2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Characterization of stability transitions and practical stability of planar singularly perturbed linear switched systems

Fouad El Hachemi, Mario Sigalotti and Jamal Daafouz

Abstract— **This paper is concerned with the stability of planar linear singularly perturbed switched systems in continuous time. Based on a necessary and sufficient stability condition, we characterize all possible stability transitions for this class of switched systems and we propose a practical stability result. We answer the questions related to what happen as ϵ, the singular perturbation parameter, grows and how many times the system can change its stability behavior (asymptotic stability, stability, instability) and which transitions are possible. Moreover, we analyze practical stability from the viewpoint of Tikhonov approach and in particular based on existing results obtained in the context of differential inclusions. We show that these approaches can be applied to singularly perturbed switched systems allowing to prove practical stability in some cases. This kind of stability focuses on the behavior of the system on compact time-intervals as ϵ tends to 0 (in particular, it does not ensure the asymptotic stability towards the origin). It is therefore different from the stability criteria where ϵ is fixed (arbitrarily small) and the asymptotic behavior for large times is considered. For planar systems, it turns out that when practical stability can be deduced from Tikhonov-type results, then global uniform asymptotic stability (for $\epsilon > 0$ small) holds true. It is an open question whether this is still true for higher dimensional singularly perturbed switched systems.**

I. INTRODUCTION

Singular perturbation methods are well known tools used when a system involves two time scale dynamics [10], [14]. They consist in decomposing the system into two subsystems, one for each time scale. Thus, a different controller is designed for each of them. As far as a linear time invariant model is considered, this time scale separation makes these two subsystems independent of each other and thereby simplify the control design problem and avoid ill-conditioning. The situation is complex when switched systems are considered [11], [18]. It has been shown that even if the slow and the fast subsystems can be computed, they cannot be considered separatly [13]. Stability of these two subsystems independently does not imply stability of the original switched system for small values of the singular perturbation parameter. To our knowledge, there are only few contributions in the context of hybrid systems and singular perturbations. In [9], singular perturbation in piecewise-linear systems are considered. A technique that allows decoupling of such systems into fast and slow subsystems is proposed. In [7], it is shown how an approximate optimal control law can be constructed from the solution of the limit control problem for a particular class of singularly pertrubed hybrid systems: the fast mode of the system is represented by deterministic state equations whereas the slow mode of the system corresponds to a jump disturbance process. In [16], considering the effect of unmodeled sensor/actuator dynamics in the closed loop, it is proved that stability is robust with respect to a class of singular perturbations. Here, we consider continuous time switched linear systems in the singular perturbation form. Our objective is to characterize all possible stability transitions and to analyze the practical stability property for this class of switched systems.

The stability of linear switched system on the plane has been actively studied in the past. A first result has been obtained by Shorten and Narendra in [17], where the authors give a characterization of planar switched systems admitting a common quadratic Lyapunov function. It is well known that, even in dimension two, the existence of a quadratic Lyapunov function is a sufficient but not necessary condition for global uniform asymptotic stability (a two-dimensional example illustrating this fact can be found, for instance, in [4]). Boscain, in collaboration with Balde and Mason, in a series of papers ([3], [1], [2]) provided a complete characterization of stability for linear planar switched system. The novelty of their approach, is that, instead of being based on Lyapunov functions, it exploits the invariants of the system and the concept of *worst trajectory*. This approach has been extended for planar singularly perturbed switched systems in [6]. It is shown in particular that this kind of systems has always a stability/instability behavior common to all switched systems corresponding to the singular perturbation parameter $\epsilon > 0$ small enough. Here, we are interested in describing what happens when ϵ grows and how many times the system can change its stability behavior (asymptotic stability, stability, instability) and which transitions are possible. We also analyze the practical stability property with respect to existing results based on Tikhonov approaches proposed in the context of differential inclusions (see [15], [19], [20], [8]). We show how these approaches can be applied to prove the practical stability of singularly perturbed switched systems. This kind of stability focuses on the behavior of the system on compact time-intervals as ϵ tends to 0 (in particular, it does not ensure the asymptotic stability towards the origin). It is therefore different from the stability criteria

This paper was partially supported by ANR project ArHyCo, Programme “Systèmes Embarqués et Grandes Infrastructures” - ARPEGE, contract number ANR-2008 SEGI 004 01-30011459.

Jamal Daafouz and Fouad El Hachemi are with Centre de Recherche en Automatique de Nancy, UMR 7039 CNRS - Nancy Université, ENSEM, 2, Avenue de la forêt de Haye 54516 Vandoeuvre-lès-Nancy, France, Jamal.Daafouz@ensem.inpl-nancy.fr, fouad.el-hachemi@ensem.inpl-nancy.fr.

