexity of stable equilibria for repulsive agents on nonlinear spaces, specifically projective spaces. It therefore exploits a link between this dynamical system and the algorithmic problem of graph coloring.

The research field of "analog computation" has long recognized the potential of relaxing the discrete computational space $\mathcal{C}$ of an algorithmic problem to a continuous manifold $\mathcal{M}$, and replacing a succession of discrete elementary operations on $\mathcal{C}$ (digital computer algorithm) by a continuous evolution on $\mathcal{M}$ whose final equilibrium would solve the original problem. Related papers in the systems&control community include [4], [6], [7]. More than a curiosity, this viewpoint yields insight into computer algorithm design and analysis, or even motivates analog implementation for some problems (see e.g. [1] and references therein). One feature of the discrete-computation / continuous-dynamics relation is task formulation as an optimization problem in a search space with specific nonlinear geometry (Lie group of rotations, Riemannian metric,... ; see examples in [4], [6]) that reflects the problem's structure. Tools for optimization in nonlinear geometry have been developed and existing algorithms have been related to this viewpoint, see e.g. [3], [10].

In view of these two developments, it is tempting to link one of the many computational problems that concern graph properties, to simple dynamical systems with agents interacting according to the same graph. This requires to identify interacting-agents settings whose (e.g.) geometry reflects the computational problem.

The specific link between graph k-coloring ($\mathcal{C}$-domain) and the disposition of mutually orthogonal vectors in $\mathbb{R}^k$ ($\mathcal{M}$-domain) has been explored in the combinatorial literature, see e.g. [9]. This has not led so far to a major breakthrough for graph coloring, which remains NP-hard for $k > 2$. The present paper (obviously) changes nothing to this point: it just illustrates the complexity of multi-agent systems on nonlinear spaces. It makes essentially the same link by matching k-coloring of a graph G with stable equilibria of agents repulsing each other according to interconnection structure G on projective space $\mathbb{P}^{k-1}\mathbb{R}$. Note that a set of repulsive agents on $\mathbb{P}^{k-1}\mathbb{R}$ has been used in e.g. [8] to solve the *continuous* optimization problem of "packing" lines. In contrast, the present link is with a *discrete* optimization problem. The stable equilibria of the agents evolving on $\mathbb{P}^{k-1}\mathbb{R}$ are then robustly — i.e. for all distance-dependent repulsion functions satisfying some bound — as difficult to characterize as graph coloring, i.e. NP-hard for $k > 2$.

The paper is organized as follows. Section II gives a motion law for repulsive agents on $\mathbb{P}^{k-1}\mathbb{R}$, with standard convergence properties. Section III defines a particular class of pairwise repulsion functions between the agents, to determine stability of particular equilibria. Section IV treats the case $k = 2$. Section V formalizes the link with graph coloring in general to establish the main result.

II. MOTION OF REPULSIVE AGENTS ON PROJECTIVE SPACE

Each point p of the $(k-1)$-dimensional projective space $\mathbb{P}^{k-1}\mathbb{R}$ can be viewed as representing a diameter of the

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sphere $S^{k-1} = \{x \in \mathbb{R}^k : x^T x = 1\}$. Equivalently, each $p \in \mathbb{P}^{k-1}\mathbb{R}$ represents a unit vector modulo overall sign, $p = \{v_p, -v_p\} \subset S^{k-1}$. $\mathbb{P}^{k-1}\mathbb{R}$ can be embedded in $\mathbb{R}^{k\times k}$ using its *projector representation*: to $p = \{v_p, -v_p\}$, associate the unique rank-one orthonormal projector $\Pi_p = v_p v_p^T$. Then $\mathbb{P}^{k-1}\mathbb{R} = \{X \in \mathbb{R}^{k\times k} : X = X^T,\ \mathrm{rank}(X) = 1,\ \ \mathrm{trace}(X) = 1\}$. This representation is used in the remainder of the paper. The *chordal distance* (see e.g. [8], [17]) between two points on $\mathbb{P}^{k-1}\mathbb{R}$ is given by

$$\begin{aligned} d_c(\Pi_1, \Pi_2) \quad &:= \quad \|\Pi_1 - \Pi_2\|_F = \sqrt{\mathrm{trace}((\Pi_1 - \Pi_2)^2)} \quad (1) \\ &= \quad \sqrt{2 - 2(v_1^T v_2)^2} = \sqrt{2\sin^2(\phi)}\ , \end{aligned}$$

where $\|\cdot\|_F$ is the Frobenius norm, ϕ is the angle between the two lines onto which Π_1 and Π_2 project, and v_1, v_2 respectively are unit vectors spanning these lines. Maximum $d_c = \sqrt{2}$ requires Π_1 and Π_2 projecting onto mutually orthogonal lines, i.e. $\Pi_1\Pi_2 = 0$ (the zero matrix). Importantly, $d_c^2(\Pi_1, \Pi_2)$ is infinitely continuously differentiable in its arguments at any point of $\mathbb{P}^{k-1}\mathbb{R}$, unlike e.g. the geodesic distance whose derivative features discontinuities.

