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Nonlinear Stabilization under Sampled and Delayed Measurements, and with Inputs Subject to Delay and Zero-Order Hold

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Abstract—**Sampling arises simultaneously with input and output delays in networked control systems. When the delay is left uncompensated, the sampling period is generally required to be sufficiently small, the delay sufficiently short, and, for nonlinear systems, only semiglobal practical stability is generally achieved. In this paper we present two general results. First, we present global asymptotic stabilizers for forward complete systems under arbitrarily long input and output delays, with arbitrarily long sampling periods, and with continuous application of the control input. Second, we consider systems with sampled measurements and with control applied through a zero-order hold, under the assumption that the system is stabilizable under sampled-data feedback for some sampling period, and then construct sampled-data feedback laws that achieve global asymptotic stabilization under arbitrarily long input and measurement delays. All the results employ "nominal" feedback laws designed for the continuous-time systems in the absence of delays, combined with "predictor-based" compensation of delays and the effect of sampling.**

I. Introduction

SAMPLING arises simultaneously with input and output delays in many control problems, most notably in control over networks. In the absence of delays, in sampled-data control of nonlinear systems semiglobal practical stability is generally guaranteed [6,27,28,29], with the desired region of attraction achieved by sufficiently fast sampling. Alternatively, global results are achieved under restrictive conditions on the structure of the system [5,8,11,12,14,31]. On the other hand, in purely continuous-time nonlinear control, input delays of arbitrary length can be compensated [15,19,20] but no sampled-data extensions of such results are available. Simultaneous consideration to sampling and delays (either physical or sampling-induced) is given in the literature on control of linear and nonlinear systems over networks [3,4,7,26,30,31,32,33] or sampled-data control [2,22], but most available results rely on delay-dependent conditions for the existence of stabilizing feedback.

Despite the remarkable accomplishments in the fields of sampled-data, networked, and nonlinear delay systems, the following example problems remain open: global stabilization of strict-feedforward systems under sampled measurements and continuous control, sampled-data stabilization of the nonholonomic unicycle with inputs applied via zero-order hold and under arbitrarily sparse sampling, and sampled-data stabilization of LTI systems over networks with long delays.

In this paper we introduce two frameworks for solving such problems:

1. We present global asymptotic stabilizers for forward complete systems under arbitrarily long input and output delays, with arbitrarily long sampling periods, and with continuous application of the control input.
2. We consider systems with sampled measurements and with control applied through a zero-order hold, under the assumption that the system is stabilizable under sampled-data feedback for some sampling period, and then construct sampled-data feedback laws that achieve global asymptotic stabilization under arbitrarily long input and measurement delays.

In both frameworks we employ "nominal" feedback laws designed in the absence of delays, combined with "predictor-based" compensation of delays.

<u>*Problem Statement.*</u> As in [15,19,20,21,23,24,25,34], we consider systems with input delay,

$$\dot{x}(t) = f(x(t), u(t-\tau)) \tag{1}$$

where $x(t) = (x_1(t),....,x_n(t))' \in \Re^n$, $u(t) \in \Re^m$, $f : \Re^n \times \Re^m \to \Re^n$ is a locally Lipschitz mapping with $f(0,0) = 0$ and $\tau \geq 0$ is a constant. In [15,19,20,21], the feedback design problem for system (1) is addressed by assuming a feedback stabilizer $u = k(x)$ for system (1) with no delay, i.e. (1) with $\tau = 0$, or

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

and applying a delay compensator (predictor) methodology based on the knowledge of the delay. In this paper, we incorporate also a consideration of measurement delay, namely, we address the problem of stabilization of (1) with output

$$y(t) = x(t-r) \in \Re^n \tag{3}$$

where $r \geq 0$ is a constant, i.e., we consider delayed measurements. The motivation for a simultaneous consideration of input and measurement delays is that in many chemical process control problems the measurement delay of concentrations of chemical species can be large.

We also assume that the output is available at discrete time instants τ_i (the sampling times) with $\tau_{i+1} - \tau_i = T > 0$, where $T > 0$ is the sampling period. Very few papers have studied this problem (an exception is [9] where input and measurement delays are considered for linear systems but the measurement is not sampled and the papers [2,22] where no measurement delay is present).