Fouad El Hachemi and Mario Sigalotti are with Institut Élie Cartan, UMR 7502 Nancy-Université/INRIA/CNRS, BP 239, Vandœuvre-lès-Nancy 54506, France and CORIDA, INRIA Nancy – Grand Est, France, mario.sigalotti@inria.fr.

discussed above, where ϵ is fixed (arbitrarily small) and the asymptotic behavior for large times is considered. For planar systems, it turns out that when practical stability can be deduced from Tikhonov-type results, then global uniform asymptotic stability (for $\epsilon > 0$ small) holds true.

The paper is organized as follows. Section II is dedicated to preliminaries and recalls the characterization of stability for singularly perturbed planar switched linear systems. Section III contains the description of all possible transitions in the stability behavior as ϵ grows. In section IV existing results for differential inclusions are applied. We end the paper by a conclusion.

II. TOOLS

In this section, we recall relevant stability notions for singularly perturbed switched systems and the invariant quantities introduced in [2], before recalling the stability characterization of planar singularly perturbed switched systems.

A. Notation

For every positive natural number d, denote by $\mathbb{M}_d(\mathbb{R})$ the space of $d \times d$ real-valued matrices. For any $X \in \mathbb{M}_d(\mathbb{R})$, let $\mathrm{tr}(X)$ and $\det(X)$ denote the trace and the determinant of X. Id_d denotes the identity matrix with dimension d. A continuous function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to be of *class* $\mathcal{KL}$ if, for every $r, s \geq 0$, $\beta(r, \cdot)$ is nonincreasing, $\beta(\cdot, s)$ is nondecreasing, and $\beta(0, r) = \lim_{\tilde{s} \to +\infty} \beta(r, \tilde{s}) = 0$. With two matrices $X, Y \in \mathbb{M}_2(\mathbb{R})$, we can associate the following parameters (independent of a common change of coordinates [2]):

$$\delta(X) = \mathrm{tr}(X)^2 - 4\det(X),$$

$$\Gamma(X,Y) = \frac{1}{2}(\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(XY)),$$

$$\tau(X,Y) = \begin{cases} \frac{\mathrm{tr}(X)}{\sqrt{|\delta(X)|}} & \text{if } \delta(X) \neq 0, \\ \frac{\mathrm{tr}(X)}{\sqrt{|\delta(Y)|}} & \text{if } \delta(X) = 0 \text{ and } \delta(Y) \neq 0, \\ \frac{\mathrm{tr}(X)}{2} & \text{if } \delta(X) = \delta(Y) = 0, \end{cases}$$

$$k(X,Y) = \frac{2\tau(X,Y)\tau(Y,X)}{\mathrm{tr}(X)\mathrm{tr}(Y)}(\mathrm{tr}(XY) - \frac{1}{2}\mathrm{tr}(X)\mathrm{tr}(Y)),$$

$$\Delta(X,Y) = 4(\Gamma(X,Y)^2 - \det(X)\det(Y)),$$

$$t(X,Y) = \begin{cases} \frac{\pi}{2} - \arctan \frac{\mathrm{tr}(X)\mathrm{tr}(Y)(k(X,Y)\tau(X,Y)+\tau(Y,X))}{2\tau(X,Y)\tau(Y,X)\sqrt{\Delta(X,Y)}}, \\ \frac{2\sqrt{\Delta(X,Y)}}{\tau(X,Y)\left(\mathrm{tr}(XY)-\frac{1}{2}\mathrm{tr}(X)\mathrm{tr}(Y)\right)}, \\ \mathrm{arctanh} \frac{2\tau(X,Y)\tau(Y,X)\sqrt{\Delta(X,Y)}}{\mathrm{tr}(X)\mathrm{tr}(Y)(k(X,Y)\tau(X,Y)-\tau(Y,X))}, \end{cases}$$

for respectively $\delta(X) < 0$, $\delta(X) = 0$ and $\delta(X) > 0$

$$\mathcal{R}(X,Y) = \frac{2\Gamma(X,Y) + \sqrt{\Delta(X,Y)}}{2\sqrt{\det(X)\det(Y)}} \times e^{\tau(X,Y)t(X,Y)+\tau(Y,X)t(Y,X)}.$$

Notice that the definitions of $\Gamma(\cdot,\cdot)$, $k(\cdot,\cdot)$, $\Delta(\cdot,\cdot)$ and $\mathcal{R}(\cdot,\cdot)$ are symmetric with respect to their two arguments, while those of $\tau(\cdot,\cdot)$ and $t(\cdot,\cdot)$ are not.

B. Stability notions

Let us recall some asymptotic stability notions for linear switched systems of the type

$$\dot{x} = \sigma(t)A_1x(t) + (1 - \sigma(t))A_2x(t) \tag{1}$$

with $A_1, A_2 \in \mathbb{M}_d(\mathbb{R})$ and $\sigma : [0, +\infty) \to \{0, 1\}$ measurable.

Definition 1: We say that the switched system (1) is unbounded if there exists a trajectory (solution of (1)) that goes to infinity as $t \to +\infty$.

