2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Stabilizing a switched linear system by sampled-data quantized feedback

Daniel Liberzon

***Abstract*— We study the problem of asymptotically stabilizing a switched linear control system using sampled and quantized measurements of its state. The switching is assumed to be slow enough in the sense of combined dwell time and average dwell time, each individual mode is assumed to be stabilizable, and the available data rate is assumed to be large enough. Our encoding and control strategy is rooted in the one proposed in our earlier work on non-switched systems, and in particular the data-rate bound used here is the data-rate bound from that earlier work maximized over the individual modes. The main technical step that enables the extension to switched systems concerns propagating over-approximations of reachable sets through sampling intervals, during which the switching signal is unknown.**

I. Introduction

We consider an asymptotic stabilization problem in which the information flow from the continuous-time plant to the feedback controller is subject to a *finite data-rate* constraint, i.e., state measurements undergo time sampling and finite-alphabet encoding (quantization). Feedback control problems with data-rate constraints have been an active research area for some time now, as surveyed in [1] (several specifically relevant works will be cited below). Besides multiple practical motivations, the questions of how much information is really needed to solve a given control problem, or what interesting control tasks can be performed with a given amount of information, are quite fundamental from the theoretical point of view. In this paper we report some results in this direction for a new class of systems, namely, systems that involve switching between several modes of operation. Such *switched systems* have been a popular subject of study in recent years, with research efforts focusing on various analysis and synthesis properties; see, e.g., the books [2], [3], the survey [4], and the many references therein. Control problems with limited information, however, do not seem to have received much attention so far in the context of switched systems (with the exception of some work on quantized Markov jump linear systems [5], [6], [7]), and the present paper is intended to begin filling this gap. We contend that a marriage of these two research areas is actually quite natural, due to close similarities between some of the technical tools that are being employed in them.

In order to understand how much information is needed—and how this information should be used—to stabilize a given system, we must understand how the uncertainty about the system's state evolves over time along its dynamics. In more precise terms, this means that we need to be able to characterize propagation of reachable sets or their suitable over-approximations. This is a crucial ingredient in the available results on rate-constrained control of non-switched systems (such as [8] which serves as the basis for the present work), and the bulk of the effort required to handle the switched system scenario is concentrated in implementing this step and analyzing its consequences. If the switching signal were precisely known to the controller, then the problem of reachable set propagation would be just a sequence of corresponding problems for the individual modes, and as such would pose very little extra difficulty. (This would essentially correspond to the situation considered, in a discrete-time stochastic setting, in [5].) On the other hand, if the switching signal were completely unknown, then the set of possible trajectories of the switched system would be too large to hope for a reasonable (not overly conservative) solution. To strike a balance between these two situations, we assume here that we have a partial knowledge of the switching signal; namely, we assume that the active mode of the switched system is known at each sampling time, and that the switching is subject to a fairly mild "slow-switching" assumption (described by a combination of a dwell time and an average dwell time). If in addition the allowed data rate is large enough, then we can design a provably correct communication and control strategy to stabilize the system.

We now outline in a bit more detail the sequence of steps that we follow. In Section II we define the switched linear system that we want to stabilize, explain what the information structure is, and state the basic assumptions and the main result. In Section III we describe the basic encoding and control strategy which assumes that appropriate bounds on reachable sets are available. Section IV is devoted to generating such reachable set bounds. With these ingredients in place, we complete the analysis through revealing a cascade structure within the closed-loop system, constructing a (mode-dependent) Lyapunov function which decreases in the absence of switching if the data rate is large enough, and invoking the average dwell-time assumption to establish global asymptotic stability; this reasoning is sketched in Section V. Sections VI and VII contain a short simulation example and some concluding remarks.

II. Problem formulation

A. Switched system

The system to be controlled is the switched linear control system

$$\dot{x} = A_\sigma x + B_\sigma u, \qquad x(0) = x_0 \tag{1}$$

D. Liberzon is with Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61821, U.S.A. Email: liberzon@uiuc.edu.
This work was supported by the NSF grant ECCS-0701676 and by the UIUC Center for Advanced Study.