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The problem of stabilization of (1) with output given by (3) is intimately related to the stabilization of system (1) alone. To see this, notice that the output $y(t)$ of (1), (3) satisfies the following system of differential equations for all $t \geq r$:

$$\dot{y}(t) = f(y(t), u(t-r-\tau))$$

Consider the comparison between two problems described by the same differential equations: the problem of stabilization of (1) with input delay $r > 0$ and no measurement delay (i.e., $\dot{x}(t) = f(x(t), u(t-r))$ for all $t \geq 0$) and the problem of stabilization of (1), (3) with no input delay and measurement delay $r > 0$ (i.e., $\dot{y}(t) = f(y(t), u(t-r))$ for all $t \geq r$). The two problems are not identical: in the first stabilization problem the applied input values for $t \in [0, r]$ are given (as initial conditions), while in the second stabilization problem the applied input values for $t \in [0, r]$ must be computed based on an arbitrary initial condition $x(\theta) = x_0(\theta)$, $\theta \in [-r, 0]$ (irrespective of the current value of the state). Therefore, serious technical issues concerning the existence of the solution for $t \in [0, r]$ arise for the second stabilization problem.

Results of the paper. We establish two general results:

1. A solution for the stabilization of (1) with output given by (3) under the assumption that system (2) is globally stabilizable and forward complete and *the input can be continuously adjusted* (Theorem 2.1). The proposed dynamic sampled-data controller uses values of the output (3) at the discrete time instants $\tau_i = t_0 + iT, i \in Z^+$, where $T > 0$ is the sampling period and $t_0 \geq 0$ is the initial time. This justifies the term "sampled-data". No restrictions for the values of the delays $r, \tau \geq 0$ or the sampling period $T > 0$ are imposed. In general, we show that there is no need for continuous measurements for global asymptotic stabilization of any stabilizable forward complete system with arbitrary input and output delays.
2. A solution for the stabilization of (1) with output given by (3) under the assumption that system (2) is globally stabilizable and forward complete and *the control action is implemented with zero order hold* (Theorem 3.2). Again, the proposed sampled-data controller uses values of the output (3) at the discrete time instants $\tau_i = t_0 + iT, i \in Z^+$, where $T > 0$ is the sampling period and $t_0 \geq 0$ is the initial time. In this case, we can solve the stabilization problem for systems with both delayed inputs and measurements provided that the user chooses the sampling period as the ratio of the input delay and any integer.

Our delay compensation methodology guarantees that any controller (continuous or sampled-data) designed for the delay-free case can be used for the regulation of the delayed system with input/measurement delays and sampled measurements. For example, all sampled-data feedback designs proposed in [5,6,11,14,27,28,29,31] which guarantee global stabilization can be exploited for the stabilization of a delayed system with input/measurement delays, sampled measurements and input applied with zero order hold. The results can be directly applied to the case of linear autonomous systems and to the case of nonlinear systems which are diffeomorphically equivalent to a chain of integrators (see [16]). Moreover, the stabilization problem for nonholonomic unicycle with arbitrarily sparse sampling is also addressed in [16].

Due to space limitations all proofs are omitted and are available upon request.

Notations Throughout this paper we adopt the following notations:

* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm, by x' its transpose.
* $\Re^+$ denotes the set of non-negative real numbers. Z^+ denotes the set of non-negative integers. For every $t \geq 0$, $[t]$ denotes the integer part of $t \geq 0$, i.e., the largest integer being less or equal to $t \geq 0$.
* For the definition of the class of functions KL, see [17].
* By $C^j(A)$ ($C^j(A;\Omega)$), where $j \geq 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
* Let $x:[a-r,b) \to \Re^n$ with $b > a \geq 0$ and $r \geq 0$. By $T_r(t)x$ we denote the "history" of x from $t-r$ to t, i.e., $(T_r(t)x)(\theta) := x(t+\theta)$; $\theta \in [-r,0]$, for $t \in [a,b)$. By $\breve{T}_r(t)x$ we denote the "open history" of x from $t-r$ to t, i.e., $(\breve{T}_r(t)x)(\theta) := x(t+\theta)$; $\theta \in [-r,0)$, for $t \in [a,b)$.
* Let $I \subseteq \Re^+ := [0,+\infty)$ be an interval. By $L^\infty(I;U)$ ($L^\infty_{loc}(I;U)$) we denote the space of measurable and (locally) bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$. Notice that we do not identify functions in $L^\infty(I;U)$ which differ on a measure zero set. For $x \in L^\infty([-r,0];\Re^n)$ or $x \in L^\infty([-r,0);\Re^n)$ we define $\|x\|_r := \sup_{\theta\in[-r,0]} |x(\theta)|$ or $\|x\|_r := \sup_{\theta\in[-r,0)} |x(\theta)|$. Notice that $\sup_{\theta\in[-r,0]} |x(\theta)|$ is not the essential supremum but the actual supremum and that is why the quantities $\sup_{\theta\in[-r,0]} |x(\theta)|$ and $\sup_{\theta\in[-r,0)} |x(\theta)|$ do not coincide in general. We will also use the notation M_U for the space of measurable and locally bounded functions $u:\Re^+ \to U$.
* We say that a system of the form (2) is forward complete if for every $x_0 \in \Re^n$, $u \in M_U$ the solution $x(t)$ of (2) with initial condition $x(0) = x_0 \in \Re^n$ corresponding to input $u \in M_U$ exists for all $t \geq 0$.