A. Agent dynamics

Motion of attracting and repulsive agents on manifolds is formalized in [17]. The following particularizes it for $\mathbb{P}^{k-1}\mathbb{R}$. Let $G(V, E)$ a graph, where V is a finite set of *nodes* denoted $V = \{1, 2, ..., N\}$ without loss of generality and E is a set of *undirected edges* between nodes, i.e. a set of unordered[1] node pairs $(a, b)\ :\ \ a, b \in V\ ,\ \ \ a \neq b$. We denote $\#E$ the number of edges. A set of N agents, labeled $1, 2, ..., N$, interact according to a graph $G(V, E)$ when each agent is identified with a node $\in V$ and the presence (absence) of edge (a, b) in E represents the presence (absence) of interaction between agents a and b. To each agent $a \in V$, associate a state $\Pi_a \in \mathbb{P}^{k-1}\mathbb{R}$. The author's understanding of repulsion between agents is to maximize their chordal distance. We therefore take any smooth strictly monotone increasing function $g : \mathbb{R} \to \mathbb{R}$ that maps $[0, 2] \to [0, B]$, $B > 0$, and associate to each edge $(a, b) \in E$ the value function $w_{(a,b)} = g(d_c(\Pi_a, \Pi_b)^2)$. Repulsion on (a, b) is modeled to maximize $w_{(a,b)}$ and the sum of repulsions in the overall network is modeled to maximize

$$W = \textstyle\sum_{(a,b)\in E}\ g(d_c(\Pi_a, \Pi_b)^2)\ . \qquad (2)$$

There are several rational ways to make agents move. Physical systems would usually follow second-order dynamics, with W playing the role of a potential. We consider a simpler first-order dynamics, widely applied in the literature when studying higher-level commands, e.g. for consensus [14], [15], [17]. Agent motion is then derived as the gradient of W. Our model thus writes

$$\begin{aligned} \tfrac{d}{dt}\Pi_a \quad &= \quad \tfrac{\alpha}{2}\ \mathrm{grad}_{\Pi_a} W \qquad (3) \\ &= \quad \tfrac{\alpha}{2} \textstyle\sum_{\{\, b:(a,b)\in E\,\}} \mathrm{grad}_{\Pi_a}(w_{(a,b)}) \end{aligned}$$

[1]This makes an abuse of notation customary in graph theory, writing (a, b) to mean an *unordered* pair. The context should avoid any confusion.

for all $a \in V$, with $\alpha \in \mathbb{R}_{>0}$ a gain and grad_{Π_a} the gradient with respect to Π_a *on nonlinear space* $\mathbb{P}^{k-1}\mathbb{R}$ embedded in $\mathbb{R}^{k\times k}$ by its projective representation. (We identify tangent and cotangent spaces using the canonical metric induced by $\mathbb{R}^{k\times k}$.) Algorithm (3) is a variant of the "anti-consensus" motion in [17], the latter being restricted to $g = \mathsf{Identity}$.

Explicit computation of grad_{Π_a}, see e.g. [13], [17], yields

$$\begin{aligned} \tfrac{d}{dt}\Pi_a \ &= -\alpha \sum_{\{\, b:(a,b)\in E\,\}} g'(d_c(\Pi_a, \Pi_b)^2)\,(\Pi_a\Pi_b\Pi_a^\perp + \Pi_a^\perp\Pi_b\Pi_a) \\ &=:\ \alpha \sum_{\{\, b:(a,b)\in E\,\}} f_{ab}\, Q_{ab}\ . \qquad (4) \end{aligned}$$

Here $g'(u) \in \mathbb{R}$ denotes first derivative of $g : \mathbb{R} \to \mathbb{R}$ evaluated at u, and $\Pi^\perp = (\mathsf{Identity} - \Pi)$ is the orthonormal projection onto the kernel of Π. The last line of (4) decomposes the effect of agent b on the motion of agent a into a "normalized direction" Q_{ab} satisfying $\|Q_{ab}\|_F = 1$, tangent to $\mathbb{P}^{k-1}\mathbb{R}$ at Π_a, and a "magnitude" $f_{ab} \in \mathbb{R}_{>0}$. Defining v_s to satisfy $\Pi_s = v_s v_s^T$ and denoting $x = (v_a^T v_b)^2$, we have $f_{ab} = \sqrt{2x(1-x)}\, g'(2-2x)$ and $Q_{ab} = (2\Pi_a - \frac{v_a v_b^T + v_b v_a^T}{v_a^T v_b})\, \frac{1}{\sqrt{2/x - 2}}$. Unsurprisingly, the direction is ill-defined when the magnitude is zero.

B. Equilibrium characterization

The following convergence property of (4) is standard for gradient systems, see e.g. [2].

Proposition 1: The system of coupled agents following (4) with $\alpha > 0$ converges to a set of equilibria, for any initial condition. The stable equilibrium set consists of all (local) maxima of W.