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $\{(A_p, B_p) : p \in \mathcal{P}\}$ is a collection of matrix pairs defining the individual control systems ("modes") of the switched system, $\mathcal{P}$ is a finite index set, and $\sigma : [0, \infty) \to \mathcal{P}$ is a right-continuous, piecewise constant function called the *switching signal* which specifies the active mode at each time. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation away from the discontinuities of σ (in particular, there are no state jumps). The switching signal σ is fixed but not known to the controller a priori. The discontinuities of σ (also called "switching times," or just "switches") are denoted by $t_1, t_2, \dots$, and we let $N_\sigma(t, s)$ stand for the number of switches on an interval $(s, t]$. Our first basic assumption is that the switching is not too fast, in the following sense.

Assumption 1 (Slow Switching)

1) There exists a number $\tau_d > 0$ (called a *dwell time*) such that $t_{i+1} - t_i \geq \tau_d$ for all i;
2) There exist numbers $\tau_a > \tau_d$ (called an *average dwell time*) and $N_0 \geq 1$ such that

$$N_\sigma(t, s) \leq N_0 + (t - s)/\tau_a \qquad \forall\, t > s \geq 0.$$

The concept of average dwell time was introduced in [9] and has since then become standard. Note that without the constraint that $\tau_a > \tau_d$, the average dwell-time condition (item 2) would be implied by the dwell-time condition (item 1). Switching signals satisfying Assumption 1 were considered in [10], where they were called "hybrid dwell-time" signals.

Our next basic assumption is stabilizability of all modes.

Assumption 2 (Stabilizability) For each $p \in \mathcal{P}$ the pair (A_p, B_p) is stabilizable, i.e., there exists a state feedback gain matrix K_p such that $A_p + B_p K_p$ is Hurwitz.

In the sequel, we assume that a family of such stabilizing gain matrices K_p, $p \in \mathcal{P}$ has been selected and fixed. We understand that (at least some of) the open-loop matrices A_p, $p \in \mathcal{P}$ are not Hurwitz.

B. Information structure

The task of the controller is to generate a control input $u(\cdot)$ based on limited information about the state $x(\cdot)$ and about the switching signal $\sigma(\cdot)$. The information to be communicated to the controller is subject to the following two constraints.

Sampling: State measurements are taken at times $k\tau_s$, $k = 0, 1, 2, \dots$, where $\tau_s > 0$ is a fixed *sampling period*.

Quantization: Each state measurement $x(k\tau_s)$ is encoded by an integer from 0 to N^n, where N is an odd positive integer, and sent to the controller. In addition, the value of $\sigma(k\tau_s) \in \mathcal{P}$ is also sent to the controller.

As a consequence, data is transmitted to the controller at the rate of $(\log_2(N^n + 1) + \log_2 |\mathcal{P}|)/\tau_s$ bits per time unit, where $|\mathcal{P}|$ is the number of elements in $\mathcal{P}$. We assume the data transmission to be noise-free and delay-free. We take the sampling period τ_s to be no larger than the dwell time from Assumption 1 (item 1):

$$\tau_s \leq \tau_d. \tag{2}$$

This guarantees that at most one switch occurs within each sampling interval. Since the average dwell time τ_a in Assumption 1 (item 2) is larger than τ_d, we know that switches actually occur less often than once every sampling period. The reason for taking the integer N to be odd is to ensure that our control strategy preserves the equilibrium at the origin.

Throughout the paper, we work with the ∞-norm $\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}$ on $\mathbb{R}^n$ and the corresponding induced matrix norm $\|A\|_\infty = \max\{\sum_{j=1}^n |A_{ij}| : 1 \leq i \leq n\}$ on $\mathbb{R}^{n \times n}$, which we denote simply by $\|\cdot\|$. To formulate our final basic assumption, we define

$$\Lambda_p := \|e^{A_p \tau_s}\|, \qquad p \in \mathcal{P}. \tag{3}$$

Assumption 3 (Data Rate) $\Lambda_p < N$ for all $p \in \mathcal{P}$.