II. Dynamic Sampled-Data Feedback For Continuously Adjusted Input

We start by presenting the assumptions for system (2). Our first assumption concerning system (2) is forward completeness.

Hypothesis (H1): *System (2) is forward complete.*

Assumption (H1) guarantees that system (1) is forward complete as well: for every $x_0 \in \Re^n$, $u \in L^\infty_{loc}([-\tau,+\infty);\Re^m)$ the solution $x(t)$ of (1) with initial condition $x(0) = x_0 \in \Re^n$ corresponding to input $u \in L^\infty_{loc}([-\tau,+\infty);\Re^m)$ exists for all $t \geq 0$. Therefore, we are in a position to define the "predictor" mapping $\Phi : \Re^n \times L^\infty\left([-r-\tau,0);\Re^m\right) \to \Re^n$ for all $r,\tau \geq 0$ with $r+\tau > 0$ in the following way:

"for every $x_0 \in \Re^n$, $u \in L^\infty\left([-r-\tau,0);\Re^m\right)$ the solution $x(t)$ of (1) with initial condition $x(-r) = x_0$ corresponding to input $u \in L^\infty\left([-r-\tau,0);\Re^m\right)$ satisfies $x(\tau) = \Phi(x_0,u)$"

By virtue of the results in [1,10], we can guarantee the existence of $a \in K_\infty$ such that

$$|\Phi(x,u)| \leq a\left(|x| + \|u\|_{r+\tau}\right),$$
$$\text{for all } (x,u) \in \Re^n \times L^\infty\left([-r-\tau,0);\Re^m\right) \qquad (4)$$

We assume next that (2) is globally stabilizable.

Hypothesis (H2) (continuously adjusted input): *There exists $k \in C^1(\Re^+ \times \Re^n;\Re^m)$, $g \in K_\infty$ with*

$$|k(t,x)| \leq g(|x|), \text{ for all } (t,x) \in \Re^+ \times \Re^n \qquad (5)$$

such that $0 \in \Re^n$ is Uniformly Globally Asymptotically Stable for system (2) with $u = k(t,x)$, i.e., there exists a function $\sigma \in KL$ such that for every $(t_0,x_0) \in \Re^+ \times \Re^n$ the solution $x(t)$ of (2) with $u = k(t,x)$ and initial condition $x(t_0) = x_0 \in \Re^n$ satisfies the following inequality:

$$|x(t)| \leq \sigma\left(|x_0|, t-t_0\right), \ \forall t \geq t_0 \qquad (6)$$

Consider system (1) under hypotheses (H1), (H2) for system (2). Our proposed dynamic sampled-data feedback has states $(z(t), T_{r+\tau}(t)u) \in \Re^n \times L^\infty\left([-r-\tau,0];\Re^m\right)$ and inputs $y(t) \in \Re^n$ and for each $t_0 \geq 0$, $(z_0,u_0) \in \Re^n \times L^\infty\left([-r-\tau,0];\Re^m\right)$ the states are computed by the interconnection of two subsystems:

1) A sampled-data subsystem (see [10]) with inputs $(y(t), T_{r+\tau}(t)u) \in \Re^n \times L^\infty\left([-r-\tau,0];\Re^m\right)$:

$$\begin{aligned}
&\dot{z}(t) = f(z(t),u(t)) \ , \ t \in [\tau_i, \tau_{i+1}), i \in Z^+ \\
&z(\tau_{i+1}) = \Phi\left(y(\tau_{i+1}), \breve{T}_{r+\tau}(\tau_{i+1})u\right) \\
&z(t_0) = z_0 \in \Re^n
\end{aligned} \qquad (7)$$

where

$$\tau_i = t_0 + iT \ , \ i \in Z^+$$

are the sampling times and $T > 0$ is the sampling period. We stress that the proposed sampled-data dynamic controller uses only values of the output $y(t) = x(t-r) \in \Re^n$ at the discrete time instants $\tau_i = t_0 + iT$, where $i \in Z^+$.