Proof: $-W$ is a (non-strict) Lyapunov function as $\frac{d}{dt}(-W)$

$$= -\textstyle\sum_a \langle \mathrm{grad}_{\Pi_a}(W), \frac{d}{dt}\Pi_a\rangle_F \ = -\alpha \sum_a \|\frac{d}{dt}\Pi_a\|_F^2 \leq 0 \qquad (5)$$

where $\langle Q_1, Q_2\rangle_F = \mathrm{trace}(Q_1^T Q_2)$ is the scalar product on tangent spaces to $\mathbb{P}^{k-1}\mathbb{R}$. LaSalle invariance principle directly shows that the limit set corresponds to equilibria, i.e. points at which $\frac{d}{dt}\Pi_a = \mathrm{grad}_{\Pi_a} W = 0$ for $a = 1, 2, ..., N$. By (5), $-W$ strictly decreases whenever the state changes, so local maxima of W must be stable and other equilibria cannot be stable. □

Definition 1: Given $k \in \mathbb{N}$, let

$$\begin{aligned} S_o \quad = \quad \{&(\Pi_1, \Pi_2, ..., \Pi_N) \in (\mathbb{P}^{k-1}\mathbb{R})^N : \qquad (6) \\ &\Pi_a\Pi_b = \Pi_b\Pi_a \quad \forall\, a, b \qquad \}\ . \end{aligned}$$

In other words the Π_a, $a = 1, ..., N$, project on m directions of an orthonormal basis of $\mathbb{R}^k$, for some $m \leq k$.

Definition 2: Given graph $G(V, E)$ and $k \in \mathbb{N}$, let

$$\begin{aligned} S_p(G) \quad = \quad \{&(\Pi_1, \Pi_2, ..., \Pi_N) \in (\mathbb{P}^{k-1}\mathbb{R})^N : \qquad (7) \\ &\Pi_a\Pi_b = 0 \quad \forall\, (a, b) \in E \qquad \}\ . \end{aligned}$$

A state belongs to S_o if *each* agent pair — irrespective of any interconnection structure — is either aligned or orthogonal. S_p in contrast is graph-dependent: each agent pair forming an edge of $G(V,E)$ must be orthogonal, but agent pairs not in E are unconstrained. $S_p(G)$ can be empty, depending on k and G. The relation between $S_p(G)$ and S_o will be clarified throughout the paper.

Proposition 2:
(a) All states in S_o are equilibria of (3) for any $G(V,E)$.
(b) If $S_p(G)$ is nonempty, then it is the set of *global* maxima of W defined by (2) with graph G; each term $w_{(a,b)}$ takes its maximal value such that $W = B\,\#E$. Therefore $S_p(G)$ is an (at least locally) asymptotically stable equilibrium set under (3), for any coupling $g(\cdot)$.

Proof: States in S_o satisfy $(\Pi_a \Pi_b \Pi_a^\perp + \Pi_a^\perp \Pi_b \Pi_a) = 0 \;\; \forall a, b$, thus a fortiori $\forall (a,b) \in E$, in the first line of (3); this proves (a). The first part of (b) holds by definition, the second follows from Proposition 1. □

Note that S_o does not necessarily contain all equilibria of (3). However, characterizing the stable equilibria of (3) includes characterizing those belonging to S_o. The following section designs $g(\cdot)$ to affect stability of points in S_o in a particular way.

III. How coupling makes $S_o \setminus S_p$ unstable

Denote $S_o \setminus S_p(G)$ the set of states belonging to S_o but not to $S_p(G)$. In particular, $(\Pi_1, \Pi_1, ..., \Pi_1) \in S_o \setminus S_p(G)t$ for any G except the trivial graph that contains no edge. Discarding the latter as obviously uninteresting for agent motion (3), ensures $S_o \setminus S_p(G) \neq \emptyset$ in all our formulations.

Given $G(V,E)$, consider $(\Pi_1^*, ..., \Pi_N^*) \in S_o \setminus S_p(G)$. Then by definition there exists $a \in V$ for which $\Pi_a^* = \Pi_c^* = v_a\, v_a^T$ for some $(a,c) \in E$. We examine how W varies with Π_a when all other agents $b \in V \setminus \{a\}$ remain fixed at their equilibrium position $\Pi_b^* = v_b\, v_b^T$. Real functions on a manifold are best evaluated by considering a parametrized curve of their argument, in order to obtain a function from $\mathbb{R}$ to $\mathbb{R}$. We therefore define a variation $\Pi_a(s) = v(s)\, v(s)^T$ with $v(s) = \cos(s) v_a + \sin(s) v_* \in \mathbb{R}^k$, for $s \in (-\gamma, \gamma)$, $\gamma \ll 1$ and some unit vector $v_* \in \mathbb{R}^k$ such that $v_*^T v_a = 0$. Then $d_c(\Pi_a(s), \Pi_b^*)^2 = 2 - 2(v_a^T v_b \cos(s) + v_*^T v_b \sin(s))^2$. A second-order Taylor development of $W(s) := W(\Pi_a(s), \{\Pi_b^* : b \neq a\})$ yields, after a few computations, $W(s) - W(0) \approx$