We can view the above inequality as a data-rate bound because it requires N to be sufficiently large relative to τ_s, thereby imposing (indirectly) a lower bound on the available data rate. A very similar data-rate bound but for the case of a single mode appears in [8], where it is shown to be sufficient for stabilizing a non-switched linear system. That bound is slightly conservative compared to known bounds that characterize the minimal data rate necessary for stabilization (see, e.g., [11], [12]). However, the control scheme of [8] can be refined by tailoring it better to the structure of the system matrix A, and then the data rate that it requires will approach the minimal data rate (see also the discussion in [13, Section V]). Therefore, it is fair to say that Assumption 3 does not introduce a significant conservatism beyond requiring that the data rate be sufficient to stabilize each individual mode of the switched system (1).

C. Main objective

The control objective is to asymptotically stabilize the system defined in Section II-A while respecting the information constraints described in Section II-B. More concretely, we want to provide a constructive proof of the following result.

Theorem 1 (Main Result) Consider the switched linear system (1) and let Assumptions 1–3 and the inequality (2) hold. If the average dwell time τ_a is large enough, then there exists an encoding and control strategy that yields the following two properties:

Exponential convergence: There exist a number $\lambda > 0$ and a function $g : [0, \infty) \to (0, \infty)$ such that for every initial condition x_0 and every time $t \geq 0$ we have

$$\|x(t)\| \leq e^{-\lambda t} g(\|x_0\|). \tag{4}$$

Lyapunov stability: For each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t)\| < \varepsilon \quad \forall\, t \geq 0. \tag{5}$$

A precise lower bound on the average dwell time τ_a will be derived in the course of the proof (see the formula (25) in Section V-C). As for the function g in the exponential convergence property, from the proof it will be clear that $g(r)$ does not go to 0 as $r \to 0$ and that, in general, g grows faster than any linear function at infinity (see the formula (28) in Section V-D and the discussion at the end of Section IV-C). For this reason, Lyapunov stability needs to be established separately, and the two properties (exponential convergence and Lyapunov stability) combined still do not give the standard global exponential stability, but just global asymptotic stability with an exponential convergence rate.

The control strategy that we will develop to prove Theorem 1 is a dynamic one: it involves an additional state denoted by $\hat{x}$. Theorem 1 only discusses the behavior of the state x, which is the main quantity of interest, but it can be deduced from the proof that the controller state $\hat{x}$ satisfies analogous bounds.

III. Basic encoding and control strategy

In this section we outline our encoding and control strategy, assuming for now that the state x satisfies known bounds at the sampling times. The problem of generating such state bounds is solved in the next section.

First, suppose that at some sampling time $k_0\tau_s$ we have

$$\|x(k_0\tau_s)\| \le E_{k_0}$$

where $E_{k_0} > 0$ is a number known to the controller. (In Section IV-C we will show how such a bound can be generated for an arbitrary initial state x_0, by using a "zooming-out" procedure.) At the first such sampling time our controller is initialized. The encoder works by partitioning the hypercube $\{x \in \mathbb{R}^n : \|x\| \le E_{k_0}\}$ into N^n equal hypercubic boxes, N per each dimension, and numbering them from 1 to N^n in some specific way. It then records the number of the box that contains[1] $x(k_0\tau_s)$ and sends it to the controller, along with the value of $\sigma(k_0\tau_s)$. We assume that the controller knows the box numbering system used by the encoder, so it can decode the box number. It lets $c_{k_0} \in \mathbb{R}^n$ be the center of the box containing $x(k_0\tau_s)$. We then have $\|x(k_0\tau_s) - c_{k_0}\| \le E_{k_0}/N$. For $t \in [k_0\tau_s, (k_0+1)\tau_s)$, the control is set to $u(t) = K_{\sigma(k_0\tau_s)}\hat{x}(t)$ where $\hat{x}$ is defined to be the solution of

$$\dot{\hat{x}} = (A_{\sigma(k_0\tau_s)} + B_{\sigma(k_0\tau_s)}K_{\sigma(k_0\tau_s)})\hat{x} = A_{\sigma(k_0\tau_s)}\hat{x} + B_{\sigma(k_0\tau_s)}u$$

with the boundary condition $\hat{x}(k_0\tau_s) = c_{k_0}$.