2) A subsystem described by Functional Difference Equations (see [13]) with inputs $z(t) \in \Re^n$:

$$\begin{aligned}
&u(t) = k(t+\tau, z(t)), t > t_0 \\
&T_{r+\tau}(t_0)u = u_0 \in L^\infty\left([-r-\tau,0];\Re^m\right)
\end{aligned} \qquad (8)$$

Our first main result is now stated.

Theorem 2.1: *Let $T > 0$, $r,\tau \geq 0$ with $r+\tau > 0$ and suppose that hypotheses (H1), (H2) hold for system (2). Then the closed-loop system (1), (3) (7), (8) is Uniformly Globally Asymptotically Stable, in the sense that there exists a function $\tilde{\sigma} \in KL$ such that for every $t_0 \geq 0$, $(x_0,z_0,u_0) \in C^0([-r,0];\Re^n) \times \Re^n \times L^\infty\left([-r-\tau,0];\Re^m\right)$, the solution $(x(t),z(t),u(t)) \in \Re^n \times \Re^n \times \Re^m$ of the closed-loop system (7), (8), (3), (1) with initial condition $z(t_0) = z_0 \in \Re^n$, $T_{r+\tau}(t_0)u = u_0 \in L^\infty\left([-r-\tau,0];\Re^m\right)$, $T_r(t_0)x = x_0 \in C^0\left([-r,0];\Re^n\right)$ satisfies the following inequality for all $t \geq t_0$:*

$$\begin{aligned}
&|z(t)| + \|T_r(t)x\|_r + \|T_{r+\tau}(t)u\|_{r+\tau} \leq \\
&\tilde{\sigma}\left(|z_0| + \|x_0\|_r + \|u_0\|_{r+\tau}, t-t_0\right)
\end{aligned} \qquad (9)$$

Remark 2.2: For the implementation of the controller (7), (8), we must know the "predictor" mapping $\Phi : \Re^n \times L^\infty\left([-r-\tau,0);\Re^m\right) \to \Re^n$. This mapping can be explicitly computed for

(i) Linear systems $\dot{x} = Ax + Bu$, with $x \in \Re^n, u \in \Re^m$. In this case (Corollary 3.4 below) the predictor mapping $\Phi : \Re^n \times L^\infty\left([-r-\tau,0);\Re^m\right) \to \Re^n$ is given by the explicit equation $\Phi(x,u) := \exp\left(A(\tau+r)\right)x + \int_{-r-\tau}^{0} \exp\left(-Aw\right)Bu(w)dw$.

(ii) Bilinear systems $\dot{x} = Ax + Bu + uCx$, with $x \in \Re^n, u \in \Re$ and $AC = CA$.

(iii) Nonlinear systems of the following form:

$$\begin{aligned}
&\dot{x}_1 = a_1(u)x_1 + f_1(u) \\
&\vdots \\
&\dot{x}_n = a_n(u,x_1,...,x_{n-1})x_n + f_n(u,x_1,...,x_{n-1})
\end{aligned}$$

where $(x,u) \in \Re^n \times \Re^m$ and all mappings a_i, f_i ($i = 1,...,n$) are locally Lipschitz. In this case the predictor mapping $\Phi : \Re^n \times L^\infty([-r-\tau,0);\Re^m) \to \Re^n$ can be constructed inductively. Example 2.3 below applies Theorem 2.1 to a three-dimensional nonlinear system of the above class.

(iv) Nonlinear systems $\dot{x} = f(x,u)$, for which there exists a global diffeomorphism $\Theta : \Re^n \to \Re^n$ such that the change of coordinates $z = \Theta(x)$ transforms the system to one of the above cases.