Definition 2: We say that the switched system (1) is *globally uniformly asymptotically stable* (GUAS, for short) if there exists a class $\mathcal{KL}$ function β such that, for every switching signal σ and every initial condition $x(0)$, the solution of (1) satisfies the inequality

$$\|x(t)\| \leq \beta(\|x(0)\|, t) \qquad \forall t \geq 0.$$

A particular case of global uniform asymptotic stability is the so-called *quadratic stability*, which indicates the existence of a common quadratic Lyapunov function.

Definition 3: If there exists a positive definite matrix P satisfying

$$A_i^T P + PA_i < 0, \qquad i = 1, 2, \tag{2}$$

then $V(x) = x^T Px$ is called a *common quadratic Lyapunov function* (CQLF, for short) for (1).

A standard stability criterion for switched systems is the following: If the switched system (1) admits a CQLF, then it is GUAS (the converse being false, see [4]).

The objective of this paper is the study of the stability of *singularly perturbed switched systems* (SPSS) of the form

$$\dot{x} = \sigma(t)A_1^\epsilon x(t) + (1 - \sigma(t))A_2^\epsilon x(t) \tag{3}$$

with $\sigma : [0, +\infty) \to \{0, 1\}$ measurable (while ϵ does not depend on time),

$$A_i^\epsilon = \begin{pmatrix} \frac{1}{\epsilon}\mathrm{Id}_{d_1} & 0 \\ 0 & \mathrm{Id}_{d_2} \end{pmatrix} M_i, \qquad i \in \{1, 2\} \tag{4}$$

and $M_1, M_2 \in \mathbb{M}_d(\mathbb{R})$, $d_1 + d_2 = d$. The above definitions lead to the following notions of stability.

Definition 4: We say that the SPSS (3) is *GUAS (respectively, quadratically stable/unbounded) as* $\epsilon \to 0^+$ if there exists ϵ_0 such that for all ϵ in $(0, \epsilon_0)$, the switched system described by (3) (with ϵ fixed) is GUAS (respectively, quadratically stable/unbounded).

C. Characterization of the stability of a planar SPSS

As we are concerned with the stability of planar SPSSs of the form (3) with $d_1 = d_2 = 1$, let us write $M_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$, for $i = 1, 2$. A first necessary condition for the stability of (3) is that $A_i^\epsilon = \left(\begin{smallmatrix} 1/\epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) M_i$ are Hurwitz matrices for all $\epsilon > 0$ small enough and for $i = 1, 2$. Hence, $\mathrm{tr}(A_i^\epsilon) = \frac{a_i}{\epsilon} + d_i$ and $-\det(A_i^\epsilon) = \frac{-\det(M_i)}{\epsilon}$ must be negative. We can therefore restrict our attention to the case in which M_1 and M_2 belong to the set

$$\Lambda = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(M) > 0 \text{ and } (a < 0 \text{ or } (a = 0 \text{ and } d < 0)) \right\}.$$

Using the definitions given in section II-A, it is possible to characterize the stability of a planar SPSS as follows.

Theorem 5 ([6]): Let M_1, M_2 belong to Λ. The stability of the singularly perturbed switched system (3) is described by the following five cases:

(SP1) System (3) is quadratically stable as $\epsilon \to 0^+$ if and only if $\Gamma(M_1, M_2) > -\sqrt{\det(M_1)\det(M_2)}$ and at least one of the following conditions is satisfied
  1) $\Gamma(M_1, M_2) \leq \sqrt{\det(M_1)\det(M_2)}$,
  2) $a_1 a_2 \neq 0$,
  3) $a_1 a_2 = 0$ with $a_1^2 + a_2^2 \neq 0$, and $b_1 c_2 + b_2 c_1 \geq -2\sqrt{\det(M_1)\det(M_2)}$.

(SP2) If $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$, $a_1 a_2 = 0$ with $a_1^2 + a_2^2 \neq 0$, and $b_1 c_2 + b_2 c_1 < -2\sqrt{\det(M_1)\det(M_2)}$, then (3) is GUAS as $\epsilon \to 0^+$.

(SP3) If $\Gamma(M_1, M_2) = -\sqrt{\det(M_1)\det(M_2)}$, then for all $\epsilon > 0$ (3) is uniformly stable but not GUAS.

(SP4) If $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$ and $a_1 = a_2 = 0$, then (3) is unbounded as $\epsilon \to 0^+$.

(SP5) If $\Gamma(M_1, M_2) < -\sqrt{\det(M_1)\det(M_2)}$, then for all $\epsilon > 0$ (3) is unbounded.

In the special case where $a_1 = a_2 = 0$ the conditions given above can be simplified as follows.

Corollary 6: Let M_1, M_2 belong to Λ. If $a_1 = a_2 = 0$ then (3) is GUAS as $\epsilon \to 0^+$ if and only if $b_1 c_2 = b_2 c_1 < 0$.