At a general sampling time $k\tau_s$, $k \ge k_0 + 1$, suppose that a point $x_k^* \in \mathbb{R}^n$ and a number $E_k > 0$ are known such that

$$\|x(k\tau_s) - x_k^*\| \le E_k. \tag{6}$$

Of course the encoder has precise knowledge of x; the quantities x_k^* and E_k have to be obtainable on the decoder/controller side, based on the knowledge of the system matrices (but not the switching signal) and previously received measurements. We explain later how such x_k^* and E_k can be generated. The encoder also computes x_k^* and E_k in the same way, to ensure that the encoder and the decoder are synchronized. The encoding is then done as follows. Partition the hypercube $\{x \in \mathbb{R}^n : \|x - x_k^*\| \le E_k\}$ into N^n equal hypercubic boxes, N per each dimension. Send the number of the box to the controller, along with the value of $\sigma(k\tau_s)$. On the decoder/controller side, let c_k be the center of the box containing $x(k\tau_s)$. This gives

$$\|x(k\tau_s) - c_k\| \le E_k/N, \tag{7}$$

$$\|c_k - x_k^*\| \le E_k(N-1)/N. \tag{8}$$

Note that the formula (8) is also valid for $k = k_0$ if we set $x_{k_0}^* := 0$, a convention that we follow in the sequel. For $t \in [k\tau_s, (k+1)\tau_s)$ define the control, along the same lines as before, by

$$u(t) = K_{\sigma(k\tau_s)}\hat{x}(t)$$

where $\hat{x}$ is the solution of

$$\dot{\hat{x}} = (A_{\sigma(k\tau_s)} + B_{\sigma(k\tau_s)}K_{\sigma(k\tau_s)})\hat{x} = A_{\sigma(k\tau_s)}\hat{x} + B_{\sigma(k\tau_s)}u \tag{9}$$

with the boundary condition

$$\hat{x}(k\tau_s) = c_k. \tag{10}$$

The above procedure is to be repeated for each subsequent value of k. Note that $\hat{x}$ is, in general, discontinuous (only right-continuous) at the sampling times, and we will use the notation $\hat{x}(k\tau_s^-) := \lim_{t\nearrow k\tau_s}\hat{x}(t)$. In the earlier work [8], x_k^* was obtained directly from $\hat{x}$ via $x_k^* := \hat{x}(k\tau_s^-)$. On sampling intervals containing a switch this construction no longer works, and the task of defining x_k^* as well as E_k becomes more challenging.

IV. Generating state bounds: over-approximations of reachable sets

Proceeding inductively, we start with known x_k^* and E_k satisfying (6), where $k \ge k_0$, and show how to find x_{k+1}^* and E_{k+1} such that

$$\|x((k+1)\tau_s) - x_{k+1}^*\| \le E_{k+1}. \tag{11}$$

Generation of E_{k_0} is addressed at the end of the section.

A. Sampling interval with no switch

We first consider the simpler case when $\sigma(k\tau_s) = \sigma((k+1)\tau_s) = p \in \mathcal{P}$. By (2) we know that no switch has occurred on $(k\tau_s, (k+1)\tau_s]$, since two switches would have been impossible. So, we know that on the whole interval $[k\tau_s, (k+1)\tau_s]$ mode p is active. We can then proceed as in [8]. It is clear from (1) and (9) that the error $e := x - \hat{x}$ satisfies $\dot{e} = A_pe$ on $[k\tau_s, (k+1)\tau_s)$, and we know from (10) and (7) that $\|e(k\tau_s)\| \le E_k/N$, hence

$$\|e((k+1)\tau_s^-)\| \le \Lambda_pE_k/N =: E_{k+1} \tag{12}$$

where Λ_p was defined in (3). It remains to let

$$x_{k+1}^* := \hat{x}((k+1)\tau_s^-) = e^{(A_p+B_pK_p)\tau_s}c_k \tag{13}$$

and recall that x is continuous to see that (11) indeed holds.