For globally Lipschitz systems, one can utilize approximate "predictor" mappings $\Phi : \Re^n \times L^\infty([-r-\tau,0);\Re^m) \to \Re^n$ as shown in [15] under additional and more restrictive hypotheses.

We next present an example which shows how the obtained results can be applied to feedforward nonlinear systems.

Example 2.3 (Control of strict-feedforward systems with arbitrarily sparse sampling): Consider the following example taken from [20]:

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t) + x_3^2(t), \\
&\dot{x}_2(t) = x_3(t) + x_3(t)u(t-\tau), \quad \dot{x}_3(t) = u(t-\tau) \\
&x(t) = (x_1(t), x_2(t), x_3(t))' \in \Re^3, u(t) \in \Re
\end{aligned} \tag{10}$$

Here, we consider the stabilization problem for (10) with output given by (3) available only at the discrete time instants τ_i (the sampling times) with $\tau_{i+1} - \tau_i = T > 0$, where $T > 0$ is the sampling period. Hypothesis (H1) holds for system (10) and the predictor mapping can be explicitly expressed by the equations:

$$\Phi(x,u) := \begin{bmatrix} \phi_1(x,u) \\ x_2 + (\tau+r)x_3 + x_3 \int_{-r-\tau}^{0} u(s)ds + \int_{-r-\tau}^{0} (1+u(s)) \int_{-r-\tau}^{s} u(q)dqds \\ x_3 + \int_{-r-\tau}^{0} u(s)ds \end{bmatrix} \tag{11}$$

where

$$\begin{aligned}
\phi_1(x,u) = {} & 3x_3 \int_{-r-\tau}^{0} \int_{-r-\tau}^{s} u(q)dq\,ds + \frac{1}{2}(\tau+r)^2 x_3 \\
& + x_1 + (\tau+r)x_2 + \int_{-r-\tau}^{0} \int_{-r-\tau}^{s} (1+u(w)) \int_{-r-\tau}^{w} u(q)dq\,dw\,ds \\
& + (\tau+r)x_3^2 + \int_{-r-\tau}^{0} \left(\int_{-r-\tau}^{s} u(q)dq \right)^2 ds
\end{aligned} \tag{12}$$

Moreover, hypothesis (H2) holds as well with the smooth, time-independent feedback law:

$$\begin{aligned}
k(x) := {} & -x_1 - 3x_2 - \frac{3}{8}x_2^2 - 3x_3 - \frac{3}{4}x_1x_3 - \frac{3}{2}x_2x_3 \\
& + \frac{3}{8}x_3\left(x_3 + x_2x_3 + \frac{5}{4}x_3^2 - \frac{1}{2}x_3^3 - \frac{3}{4}\left(x_2 - \frac{1}{2}x_3^2 \right)^2 \right)
\end{aligned} \tag{13}$$

It follows from Theorem 2.1 that the dynamic sampled-data controller $u(t) = k(z(t))$ with

$$\begin{aligned}
&\dot{z}_1(t) = z_2(t) + z_3^2(t), \\
&\dot{z}_2(t) = z_3(t) + z_3(t)u(t), \quad \dot{z}_3(t) = u(t), \text{ for } t \in [\tau_i, \tau_{i+1}) \\
&z(t) = (z_1(t), z_2(t), z_3(t))' \in \Re^3
\end{aligned} \tag{14}$$

and

$$z(\tau_{i+1}) = \Phi(y(\tau_{i+1}), \breve{T}_{r+\tau}(\tau_{i+1})u), \; i \in Z^+ \tag{15}$$

where $\Phi : \Re^3 \times L^\infty([-r-\tau,0);\Re^m) \to \Re^3$ is defined by (11), (12) and $k : \Re^3 \to \Re$ is defined by (13), guarantees global asymptotic stability for system (10). ◁

III. SAMPLED-DATA FEEDBACK FOR INPUT APPLIED WITH ZERO-ORDER HOLD

This section is devoted to the case where the input is applied with zero order hold. In this section we assume that (2) is globally stabilizable with feedback applied with zero order hold.