$$\sum_{\{b:(a,b)\in E\}} \quad -4\, g'(d_c(\Pi_a^*, \Pi_b^*)^2)\; (v_a^T\, \Pi_b^*\, v_*)\, s$$
$$+2\, g''(d_c(\Pi_a^*, \Pi_b^*)^2)\; (v_a^T\, \Pi_b^*\, v_*)^2\, s^2$$
$$+2\, g'(d_c(\Pi_a^*, \Pi_b^*)^2)\, (\, (v_a^T v_b)^2 - (v_*^T v_b)^2\,)\, s^2 \,.$$

Since the equilibrium state is in S_o, for any $b \in V$ either $v_a^T \Pi_b^* = 0$ or $\Pi_b^* = \Pi_a$, such that $v_a^T\, \Pi_b^*\, v_* = v_a^T\, v_* = 0$. The first two lines of the Taylor expansion therefore vanish. The absence of first-order term is expected since, for a state in S_o, the value function associated to every (potential) edge (a,b) is at an extremum — minimum if $\Pi_a^* = \Pi_b^*$, maximum if $\Pi_a^* \Pi_b^*$ = 0. From these developments, the sign of $W(s) - W(0)$ is equal to the sign of

$$D := \sum_{\{b:(a,b)\in E\}} g'(2 - 2(v_a^T v_b)^2)\, (\, (v_a^T v_b)^2 - (v_*^T v_b)^2\,)\,.$$

Defining $A_1 = \{b \in V : \Pi_b = \Pi_a,\, b \neq a\}$ and $A_2 = V \setminus (A_1 \cup \{a\})$, we have

$$D \;=\; \textstyle\sum_{b\in A_1} g'(0) \;-\; \sum_{b\in A_2} g'(2)(v_*^T v_b)^2\,. \qquad (8)$$

Note that $g'(\cdot)$ is strictly positive by definition. Expression (8) allows to give the following proposition.

Proposition 3: If $N \leq k$ or $g'(0)/g'(2) > \lfloor \frac{N}{k} \rfloor / (\lceil \frac{N}{k} \rceil - 1)$, then for any graph $G(V,E)$ on N nodes, all equilibria in $S_o \setminus S_p(G)$ are unstable for dynamics (3) on $\mathbb{P}^{k-1}\mathbb{R}$.

Proof: By Proposition 1 a state is necessarily unstable if it is not a local maximum of W. We therefore show that the assumptions ensure, for any G and associated $S_o \setminus S_p(G)$, existence of a v_* in the above construction for which $D > 0$ in (8). Then indeed, the constructed variation of $\Pi_a(s)$ allows to increase W in the neighborhood of the equilibrium state, so the latter is not a local maximum of W, hence unstable.

Given G and $(\Pi_1^*, ..., \Pi_N^*) \in S_o \setminus S_p(G)$, let $n_G(b)$ the number of agents with same state as b. Select $a \in \mathrm{argmax}_b(\, n_G(b)\,)$; then $n_G(a) > 1$, by definition of $S_o \setminus S_p(G)$. If $m < k$ in Definition 1 for the chosen state, then there exists a unit vector $x \in \mathbb{R}^k$ such that $\Pi_b x = 0 \;\forall b \in V$ and taking $v_* = x$ in (8) yields $D = (n_G(a) - 1) g'(0) > 0$ for any $g(\cdot)$. If $m = k$ in Definition 1 for the chosen state, then select[2] $\bar{a} \in \mathrm{argmin}_b(\, n_G(b)\,)$ and set $v_* \in \{\pm v_{\bar{a}}\}$. Then $D = (n_G(a) - 1) g'(0) - (n_G(\bar{a})) g'(2)$. The condition on $g'(\cdot)$ ensures $D > 0$ for this choice as well. Given the previous paragraph, this concludes the proof. □

Thus choosing $g'(0)/g'(2)$ sufficiently large ensures that *the stable equilibria in S_o all belong to $S_p(G)$*. The condition on $g(\cdot)$ is easy to satisfy independently of N, since $\lfloor \frac{N}{k} \rfloor / (\lceil \frac{N}{k} \rceil - 1) \leq \frac{N/k}{N/k - 1}$ which monotonically converges to 1 as N increases.