[1] In case $x(k_0\tau_s)$ lies on the boundary of several boxes, either one of these boxes can be chosen.

B. Sampling interval with a switch

Suppose now that $\sigma(k\tau_s) = p$ and $\sigma((k+1)\tau_s) = q \neq p$. Then the controller knows, again by (2), that exactly one switch from mode p to mode q has occurred somewhere on the interval $(k\tau_s, (k+1)\tau_s]$, but it does not know exactly where. This case is more challenging. Let the (unknown) time of the switch from p to q be $k\tau_s + \bar{t}$, where $\bar{t} \in (0, \tau_s]$.

1) Analysis before the switch: On $[k\tau_s, k\tau_s + \bar{t})$ mode p is active, and we can derive as before that $\|x(k\tau_s + \bar{t}) - \hat{x}(k\tau_s + \bar{t})\| \le \|e^{A_p \bar{t}}\| E_k/N$. But $\hat{x}(k\tau_s + \bar{t})$ is unknown, so we need to describe a set that contains it. Choose an arbitrary $t' \in [0, \tau_s]$ (which may vary with k). By (9) and (10) we have[2]

$$\hat{x}(k\tau_s + t') = e^{(A_p + B_p K_p)t'} c_k \tag{14}$$

and $\hat{x}(k\tau_s + \bar{t}) = e^{(A_p + B_p K_p)(\bar{t} - t')} \hat{x}(k\tau_s + t')$, hence $\|\hat{x}(k\tau_s + \bar{t}) - \hat{x}(k\tau_s + t')\| \le \|e^{(A_p + B_p K_p)(\bar{t} - t')} - I\| \|e^{(A_p + B_p K_p)t'}\| \|c_k\|$. We also have from (8) that

$$\|c_k\| \le \|x_k^*\| + E_k(N-1)/N. \tag{15}$$

By the triangle inequality, we obtain

$$\begin{aligned}
&\|x(k\tau_s + \bar{t}) - \hat{x}(k\tau_s + t')\| \\
&\le \|e^{(A_p + B_p K_p)(\bar{t} - t')} - I\| \|e^{(A_p + B_p K_p)t'}\| \\
&\times (\|x_k^*\| + E_k(N-1)/N) + \|e^{A_p \bar{t}}\| E_k/N =: D_{k+1}(\bar{t}).
\end{aligned}$$

2) Analysis after the switch: On the interval $[k\tau_s + \bar{t}, (k+1)\tau_s)$, the closed-loop dynamics are

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A_q & B_q K_p \\ 0 & A_p + B_p K_p \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}. \tag{16}$$

Letting $z := \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$, $\bar{A}_{pq} := \begin{pmatrix} A_q & B_q K_p \\ 0 & A_p + B_p K_p \end{pmatrix}$, we can write (16) more compactly as $\dot{z} = \bar{A}_{pq} z$. The previous analysis shows that $\left\| z(k\tau_s + \bar{t}) - \begin{pmatrix} \hat{x}(k\tau_s + t') \\ \hat{x}(k\tau_s + t') \end{pmatrix} \right\| \le D_{k+1}(\bar{t})$ (noting the property $\|(a^T, b^T)^T\| \le \max\{\|a\|, \|b\|\}$ of the ∞-norm). Consider the auxiliary system copy (on $\mathbb{R}^{2n}$)

$$\dot{\bar{z}} = \bar{A}_{pq} \bar{z}, \qquad \bar{z}(0) = \begin{pmatrix} \hat{x}(k\tau_s + t') \\ \hat{x}(k\tau_s + t') \end{pmatrix}.$$

We have $\|z((k+1)\tau_s^-) - \bar{z}(\tau_s - \bar{t})\| \le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| D_{k+1}(\bar{t})$. We now need to generate a bound for the unknown $\bar{z}(\tau_s - \bar{t})$. Similarly to what we did before, pick a $t'' \in [0, \tau_s]$. Then $\bar{z}(t'') = e^{\bar{A}_{pq} t''} \bar{z}(0)$, $\bar{z}(\tau_s - \bar{t}) = e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} \bar{z}(t'')$, and

$$\begin{aligned}
&\|\bar{z}(\tau_s - \bar{t}) - \bar{z}(t'')\| \le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|\bar{z}(t'')\| \\
&\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|e^{\bar{A}_{pq} t''}\| \|\hat{x}(k\tau_s + t')\| \\
&\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p)t'}\| \\
&\times (\|x_k^*\| + E_k(N-1)/N)
\end{aligned}$$