Hypothesis (H3) (input applied with zero order hold): *There exists $k : \Re^n \to \Re^m$, $g \in K_\infty$, $T > 0$ such that*

$$|k(x)| \le g(|x|), \text{ for all } x \in \Re^n \tag{16}$$

and such that $0 \in \Re^n$ is Uniformly Globally Asymptotically Stable for the sampled-data system

$$\begin{aligned}
&\dot{x}(t) = f(x(t), k(x(\tau_i))), \; t \in [\tau_i, \tau_{i+1}) \\
&x(\tau_{i+1}) = \lim_{t \to \tau_{i+1}^-} x(t) \\
&\tau_{i+1} = \tau_i + T \\
&\tau_0 = 0 \ge 0, \; x(0) = x_0 \in \Re^n
\end{aligned} \tag{17}$$

in the sense that there exists a function $\sigma \in KL$ such that for every $x_0 \in \Re^n$ the solution $x(t)$ of (17) with initial condition $x(0) = x_0 \in \Re^n$ satisfies inequality (6) with $t_0 = 0$ for all $t \ge 0$.

Remark 3.1: Hypothesis (H3) seems like a restrictive hypothesis, because it demands global stabilizability by means of sampled-data feedback with positive sampling rate. However, hypothesis (H3) can be satisfied for:

(i) Linear stabilizable systems, where $f(x,u) = Ax + Bu$, $A \in \Re^{n \times n}, B \in \Re^{n \times m}$,

(ii) Nonlinear systems of the form $\dot{x} = f(x) + g(x)u$, $x \in \Re^n, u \in \Re$, where the vector field $f : \Re^n \to \Re^n$ is globally Lipschitz and the vector field $g:\Re^n \to \Re^n$ is locally Lipschitz and bounded, which can be stabilized by a globally Lipschitz feedback law $u = k(x)$ (see [8]).

(iii) Nonlinear systems of the form $\dot{x}_i = f_i(x,u) + g_i(x,u)x_{i+1}$ for $i = 1,...,n-1$ and $\dot{x}_n = f_n(x,u) + g_n(x,u)u$, where the drift terms $f_i(x,u)$ ($i = 1,...,n$) satisfy the linear growth conditions $|f_i(x)| \le L|x_1| + ... + L|x_i|$ ($i = 1,...,n$) for certain constant $L \ge 0$ and there exist constants $b \ge a > 0$ such that $a \le g_i(x,u) \le b$ for all $i = 1,...,n$, $x \in \Re^n, u \in \Re$ (see [12]).

(iv) Asymptotically controllable homogeneous systems with positive minimal power and zero degree (see [5]).

(v) Systems satisfying the reachability hypotheses of Theorem 3.1 in [14], or hypotheses (A1), (A2), (A3) in Section 4 of [11],

(vi) Nonlinear systems $\dot{x} = f(x,u)$, for which there exists a global diffeomorphism $\Theta:\Re^n \to \Re^n$ such that the change of coordinates $z = \Theta(x)$ transforms the system to one of the above cases.

Consider system (1) under hypotheses (H1), (H3) for system (2). In this case we propose a feedback law that is simply a composition of the feedback stabilizer and the delay compensator:

$$u(t) = k\left(\Phi\left(y(\tau_i), \breve{T}_{r+\tau}(\tau_i)u\right)\right),\ t \in [\tau_i, \tau_{i+1}) \quad (18)$$

where $\tau_i = iT, i \in Z^+$ are the sampling times and $\Phi:\Re^n \times L^\infty\left([-r-\tau,0);\Re^m\right) \to \Re^n$ is the predictor mapping involved in (4), (2.2). The control action is applied with zero order hold, i.e., it is constant on $[\tau_i, \tau_{i+1})$; however the control action affecting system (1) remains constant on the interval $[\tau_i + \tau, \tau_{i+1} + \tau)$.

Our main result is stated next.

Theorem 3.2: *Let $T > 0$, $r,\tau \ge 0$ with $r + \tau > 0$ and suppose that there exists $l \in Z^+$ such that $\tau = lT$. Moreover, suppose that hypotheses (H1), (H2) hold for system (2). Then the closed-loop system (1) with (18), i.e., the following sampled-data system*