IV. Special case $k = 2$

Projective space $\mathbb{P}^1\mathbb{R}$ is diffeomorphic to the unit circle S^1. Indeed, $\Pi = v\, v^T$ with $v = \pm (\, \cos(\phi) \;\; \sin(\phi)\,)^T$ yields

$$\Pi = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \begin{pmatrix} \frac{\cos(2\phi)}{2} & \frac{\sin(2\phi)}{2} \\ \frac{\sin(2\phi)}{2} & -\frac{\cos(2\phi)}{2} \end{pmatrix} \qquad (9)$$

and mapping $\Pi \rightarrow 2\phi = \theta \in S^1$ establishes an equivalence with the circle. Geometrically, to $p \in \mathbb{P}^1\mathbb{R}$ representing a diameter of S^1 (see first paragraph of Section II) that makes angle ϕ with $(1 \;\; 0)^T \in \mathbb{R}^2$, we associate the unit vector $\mathbf{e}_\theta$ that makes the angle $\theta = 2\phi$ with $(1 \;\; 0)^T$. Orthogonal diameters become opposite unit vectors on the circle; the two vectors $\pm v$ of $\Pi = v\, v^T$ are naturally mapped to the same point of S^1 as their $\pi-$difference in ϕ becomes a 2π-difference on θ. Dynamics (4) thus characterizes interacting

[2] If $\frac{N}{k}$ is integer, then we might have $n_G(b)$ constant over b. In this case, it is required to choose $a, \bar{a}$ such that $\Pi_a \Pi_{\bar{a}} = 0$.

agents on the circle. Using (9) and the distance expressions (1), it rewrites

$$\begin{aligned} \tfrac{d}{dt}\theta_a &= \alpha \sum_{\{\, b:(a,b)\in E\}} g'\big(\, 2\sin^2(\tfrac{\theta_a-\theta_b}{2})\,\big)\ \sin(\theta_a-\theta_b) \quad (10)\\ &=: \alpha \sum_{\{\, b:(a,b)\in E\}} f(\theta_a-\theta_b)\ . \end{aligned}$$

The Kuramoto coupling [12], that is (10) with $g = \mathsf{Identity}$ i.e. $f(\cdot) = \sin(\cdot)$, has been studied under various graph assumptions, mostly with attractive agents ($\alpha < 0$). It highlights rich behavior; see [16] and references therein. For fixed undirected G, it features "spurious" local equilibria. For $\alpha > 0$ and general $g(\cdot)$, (10) can still feature spurious local equilibria, irrespective of the condition on $g'(0)/g'(2)$.

Proposition 4: There exist G such that $S_p(G) \neq \emptyset$ for $k = 2$ but for which dynamics (10) can have locally asymptotically stable equilibria where $W < B\,\#E$, with $g(\cdot)$ satisfying the conditions of Proposition 3.

Proof: Consider G with even number $N > 5$ of nodes and $(a, b) \in E$ if and only if $|(a-b)\text{modulo}(N)| = 1$, that is a *ring graph*. Then $S_p = \{(\theta_1, ..., \theta_N) : \theta_a = a\pi + c\ \ \forall a,\ \ c \text{ constant } \in \mathbb{R}\}$ is nonempty. Now take a configuration $\theta_a = a\theta_0\ \ \forall a$, with $\theta_0 < \pi$ such that $N\theta_0$ is a nonzero integer multiple of 2π. Then $|\theta_a - \theta_b| = \theta_0$ for all $(a, b) \in E$. Computing the Hessian of $W(\theta_1, ..., \theta_N)$ for this configuration yields the Laplacian matrix of G multiplied by $-r\, f'(\theta_0)$, with r a positive constant. The Laplacian of an undirected connected graph has all positive eigenvalues, except one zero eigenvalue corresponding to invariance direction $(\theta_1, ..., \theta_N) \rightarrow (\theta_1 + \beta, ..., \theta_N + \beta)$, for $\beta \in \mathbb{R}$. Then taking $f'(\theta_0) < 0$ suffices to make the spurious configuration a local maximum of W, thus stable under (10). This is indeed possible: e.g. $g = \mathsf{Identity}$ makes $f'(\theta_0) < 0$ for $\theta_0 < \pi/2$. The *local* conditions, on $f'(\cdot)$ at θ_0 and on $g'(\cdot)$ at 0 and 2 in Proposition 3, do not interfere (although Proposition 3 does not allow $g = \mathsf{Identity}$). □

Proposition 4 shows that a non-empty $S_p(G)$ does not necessarily cover the whole stable equilibrium set. Determining the latter is in fact currently an open question even for $g = \mathsf{Identity}$ and $k = 2$, see [16]. In contrast, $S_p(G)$ and its relation with S_o are easy to characterize for $k = 2$.

Proposition 5: For $k = 2$,
(a) $S_p(G) \neq \emptyset$ if and only if G is bipartite.
(b) $S_p(G) \cap S_o \neq \emptyset$ if and only if G is bipartite.
(c) If G is bipartite and connected, then $S_p(G) \subset S_o$.