[2]In case either $k\tau_s + t'$ or $k\tau_s + \bar{t}$ equals $(k+1)\tau_s$, the value of $\hat{x}$ at that time should be replaced by the left limit $\hat{x}(k\tau_s + t'^-)$ or $\hat{x}(k\tau_s + \bar{t}^-)$, respectively.

where we used (14) and (15) in the last step. By the triangle inequality,

$$\begin{aligned}
&\|z((k+1)\tau_s^-) - \bar{z}(t'')\| \le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \\
&\times \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p)t'}\| \big(\|x_k^*\| + E_k(N-1)/N\big) \\
&+ \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| D_{k+1}(\bar{t}) =: E_{k+1}(\bar{t}).
\end{aligned}$$

To eliminate the dependence on the unknown $\bar{t}$, we take the maximum over $\bar{t}$:

$$\begin{aligned}
&E_{k+1} := \max_{0 \le \bar{t} \le \tau_s} E_{k+1}(\bar{t}) = \max_{0 \le \bar{t} \le \tau_s} \Big\{ \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \\
&\times \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p)t'}\| \big(\|x_k^*\| + E_k(N-1)/N\big) \\
&+ \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \Big(\|e^{(A_p + B_p K_p)(\bar{t} - t')} - I\| \|e^{(A_p + B_p K_p)t'}\| \\
&\times \big(\|x_k^*\| + E_k(N-1)/N\big) + \|e^{A_p \bar{t}}\| E_k/N \Big) \Big\}.
\end{aligned}$$

We can use the inequalities

$$\|M - I\| \le \|M\| + 1, \qquad \|e^{As}\| \le e^{\|A\|s} \tag{17}$$

to obtain a more conservative upper bound which is more useful for computations. This formula further simplifies if we set $t' = t'' = 0$, but the original expression for E_{k+1} is not necessarily minimized with this choice of t' and t''. Finally, x_{k+1}^* is defined by projecting $\bar{z}(t'')$ onto the x-component:

$$\begin{aligned}
x_{k+1}^* &:= \begin{pmatrix} I_{n\times n} & 0_{n\times n} \end{pmatrix} \bar{z}(t'') \\
&= \begin{pmatrix} I_{n\times n} & 0_{n\times n} \end{pmatrix} e^{\bar{A}_{pq} t''} \begin{pmatrix} \hat{x}(k\tau_s + t') \\ \hat{x}(k\tau_s + t') \end{pmatrix} \\
&= \begin{pmatrix} I_{n\times n} & 0_{n\times n} \end{pmatrix} e^{\bar{A}_{pq} t''} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} e^{(A_p + B_p K_p)t'} c_k.
\end{aligned} \tag{18}$$

C. Generating an initial state bound E_{k_0}

Initially, set the control to $u \equiv 0$. At time 0, choose an arbitrary $E_0 > 0$ and partition the hypercube $\{x \in \mathbb{R}^n : \|x\| \le E_0\}$ into N^n equal hypercubic boxes, N per each dimension. If x_0 belongs to one of these boxes, then send the number of the box to the controller. Otherwise send 0 (the "overflow" symbol). Choose an increasing sequence $E_1, E_2, \ldots$ that grows fast enough to dominate the rate of growth of the open-loop dynamics. For example, we can pick a small $\varepsilon > 0$ and let

$$E_k := e^{(2+\varepsilon)\max_{p\in\mathcal{P}} \|A_p\| k\tau_s} E_0, \qquad k = 1, 2, \ldots \tag{19}$$

There are other options but for concreteness we assume that the specific "zooming-out" sequence (19) is implemented. Repeat the above encoding procedure at each step. (As long as the quantization symbol is 0, there is no need to send the value of σ to the controller.) Then we claim that there will be a time $k_0\tau_s$ such that, for the corresponding value E_{k_0}, the symbol received by the controller will not be 0. At this time, the encoding strategy of Section III can be initialized.