Proof: Since $a_1 = a_2 = 0$, then $\Gamma(M_1, M_2) = -(b_1 c_2 + b_2 c_1)/2$ and $\sqrt{\det(M_1)\det(M_2)} = \sqrt{b_1 b_2 c_1 c_2}$. Since, moreover, M_1 and M_2 belong to Λ, then $b_1 c_1, b_2 c_2 < 0$ and, in particular, $\mathrm{sign}(b_i) = -\mathrm{sign}(c_i)$ for $i = 1, 2$. Thus, $\mathrm{sign}(b_1 c_2) = \mathrm{sign}(b_2 c_1)$. If $b_1 c_2 < 0$, then $\Gamma(M_1, M_2) \geq \sqrt{\det(M_1)\det(M_2)}$, as it follows from the inequality $(\sqrt{-b_1 c_2} - \sqrt{-b_2 c_1})^2 \geq 0$. Similarly, if $b_1 c_2 > 0$, then $\Gamma(M_1, M_2) \leq -\sqrt{\det(M_1)\det(M_2)}$, thanks to $(\sqrt{b_1 c_2} - \sqrt{b_2 c_1})^2 \geq 0$, and we conclude by (SP1). ■

III. TRANSITIONS BETWEEN STABILITY BEHAVIORS

The classification given in Theorem 5 guarantees that for ϵ in a right neighborhood of 0, system (3) belongs to one of the classes identified by Balde, Boscain and Mason in [2], where it was proved that, for $A_1, A_2 \in \mathbb{M}_2(\mathbb{R})$ Hurwitz, the stability of the switched system (1) is determined by the following four statements:

(S1) System (1) is quadratically stable if and only if $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ and $\mathrm{tr}(A_1 A_2) > -2\sqrt{\det(A_1)\det(A_2)}$;

(S2) If $\Gamma(A_1, A_2) < -\sqrt{\det(A_1)\det(A_2)}$, then (1) is unbounded;

(S3) If $\Gamma(A_1, A_2) = -\sqrt{\det(A_1)\det(A_2)}$, then (1) is uniformly stable but not GUAS;

(S4) $\Gamma(A_1, A_2) > \sqrt{\det(A_1)\det(A_2)}$ and $\mathrm{tr}(A_1 A_2) \leq -2\sqrt{\det(A_1)\det(A_2)}$, then (1) is GUAS, uniformly stable, or unbounded if $\mathcal{R}(A_1, A_2) < 1$, $\mathcal{R}(A_1, A_2) = 1$ or $\mathcal{R}(A_1, A_2) > 1$, respectively.

In particular, it cannot happen that, as ϵ converges to 0, the stability of (3) changes infinitely many times.It is nevertheless possible that, as $\epsilon > 0$ increases beyond the right neighborhood of 0 on which the stability of (3) does not depend on ϵ (whose existence is guaranteed by Theorem 5), (3) changes its stability behavior, passing to a different chart of the atlas (S1–4) given above. However, only some transitions are possible. We discuss below which and how many of them are possible as ϵ increases.

First of all, notice that speaking of the charts (S1–4) makes sense only as long as A_1^ϵ and A_2^ϵ stay Hurwitz. If $d_i \leq 0$ then A_i^ϵ is Hurwitz for every $\epsilon > 0$, since $\mathrm{tr}(A_i^\epsilon)$ is negative for all $\epsilon > 0$. On the other hand, if $d_i > 0$ then A_i^ϵ is Hurwitz for $\epsilon < -a_i/d_i$.

Since $\epsilon \mapsto \Gamma(A_1^\epsilon, A_2^\epsilon)$ and $\epsilon \mapsto \sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$ are (-1)-homogeneous, transitions can happen only between cases (S1) and (S4) and they require that $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$. Each transition is triggered by a change of sign of

$$\begin{aligned}\eta(\epsilon) &= \mathrm{tr}(A_1^\epsilon A_2^\epsilon) + 2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)} \\ &= \frac{a_1 a_2}{\epsilon^2} + \frac{b_1 c_2 + b_2 c_1 + 2\sqrt{\det(M_1)\det(M_2)}}{\epsilon} + d_1 d_2.\end{aligned}$$

We distinguish three main cases.

A. Case where $a_1 = a_2 = 0$

In this case, $d_1 d_2 > 0$ and $\epsilon\eta(\epsilon)$ is affine with respect to ϵ, with a positive coefficient multiplying ϵ and $\lim_{\epsilon \to 0} \epsilon\eta(\epsilon) < 0$. Hence, only one transition happens, from (S4) to (S1).

For instance, the planar SPSS characterized by the matrices

$$M_1 = \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$$

illustrates this. Indeed, we compute easily $\epsilon_0 = 7/3$ such that for all $\epsilon \in (0, \epsilon_0)$ the system is of type (S4) and for all $\epsilon \in (\epsilon_0, +\infty)$ it is of type (S1) (see Figure 1).

B. Case where $a_1 a_2 = 0$ and $a_1^2 + a_2^2 \neq 0$

In this case, the possible transition patterns are: a single transition from (S4) to (S1), or a single transition from (S1) to (S4).