Proof: Satisfying (7) requires $(\theta_a - \theta_b)\text{modulo}(2\pi) = \pi$ whenever $(a, b) \in E$. For G connected, this partitions V into two subsets: one A_1 whose agents all have state (say) θ_* and the other A_2 whose agents all have state $\theta_* + \pi$; this joint state belongs to S_o. Moreover, to be in $S_p(G)$ no two agents belonging to the same subset may be connected in $G(V, E)$. The existence of a partition satisfying this last property is the definition of a bipartite graph. This proves (c) and (a),(b) for connected graphs. An adaptation for G disconnected is straightforward, as G is bipartite if and only if all its connected components are bipartite. □

A bipartite graph is the same as a 2-colorable graph. The following completes the link between repulsive agents on $\mathbb{P}^{k-1}\mathbb{R}$, the sets S_o, $S_p(G)$, and graph coloring, for $k > 2$.

V. From projective geometry to graph coloring

A. Graph k-coloring

Definition 3: Given a graph $G(V, E)$ and k colors $C = \{c_1, c_2, ..., c_k\}$, $k \in \mathbb{N}$, a graph k-coloring is a mapping $\Phi : V \rightarrow C$ such that $\Phi(a) \neq \Phi(b)$ for all $(a, b) \in E$.

Problems equivalent to graph coloring include assigning different colors to neighboring countries on a map (hence the name) and the game of Sudoku. k-coloring is not feasible for any k and any G. The reader will easily find basic properties like the following.

Property 1:
(a) If G contains a subgraph consisting of m nodes that are all-to-all connected (that is an m-clique), then k-coloring is infeasible for $k < m$;
(b) If k-coloring is feasible for G, then j-coloring is feasible with any $j \geq k$ for any subgraph of G.
(c) If k-coloring is feasible for G, then it is feasible for any graph obtained from G by sequentially adding nodes $(N+1), (N+2), ..., (N+M)$ and edges such that, for all $m > 0$, node $(N+m)$ is connected to the set of nodes $\{1, 2, ..., (N+m-1)\}$ by no more than $(k-1)$ edges.

Explicit expressions of the *chromatic number* — that is the minimal k for which G is k-colorable — are known for particular types of graphs. However, obtaining exact or tight decisions about k-colorability for general graphs is NP-hard in the number of nodes.

Property 2: Determining if a graph G is k-colorable for a given $k > 2$ is NP-complete. Determining the chromatic number of G is NP-hard. Determining the chromatic number within $N^{1-\varepsilon}$ is NP-hard for any $\varepsilon > 0$.

B. Link with repulsive agents on $\mathbb{P}^{k-1}\mathbb{R}$

"Vector coloring" a graph, as studied e.g. in [9], in fact corresponds to constructing states in S_p. The link between vector coloring and traditional graph coloring therefore shows that graph coloring naturally links to the projective space geometry $\mathbb{P}^{k-1}\mathbb{R}$ (and not e.g. the sphere of $\mathbb{R}^k$). The following makes this point explicit.

Definition 4: Given $G(V, E)$ and $k \in \mathbb{N}$, let

$$\begin{aligned} S_c(G) \ = \ \{ \ & (\Pi_1, \Pi_2, ..., \Pi_N) \in (\mathbb{P}^{k-1}\mathbb{R})^N \ : \qquad (11)\\ & \Pi_a \in \{\Gamma_1, ..., \Gamma_m\} \subset \mathbb{P}^{k-1}\mathbb{R} \ \ \forall a \in V \\ & \text{with } \Gamma_c\Gamma_d = 0 \ \ \forall c \neq d \in \{1, ..., m\} \\ & \Pi_a \neq \Pi_b \ \ \forall (a, b) \in E \qquad \} \end{aligned}$$

i.e. in S_c, each Π_a takes one of $m \leq k$ values that project on m directions of an orthonormal basis of $\mathbb{R}^k$, and connected nodes have different projectors.

Property 3: Given $G(V, E)$ and $k \in \mathbb{N}$, the set $S_c(G)$ is non-empty if and only if G is k-colorable.

Proof: Assume $S_c(G) \neq \emptyset$. Select a joint state in $S_c(G)$ and denote Ψ the map associating to $a \in V$ its state $\Pi_a \in \{\Gamma_1, ..., \Gamma_m\}$, according to line 2 of (11). Let Ξ any injective map from $\{\Gamma_1, ..., \Gamma_m\}$ to $\{c_1, ..., c_k\}$. Then $\Phi = \Xi \circ \Psi$ solves the k-coloring of G. (It actually solves m-coloring, which implies j-coloring for all $j \geq m$, see Property 1(b).) Conversely, assume G is k-colorable and denote $\Phi : V \rightarrow \{c_1, ..., c_k\}$ a solution map. Let $(\mathbf{e}_1, ..., \mathbf{e}_k)$ any orthonormal basis of $\mathbb{R}^k$ and $\Gamma_j = \mathbf{e}_j \, \mathbf{e}_j^T$ for $j = 1, 2, ..., k$. Let Ξ^* any bijective map from $\{c_1, ..., c_k\}$ to $\{\Gamma_1, ..., \Gamma_k\}$. Then by construction the joint state of the agents defined by $\Pi_a = \Xi^* \circ \Phi(a)$ for $a = 1, ..., N$ belongs to $S_c(G)$. □

Thus k-colorability of G is in direct correspondence with the set $S_c(G)$ for agents on $\mathbb{P}^{k-1}\mathbb{R}$. It remains to link $S_c(G)$ to the sets considered in the previous sections.