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t-\tau)) \\ &u(t) = k\left(\Phi(x(\tau_i - r), \breve{T}_{r+\tau}(\tau_i)u)\right),\ t \in [\tau_i, \tau_{i+1}), i \in Z^+ \quad (19) \\ &\tau_{i+1} = \tau_i + T, \tau_0 = 0 \end{aligned}$$

is Uniformly Globally Asymptotically Stable, in the sense that there exists a function $\tilde{\sigma} \in KL$ such that for every $(x_0,u_0) \in C^0([-r,0];\Re^n) \times L^\infty\left([-r-\tau,0);\Re^m\right)$, the solution $(x(t),u(t)) \in \Re^n \times \Re^m$ of system (19) with initial condition $\breve{T}_{r+\tau}(0)u = u_0 \in L^\infty\left([-r-\tau,0);\Re^m\right)$, $T_r(0)x = x_0 \in C^0\left([-r,0];\Re^n\right)$ satisfies the following inequality for all $t \ge 0$:

$$\|T_r(t)x\|_r + \left\|\breve{T}_{r+\tau}(t)u\right\|_{r+\tau} \le \tilde{\sigma}\left(\|x_0\|_r + \|u_0\|_{r+\tau}, t\right) \quad (20)$$

Finally, if system (17) satisfies the dead-beat property of order jT, where $j \in Z^+$ is positive, i.e., for all $x_0 \in \Re^n$ the solution $x(t)$ of (17) with initial condition $x(0) = x_0 \in \Re^n$ satisfies $x(t) = 0$ for all $t \ge jT$, then system (19) satisfies the dead-beat property of order $\left(j + l + \left[\frac{r}{T}\right] + 1\right)T$, where $\left[\frac{r}{T}\right]$ is the integer part of $\frac{r}{T}$, i.e., for every $(x_0,u_0) \in C^0([-r,0];\Re^n) \times L^\infty\left([-r-\tau,0);\Re^m\right)$, the solution $(x(t),u(t)) \in \Re^n \times \Re^m$ of system (19) with initial condition $\breve{T}_{r+\tau}(0)u = u_0 \in L^\infty\left([-r-\tau,0);\Re^m\right)$, $T_r(0)x = x_0 \in C^0\left([-r,0];\Re^n\right)$ satisfies $x(t) = 0$ for all $t \ge \left(j + l + \left[\frac{r}{T}\right] + 1\right)T$.

Example 3.3: Dead-beat control with a predictor can be applied to any delayed 2-dimensional strict feedforward system, i.e., any system of the form:

$$\dot{x}_1(t) = x_2(t) + p(x_2(t))u(t-\tau), \quad \dot{x}_2(t) = u(t-\tau) \quad (21)$$

where $p:\Re \to \Re$ is a smooth function and the measurements are sampled and given by (3). The diffeomorphism given by (see [18])

$$\Theta(x) = \left[x_1 - \int_0^{x_2} p(w)dw \quad , \quad x_2 \right]' \quad (22)$$

transforms system (21) with $\tau = 0$ to a chain of two integrators. Therefore, the feedback law

$$u = -\frac{1}{T^2}x_1 + \frac{1}{T^2}\int_0^{x_2} p(w)dw - \frac{3}{2T}x_2 \quad (23)$$

applied with zero order hold and sampling period $T > 0$ achieves global stabilization of system (21) with $\tau = 0$ when no measurement delays are present. Moreover, the dead-beat property of order $2T$ is guaranteed for the corresponding closed-loop system.

We next consider the case where we have measurement delay $r > 0$ satisfying $r < T$. In this case ($l = q = 0$, $\tilde{r} = r$) we apply Theorem 3.2 and we can conclude that the feedback law

$$u(t) = u_i,\ t \in [iT, (i+1)T),\ i \in Z^+ \quad (24)$$

with

$$\begin{aligned} u_i = &-\frac{1}{T^2}x_1(\tau_i - r) + \frac{1}{T^2}\int_0^{x_2(\tau_i - r)} p(w)dw \\ &-\frac{3T + 2r}{2T^2}x_2(\tau_i - r) - \frac{r(r + 3T)}{2T^2}u_{i-1} \end{aligned} \quad (25)$$

guarantees the dead-beat property of order $3T$ for the corresponding closed-loop system. ◁

IV. CONCLUSIONS

Stabilization is studied for nonlinear systems with input and measurement delays, and with measurements available only at discrete time instants (sampling times). Two different cases are considered: the case where the input can be continuously adjusted and the case where the input is applied with zero order hold. Under the assumption of forward completeness and certain additional stabilizability assumptions, it is shown that sampled-data feedback laws with a predictor-based delay compensation can guarantee global asymptotic stability for the closed-loop system with no restrictions for the magnitude of the delays. Additionally, when the control is applied continuously and only the measurements are sampled, the sampling time can be arbitrarily long.

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