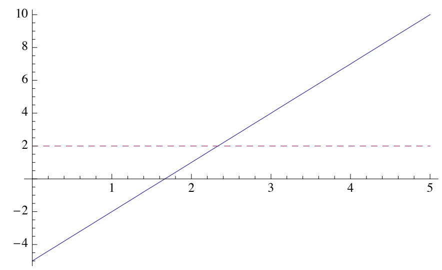

Fig. 1. The continuous and dashed line are the graph of $\epsilon \mapsto \epsilon\,\mathrm{tr}(A_1^\epsilon A_2^\epsilon)$ and $\epsilon \mapsto -2\epsilon\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$, respectively.

As an example, the planar SPSS characterized by the matrices

$$M_1 = \begin{pmatrix} 0 & 2 \\ -4 & -6 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & -0.12 \\ 2 & 0.2 \end{pmatrix}$$

illustrates the transition from (S1) to (S4). Indeed, we compute easily $\epsilon_0 = \frac{8-\sqrt{6}}{3}$ such that for all $\epsilon \in (0, \epsilon_0)$ the system is of type (S1) and for all $\epsilon \in (\epsilon_0, +\infty)$ it is of type (S4) (see Figure 2).

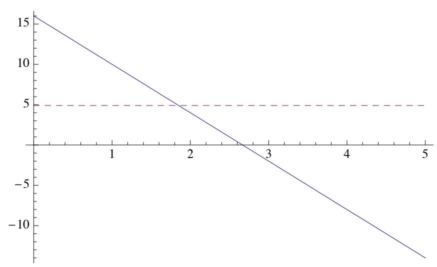

Fig. 2. The continuous and dashed line are the graph of $\epsilon \mapsto \epsilon\,\mathrm{tr}(A_1^\epsilon A_2^\epsilon)$ and $\epsilon \mapsto -2\epsilon\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$, respectively.

C. *Case where $a_1a_2 \neq 0$*

In this case, $\eta(\epsilon)$ can change sign zero, one, or two times as ϵ varies in $(0, +\infty)$. Hence, (3) can pass from case (S1) to (eventually) case (S4) and then (eventually) back to case (S1) as ϵ increases.

The example below shows a double transition, first from (S1) to (S4) and then back to case (S1), where, in addition, all three subcases of (S4) actually show up as ϵ varies.

Consider the planar SPSS characterized by the matrices

$$M_1 = \begin{pmatrix} -1 & 0.01 \\ -9 & -1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 2 \\ -2 & -2 \end{pmatrix}.$$

The solutions of $\mathrm{tr}(A_1^\epsilon A_2^\epsilon) = -2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$ are $\epsilon_0 = 0.0784$ and $\epsilon_1 = 6.3742$, so that for all $\epsilon \in (0, \epsilon_0)\bigcup(\epsilon_1, \infty)$ the system is of the type (S1), while for $\epsilon \in (\epsilon_0, \epsilon_1)$ it is of type (S4) (see Figure 3).

Analyzing $\mathcal{R}^\epsilon$, we compute $\epsilon_2 = 0.32$ and $\epsilon_3 = 1.62$, solutions of $\mathcal{R}^\epsilon - 1 = 0$, so that for all $\epsilon \in (\epsilon_0, \epsilon_2)\bigcup(\epsilon_3, \epsilon_1)$ we have $\mathcal{R}^\epsilon - 1 < 0$ and for all $\epsilon \in (\epsilon_2, \epsilon_3)$ the same quantity is positive (see Figure 4). Hence, as ϵ varies in (ϵ_0, ϵ_1) all three subcases of (S4) show up.

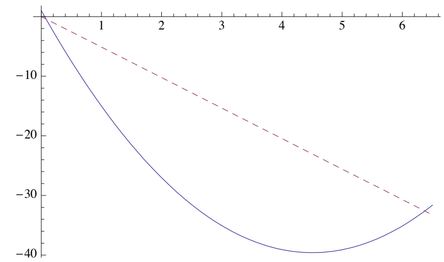

Fig. 3. The continous and dashed line are the graph of $\epsilon \mapsto \epsilon^2\mathrm{tr}(A_1^\epsilon A_2^\epsilon)$ and $\epsilon \mapsto -2\epsilon^2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$, respectively.

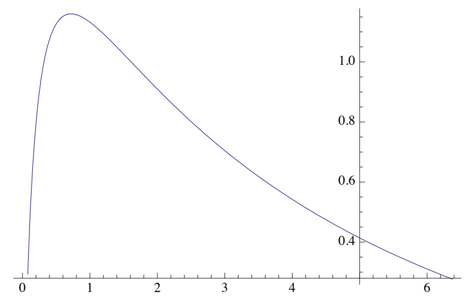

Fig. 4. Graph of $(\epsilon_0, \epsilon_1) \ni \epsilon \mapsto \mathcal{R}^\epsilon$.