To see why the above claim is true, consider a sampling interval $[k\tau_s, (k+1)\tau_s]$ on which $u \equiv 0$. If $\sigma \equiv p \in \mathcal{P}$ on this interval, then the dynamics are $\dot{x} = A_p x$ from which it follows that $\|x(t)\| \le \bar{\Lambda}_p \|x(k\tau_s)\|$ on this interval, where

$$\bar{\Lambda}_p := \max_{0 \le s \le \tau_s} \|e^{A_p s}\|. \tag{20}$$

If, on the other hand, the sampling interval contains a switch from mode p to another mode q, then the dynamics become $\dot{x} = A_q x$ after the switch and a (conservative) bound is now $\|x(t)\| \leq \bar{\Lambda}_q \bar{\Lambda}_p \|x(k\tau_s)\|$ for $t \in [k\tau_s, (k+1)\tau_s]$. Iterating, we obtain that $\|x(t)\| \leq \max_{p\in\mathcal{P}} \bar{\Lambda}_p^{2k} \|x_0\|$ on $[0, k\tau_s]$ as long as $u \equiv 0$ there. Since $\bar{\Lambda}_p \leq e^{\|A_p\|\tau_s}$, the values of E_k in (19) grow faster than the largest values that $\|x(t)\|$ can attain on the intervals $[(k-1)\tau_s, k\tau_s]$ under zero control. It follows that k_0 is indeed well defined and there exist functions $\eta : [0,\infty) \to \mathbb{Z}_{\geq 0}$ and $\gamma : [0,\infty) \to (0,\infty)$ such that

$$k_0 \leq \eta(\|x_0\|), \qquad E_{k_0} \leq \gamma(\|x_0\|), \tag{21}$$

$$\|x(t)\| \leq \gamma(\|x_0\|) \qquad \forall\, t \in [0, k_0\tau_s]. \tag{22}$$

Both functions depend on the initial choice of E_0. Note that we can pick them so that $\eta(r) = 0$ and $\gamma(r) = E_0$ for all $r \leq E_0$. For large values of its argument, $\gamma(\cdot)$ is in general super-linear. In fact, we can calculate that $\gamma(r)$ is of the order of r^2/E_0, and $\eta(r)$ is of the order of $(\max_{p\in\mathcal{P}} \|A_p\|\tau_s)^{-1} \log(r/E_0)$, for large values of r.

V. Stability analysis

In this section we prove that the encoding and control strategy developed in Sections III and IV fulfills the properties listed in Theorem 1. Due to the length limitations, the calculations are only sketched here.

A. Sampling interval with no switch

Consider an interval $[k\tau_s, (k+1)\tau_s]$, $k \geq k_0$ on which $\sigma \equiv p \in \mathcal{P}$, as in Section IV-A. Rewrite (13) as $x^*_{k+1} = e^{(A_p+B_pK_p)\tau_s}(x^*_k + \Delta_k) = S_p x^*_k + S_p\Delta_k$ where

$$\Delta_k := c_k - x^*_k, \qquad S_p := e^{(A_p+B_pK_p)\tau_s}. \tag{23}$$

We know from (8) that

$$\|\Delta_k\| \leq E_k(N-1)/N \tag{24}$$

and we know that S_p is Schur stable because $A_p + B_pK_p$ is Hurwitz. Also, (12) and Assumption 3 give us $E_{k+1} = \Lambda_p E_k/N < E_k$. We see that, as long as there are no switches, E_k decays exponentially and x^*_k evolves according to an exponentially stable discrete-time linear system whose input Δ_k is bounded in terms of E_k. It is then well known that the overall "cascade" system describing the joint evolution of x^*_k and E_k is exponentially stable. We now formalize this fact by constructing a Lyapunov function in the form of a weighted sum of a quadratic form in x^*_k and E_k^2, along standard lines. This Lyapunov function will depend on p, the currently active mode. Let $P_p = P_p^T > 0$ and $Q_p = Q_p^T > 0$ be such that