Proposition 6: $S_c(G) = S_o \cap S_p(G)$ for any G.

Proof: By definition. □

C. Complexity of multi-agent stable equilibria

It is now possible to state the main observation of the paper. Consider the following decision problem about the motion of N repulsive agents on $\mathbb{P}^{k-1}\mathbb{R}$ with dynamics (3).

Q.1: Given an interconnection graph $G(V, E)$, is any point in S_o a stable equilibrium of the multi-agent system ?

Theorem (main result): Answering Q.1 is at least as difficult as deciding k-colorability of the graph $G(V, E)$, at least if $N \leq k$ or $g'(0)/g'(2) > \lfloor \frac{N}{k} \rfloor / (\lceil \frac{N}{k} \rceil - 1)$.

Proof: If S_o contains no stable equilibrium, then $S_o \cap S_p(G) = \emptyset$ since by Proposition 2(b) all points of $S_p(G)$ are stable equilibria. Then by Property 3 and Proposition 6, G is not k-colorable. Conversely, if S_o contains a stable equilibrium, then by Proposition 3 this equilibrium must also belong to $S_p(G)$. Hence $S_o \cap S_p(G) \neq \emptyset$ and by Property 3 and Proposition 6, G is k-colorable. A classical complexity argument concludes: deciding k-colorability cannot be more difficult than answering Q.1, since the above shows that an answer to Q.1 automatically answers k-colorability. □

The theorem thus essentially says that *deciding if a given set (i.e. S_o) contains any stable equilibrium point for G is as difficult as deciding if G is k-colorable*. This complexity holds for characterizing the stable equilibria corresponding to global maxima of W only. Depending on $g(\cdot)$ there can be additional locally stable equilibria, see Proposition 4. However, adding those can only increase the complexity of stable equilibrium characterization. *The full characterization of stable equilibria for* (3) *therefore inherits the complexity of graph k-coloring, that is NP-hard for $k > 2$.*

Note that the whole argument still holds, with a slightly adapted bound condition, if the coupling function ($g(\cdot)$ above) differs from edge to edge.

D. An incomplete tool for the graph coloring problem

Given the above developments, one may be tempted to use multi-agent behavior for solving the difficult graph k-coloring problem, by maximizing W under pairwise repulsions of the agents on $\mathbb{P}^{k-1}\mathbb{R}$. Indeed, if G is k-colorable, then there is no "competition" between repulsions on different edges and all pairwise distances can take their maximal value. Formally, if G is k-colorable, then $S_c(G) \neq \emptyset$ (Property 3), hence $S_p(G) \neq \emptyset$ (Proposition 6).

A first problem in this regard is to avoid spurious stable local maxima where $W < B \,\#E$ for k-colorable G. A tailored $g(\cdot)$, as proposed in [16] to avoid spurious stable equilibria for synchronization ($\alpha < 0$), can be adapted to repulsive agents for this purpose. Although, the presence of many saddle points then still complicates a global convergence analysis for the dynamics.

Assuming this point solved, the multi-agent system for k-colorable G will converge to $S_p(G)$ — but not necessarily to $S_c(G)$, see e.g. the following simulations. To deduce colorability from a final state, it therefore remains to answer the converse of our motivating observation: Does G not k-colorable imply $S_p(G) = \emptyset$? It is known that unfortunately, this is not true in general, except for the algorithmically uninteresting case $k = 2$. See e.g. [9].

Proposition 7: For $k > 2$, there exist G with chromatic number $> k$ but for which $S_p(G) \subset (\mathbb{P}^{k-1}\mathbb{R})^N$ is nonempty. For $k = 2$, $S_p(G) \neq \emptyset$ if and only if G is 2-colorable.

Proof: The second part is a rewording of Proposition 5(a). The proof for the first part comes from the Bell-Kochen-Specker Theorem in the context of quantum systems [5], [11], see [9] and references therein; [9] also gives a simple counterexample with $k = 4$ and $N = 17$. □

Even with this negative answer, it might still be possible to distinguish k-colorability from the structure of S_p, in a way that makes repulsive agents useful. Exploring related algorithmic strategies however goes beyond the purpose and argumentation of the present paper.