Remark 7: The case (S4) gives rise, for SPSSs, to the two cases (SP2) and (SP4) described in Theorem 5. Notice that in case (S4) the system can be asymptotically stable, stable or unbounded, depending on the value of $\mathcal{R}$. Cases (SP2) and (SP4) correspond, respectively, to the situations in which $\mathcal{R}^\epsilon$ converges to 1 from below and has limit larger than 1 (see [6]). The case $\mathcal{R} = 1$ does not give rise to any subcase in Theorem 5, since it turns out to be impossible for $\mathcal{R}^\epsilon$ to be identically equal to 1 as ϵ varies in a right neighborhood of 0.

IV. Time scale separation approach

A. *Classical approach*

In the case of dynamical systems without switching, singular perturbation theory provides a set of tools to analyze stability by time-scale separation (see, e.g., [10]). Tikhonov theorem epitomizes this approach, by identifying two systems, corresponding to the fast and the slow dynamics of the system, whose separate stabilities are equivalent to the stability of the overall system for small values of the perturbation parameter. By formally applying this classical approach to system (3), that is decoupling into slow and fast dynamics ignoring the switched nature of the system under consideration, we would end up with the following two independent switched systems

$$\begin{aligned} \dot{x}_1 &= a_\sigma x_1, & \sigma \in \{1, 2\}, \\ \dot{x}_2 &= \left(d_\sigma - \frac{b_\sigma c_\sigma}{a_\sigma}\right) x_2, & \sigma \in \{1, 2\}. \end{aligned}$$

The stability of both such switched systems is equivalent to

$$a_i < 0, \tag{5}$$

$$a_i d_i - b_i c_i > 0, \tag{6}$$

for $i = 1, 2$. Such decomposition is not correct in general, as explained below.

As follows from Theorem 5, the necessary and sufficient conditions for the asymptotic stability as $\epsilon \to 0^+$ of the SPSS (3) are that the elements of the matrices M_1 and M_2 satisfy

$$\begin{cases} \bullet\ a_i < 0 \text{ or } (a_i = 0 \text{ and } d_i < 0), \quad i = 1, 2 \\ \bullet\ a_i d_i - b_i c_i > 0, \quad i = 1, 2 \\ \bullet\ \Gamma(M_1, M_2) > -\sqrt{\det(M_1)\det(M_2)} \\ \bullet\ \text{either } \Gamma(M_1, M_2) \le \sqrt{\det(M_1)\det(M_2)} \\ \quad \text{or} \quad a_1 a_2 \neq 0 \\ \quad \text{or} \quad a_1 a_2 = 0 \text{ and } a_1^2 + a_2^2 \neq 0. \end{cases}$$

So, conditions (5) and (6) are neither necessary nor sufficient for the stability of (3). In order to show that (5) and (6) are not necessary it suffices to choose a SPSS with $a_1 a_2 = 0$ that is GUAS as $\epsilon \to 0^+$. One could take, for instance,

$$A_1^\epsilon = A_2^\epsilon = \begin{pmatrix} 0 & -\frac{1}{\epsilon} \\ 1 & -1 \end{pmatrix}.$$

The fact that (5) and (6) are not sufficient for the asymptotic stability of (3) as $\epsilon \to 0^+$ was already mentioned in [12], [13], where the following choice of M_1, M_2 was introduced:

$$M_1 = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 0 \\ \alpha & -1 \end{pmatrix}. \tag{7}$$

Conditions (5) and (6) are satisfied for every $\alpha \in \mathbb{R}$. Using (SP1) we obtain that system (7) admits a CQLF as $\epsilon \to 0^+$ if and only if $\alpha \in (-2, 2)$. Indeed, condition $\Gamma(M_1, M_2) > -\sqrt{\det(M_1)\det(M_2)}$ reads $\frac{2-\alpha^2}{2} > -1$ which is equivalent to $4 - \alpha^2 > 0$. (Notice that, using the sufficient conditions for the existence of a CQLF as $\epsilon \to 0^+$ obtained in [12] and [13] by linear matrix inequalities, system (7) is shown to be quadratically stable for $-1 < \alpha < 1$. Such conditions are, therefore, conservative.) When $|\alpha| = 2$ and $|\alpha| > 2$, conditions (SP3) and (SP5) imply that system (7) is uniformly stable and unbounded, respectively, disproving the sufficiency of (5) and (6).

B. Differential inclusions

Extensions of Tikhonov theory to differential inclusions have already been considered by several authors (see, in particular, [5], [15], [19], [20]).

Such results, in the context SPSSs, can be used to prove Theorem 8 below.

We recall that a SPSS of the type (3) is said to be *practically stable* if there exists a function β of class $\mathcal{KL}$ such that, for every $\delta > 0$, there exists $\epsilon_0 > 0$ such that every trajectory of (3) with $\epsilon \in (0, \epsilon_0)$ satisfies

$$\|(x_1(t), x_2(t))\| \le \beta(\|(x_1(0), x_2(0))\|, t) + \delta, \quad t \ge 0. \tag{8}$$

The notion of practical stability given in (8) is not comparable with the notion of global uniform asymptotic stability as $\epsilon \to 0^+$ used in the previous section. Indeed, none of the two notions is *a priori* stronger than the other. While practical stability focuses on the uniformity with respect to ϵ, the uniformity in the notion of GUAS as $\epsilon \to 0^+$ is only with respect to the initial condition and the switching function and holds at ϵ fixed. On the other hand, GUAS as $\epsilon \to 0^+$ guarantees the convergence towards the origin as $t \to +\infty$ for small values of ϵ, which is not the case for practical stability.

Theorem 8: Assume that

$$a_1, a_2 < 0 \tag{9}$$

and

$$-\frac{b_i c_j}{a_i} + d_j < 0, \quad \text{for } i, j = 1, 2. \tag{10}$$

Then (3) is quadratically stable as $\epsilon \to 0^+$ and practically stable.

Proof: We start by the first part of the statement, proving that (9) and (10) imply that (3) is of type (SP1), in the classification proposed in Theorem 5.

Notice that (9) and (10) imply that both $a_1 d_2 - b_1 c_2$ and $a_2 d_1 - b_2 c_1$ are positive. Hence, $2\Gamma(M_1, M_2) = a_1 d_2 + a_2 d_1 - b_1 c_2 - b_2 c_1$ is positive and, in particular, $\Gamma(M_1, M_2) > 0 > -\sqrt{\det(M_1)\det(M_2)}$. Since $a_1 a_2 \neq 0$, we are then in the case (SP1).

The second part of the statement is proved using a general result proved by Watbled in [20] in the framework of nonlinear singularly perturbed differential inclusions. In order to do so, let us consider the differential inclusion

$$\begin{pmatrix} \epsilon \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} \in F(x_1(t), x_2(t)) \tag{11}$$

where

$$F(x_1(t), x_2(t)) := \text{co}\left\{ M_1 \left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right), M_2 \left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right) \right\}$$

and $\text{co}(\cdot)$ denotes the convex hull operator. Equation (11) can be seen as obtained from (3) by convexification. By construction, the set-valued map F has convex and compact values.

Define P_1 (respectively, P_2) as the projection of $\mathbb{R} \times \mathbb{R}$ on the first (respectively, second) component $\mathbb{R}$. For every $x_2 \in \mathbb{R}$, consider the differential inclusion

$$\dot{x}_1 \in P_1 F(x_1, x_2) = \text{co}\{a_1 x_1 + b_1 x_2, a_2 x_1 + b_2 x_2\}. \tag{12}$$

The set of equilibria of the differential inclusion (12) is

$$S(x_2) = \text{co}\left\{ -\frac{b_1}{a_1} x_2, -\frac{b_2}{a_2} x_2 \right\}.$$

Let

$$\begin{aligned} D(x_2) &= \text{co} P_2(F(S(x_2), x_2)) = \text{co}\{c_i y + d_i x_2 \mid i = 1, 2, \\ &\quad y \in S(x_2)\} \\ &= \text{co}\left\{ \left(-\frac{b_1 c_1}{a_1} + d_1\right) x_2, \left(-\frac{b_2 c_1}{a_2} + d_1\right) x_2, \right. \\ &\quad \left. \left(-\frac{b_1 c_2}{a_1} + d_2\right) x_2, \left(-\frac{b_2 c_2}{a_2} + d_2\right) x_2 \right\}. \end{aligned}$$

Theorem 3.1 in [20] guarantees that if $S(x_2)$ is asymptotically stable for (12) and if the point $\{0\}$ is an asymptotically stable equilibrium for the differential inclusion $\dot{x}_2 \in D(x_2)$ then system (3) is practically stable. (Although Watbled's result is local, global stability follows from the homogeneity of the linear system (3).)

The asymptotically stability of $S(x_2)$ for (12) is ensured by assumption (9). On the other hand, (10) implies that $\{0\}$ is asymptotically stable for the differential inclusion $\dot{x}_2 \in D(x_2)$, concluding the proof of Theorem 8. ■

Let us notice that the hypotheses of the theorem, namely, conditions (9) and (10), are more restrictive than those identified in case (SP1) of Theorem 5. This is illustrated, for instance, by the SPSS (7), for which $a_1 d_2 - b_1 c_2 = 1 - \alpha^2$ becomes negative when $|\alpha| > 1$. Hence, (9) and (10) hold for $|\alpha| < 1$, while the system is of type (SP1) (with $a_1 a_2 \neq 0$) for $|\alpha| < 2$.

For $|\alpha| < 1$ the SPSS (7) is both quadratically stable as $\epsilon \to 0^+$ and practically stable. The convergence to the origin of its trajectories is then uniform with respect to ϵ in the sense of (8). It should be stressed, nevertheless, that (7) admits no CQLF independent of ϵ if $\alpha \neq 0$. Indeed, let

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0$$

and assume that $V(x) = x^T P x$ is such a CQLF.