E. Illustrative simulations

Figure 1 shows simulation results for repulsive agent evolution on $\mathbb{P}^{k-1}\mathbb{R}$ with $k = 3$. Each agent state $\Pi_a \in \mathbb{P}^2\mathbb{R}$ represents a diameter of the sphere of $\mathbb{R}^3$. We consider two representative graphs: the *Petersen graph* G_1 on the top is 3-colorable, the *Grötzsch graph* G_2 on the bottom is not. In both graphs, each node has at least 3 neighbors; nodes with less than 3 neighbors could be discarded for 3-coloring by Property 1(c). For both graphs, we run (4) with $g(x) = \arctan(x/2)$, for which $g'(0)/g'(2) > \lfloor \frac{N}{3} \rfloor / (\lceil \frac{N}{3} \rceil - 1)$ when $N > 6$. Top left panels show the evolution of $W/(B \,\#E)$. The final configurations are depicted on the right, each state Π_a as a diameter of the unit sphere in $\mathbb{R}^3$. For G_1, an initial state of the agents close to $S_c(G_1) \supset S_o$ seems to converge towards $S_c(G_1) \supset S_o$; indeed $W/(B \,\#E)$ tends to 1 and final diameters are nearly on the axes of an orthonormal basis of $\mathbb{R}^3$. For G_2, starting close to a high-W state of S_o, the

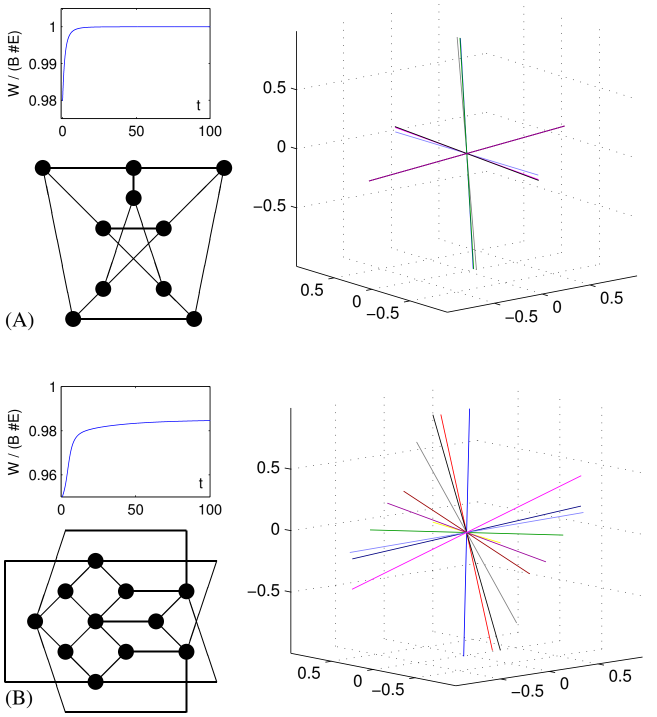

Fig. 1. Simulations illustrating (4) with $k = 3$ and $g(x) = \arctan(x/2)$ for (A) the Petersen graph, G_1 in the text, and (B) the Grötzsch graph, G_2 in the text. For each case we show the graph (bottom left), evolution of $\frac{W}{B\,\#E}$ (top left) and the final configuration (right).

agents move away from S_o while W increases (but not up to $B\,\#E$): final diameters indeed take many different directions in $\mathbb{R}^3$. This supports that equilibria in S_o are unstable. These two agent evolutions illustrate our Theorem.

The situation is however not as simple as it appears. The reported simulation for G_1 does in fact not completely reach $S_c(G_1)$ asymptotically. Analysis quickly shows that starting from a configuration in $S_c(G_1)$, one agent can in fact freely move on a curve while maintaining $W/(B\,\#E) = 1$. Due to this invariance direction, the configuration in $S_c(G_1)$ is stable but not asymptotically stable. Starting from an arbitrary initial state, the system with G_1 converges to an arbitrary state in $S_p(G_1)$ that can generally be far from $S_c(G_1)$, such that the position of diameters may be indistinguishable at first sight from the final state reached with G_2. The fact that $S_p(G_1)$ is reached while $S_p(G_2)$ is not can of course be checked on the value of W. However, as states in $S_p(G)$ may exist even if G is not 3-colorable (see Proposition 7), computing W is of little help to assess existence of points in $S_c(G)$. This illustrates that the link with graph coloring ensures complexity of multi-agent stable equilibria, but conversely the multi-agent system cannot be used (directly at least) to decide graph coloring.

VI. Conclusion

This paper considers a link between repulsive agents evolving on projective space $\mathbb{P}^{k-1}\mathbb{R}$ and the computational task of graph k-coloring. The geometric link implies that characterizing the stable equilibria of a repulsive multi-agent system on this nonlinear space is at least NP-hard for any $k > 2$ and for a large class of repulsive coupling functions. Known results imply that the considered link is incomplete to, conversely, solve k-coloring by letting repulsive agents evolve on $\mathbb{P}^{k-1}\mathbb{R}$. There are certainly other possibilities to link multi-agent equilibria to graph-coloring or other NP-hard algorithmic tasks, but the present one appears to be robust and rather natural. The stability criterion appears to be an important aspect to retain in the complexity argumentation. The result does not explain difficulties that can appear to characterize all locally stable equilibria for interacting agents on the circle ($k = 2$), see [16]. The complexity of such characterization for general distance-dependent coupling thus remains open.

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