Then

$$0 \geq \lim_{\epsilon \to 0} P A_2^\epsilon + (A_2^\epsilon)^T P = \begin{pmatrix} -2p_{11} & -p_{12} \\ -p_{12} & 0 \end{pmatrix},$$

leading to $p_{12} = 0$ and

$$0 \geq \lim_{\epsilon \to 0} P A_1^\epsilon + (A_1^\epsilon)^T P = \begin{pmatrix} -2p_{11} & \alpha p_{11} \\ \alpha p_{11} & 0 \end{pmatrix},$$

which leads to a contradiction, since $p_{11} > 0$.

V. Conclusion

In [6], a complete classification of quadratically stable, GUAS, stable, and unbounded singularly perturbed planar switched systems was proposed leading to a characterization of the asymptotic stability behavior of such singularly perturbed switched systems as the perturbation parameter goes to zero. As a complementary study, here we answer the questions related to what happens as ϵ grows and how many times the system can change its stability behavior (asymptotic stability, stability, instability) and which transitions are possible. Moreover, the practical stability property has been analyzed with respect to existing results based on Tikhonov approaches proposed in the context of differential inclusions. For planar systems, it turns out that when practical stability can be deduced from Tikhonov-type results, then global uniform asymptotic stability (for $\epsilon > 0$ small) holds true. It is an open question whether this is still true for higher dimensional singularly perturbed switched systems. An important aspect in control problems of singularly perturbed systems, which is not raised in this article, concerns the evaluation of the maximum value of ϵ that guarantees the stability for any $\epsilon \in (0, \epsilon_{max}]$. Looking for the exact value of ϵ_{max} is a challenging and a difficult problem, known as the ϵ-bound problem, and it remains an open problem in the classical LTI case.

References

[1] M. Balde and U. Boscain. Stability of planar switched systems: the nondiagonalizable case. *Communications on Pure and Applied Analysis*, 7(1):1–21, 2008.

[2] M. Balde, U. Boscain, and P. Mason. A note on stability conditions for planar switched systems. *International Journal of Control*, 82(10):1882–1888, 2009.

[3] U. Boscain. Stability of planar switched systems: the linear single input case. *SIAM Journal on Control and Optimization*, 41(1):89–112, 2002.

[4] W. P. Dayawansa and C. F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, 44(4):751–760, 1999.

[5] A. Dontchev, T. Donchev, and I. Slavov. A Tikhonov-type theorem for singularly perturbed differential inclusions. *Nonlinear Anal.*, 26(9):1547–1554, 1996.

[6] F. El Hachemi, M. Sigalotti, and J. Daafouz. Stability of singularly perturbed switched systems. In *Proceedings of IEEE American Control Conference*, pages 4061–4066, San Francisco, USA, 2011.

[7] J. A. Filar, V. Gaitsgory, and A. B. Haurie. Control of singularly perturbed hybrid stochastic systems. *IEEE Transactions on Automatic Control*, 46(2):179–190, 2001.

[8] G. Grammel. Exponential stability of nonlinear singularly perturbed differential equations. *SIAM Journal on Control and Optimization*, 44(5):1712–1724, 2005.

[9] B. S. Heck and A. H. Haddad. Singular perturbation in piecewise-linear systems. *IEEE Transactions on Automatic Control*, 34(1):87–90, 1989.

[10] P. V. Kokotović, H. K. Khalil, and J. O'Reilly. *Singular perturbation methods in control: analysis and design*. Academic Press Inc., London, 1986.

[11] D. Liberzon. *Switching in systems and control*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 2003.

[12] I. Malloci. *Two time scale switched systems: An application to steering control in hot strip mills*. PhD thesis, Institut National Polytechnique de Lorraine, 2009.

[13] I. Malloci, J. Daafouz, and C. Iung. Stabilization of continuous-time singularly perturbed switched systems. In *48th IEEE Conference on Decision and Control*, pages 6371–6376, Shanghai, China, 2009.

[14] D. S. Naidu. *Singular perturbation methodology in control systems*, volume 34 of *IEE Control Engineering Series*. Peter Peregrinus Ltd., Hitchin, 1988.

[15] M. Quincampoix. Singular perturbations for differential equations and inclusions: an approach through constrained systems. *Nonlinear Anal.*, 32(5):697–710, 1998.

[16] R. Sanfelice, A. Teel, R. Goebel, and C. Prieur. On the robustness to measurement noise and unmodeled dynamics of stability in hybrid systems. In *Proceedings of IEEE American Control Conference*, pages 4061–4066, Minneapolis, USA, 2006.

[17] R. Shorten and K. Narendra. Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order liner time-invariant system. In *Proceedings of IEEE American Control Conference*, pages 1410–1414, San Diego, USA, 1999.

[18] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49(4):545–592, 2007.

[19] V. Veliov. A generalization of the Tikhonov theorem for singularly perturbed differential inclusions. *Journal of Dynamical and Control Systems*, 3(3):291–319, 1997.

[20] F. Watbled. On singular perturbations for differential inclusions on the infinite interval. *Journal of Mathematical Analysis and Applications*, 310(2):362–378, 2005.