$$S_p^T P_p S_p - S_p = -Q_p < 0.$$

We let $\underline{\lambda}$ and $\overline{\lambda}$ denote the smallest and the largest eigenvalue of a matrix, respectively. Proceeding similarly to [14, Example 3.4] and then using (24), we can show that

$$\begin{aligned}
&(x^*_{k+1})^T P_p x^*_{k+1} - (x^*_k)^T P_p x^*_k \\
&\leq -\frac{1}{2}\underline{\lambda}(Q_p)\|x^*_k\|^2 + \big(2n^2\|S_p^T P_p S_p\|^2/\underline{\lambda}(Q_p) \\
&+ n\|S_p^T P_p S_p\|\big)E_k^2((N-1)/N)^2 =: -\alpha_{1,p}\|x^*_k\|^2 + \beta_{1,p}E_k^2.
\end{aligned}$$

We now define

$$V_p(x^*_k, E_k) := (x^*_k)^T P_p x^*_k + \rho_p E_k^2$$

where ρ_p is a positive constant large enough to satisfy $(\beta_{1,p}/\rho_p) + (\Lambda_p/N)^2 < 1$ (such a ρ_p exists because the second fraction is less than 1 by Assumption 3). By the previous calculations, $V_p(x^*_{k+1}, E_{k+1}) \leq \nu_p V_p(x^*_k, E_k) \leq \nu V_p(x^*_k, E_k)$ where $\nu := \max_{p\in\mathcal{P}} \nu_p$ and

$$\nu_p := \max\big\{1 - \alpha_{1,p}/(n\overline{\lambda}(P_p)), (\beta_{1,p}/\rho_p) + (\Lambda_p/N)^2\big\} < 1.$$

B. Sampling interval with a switch

Next, consider an interval $[k\tau_s, (k+1)\tau_s]$, $k \geq k_0$ which contains a switch from mode p to mode q, as in Section IV-B. We know from (18) that $x^*_{k+1} = H_{pq}c_k = H_{pq}(x^*_k + \Delta_k)$ where H_{pq} is a matrix defined by

$$H_{pq} := \begin{pmatrix} I_{n\times n} & 0_{n\times n} \end{pmatrix} e^{\bar{A}_{pq}t''} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} e^{(A_p+B_pK_p)t'}$$

(note that $H_{pq} = I$ if $t' = t'' = 0$) and Δ_k is defined in (23) and satisfies (24). This gives $\|x^*_{k+1}\| \leq h_{pq}(\|x^*_k\| + E_k(N-1)/N)$ where $h_{pq} := \|H_{pq}\|$. We also have from Section IV-B that $E_{k+1} \leq \alpha_{2,pq}\|x^*_k\| + \beta_{2,pq}E_k$, where

$$\begin{aligned}
\alpha_{2,pq} := &\big(e^{\|\bar{A}_{pq}\|\max\{t'',\tau_s-t''\}} + 1\big)\|e^{\bar{A}_{pq}t''}\|\|e^{(A_p+B_pK_p)t'}\| \\
&+ e^{\|\bar{A}_{pq}\|\tau_s}\big(e^{\|A_p+B_pK_p\|\max\{t',\tau_s-t'\}} + 1\big)\|e^{(A_p+B_pK_p)t'}\|
\end{aligned}$$

(this simplifies if $t' = t'' = 0$) and $\beta_{2,pq} := \big(\alpha_{2,pq}(N-1) + e^{\|\bar{A}_{pq}\|\tau_s}e^{\|A_p\|\tau_s}\big)/N$. Therefore, $V_q(x^*_{k+1}, E_{k+1}) \leq \mu_{pq}V_p(x^*_k, E_k) \leq \mu V_p(x^*_k, E_k)$ where $\mu := \max_{p,q\in\mathcal{P}} \mu_{pq}$ and

$$\begin{aligned}
\mu_{pq} := \max\big\{&(2n\overline{\lambda}(P_q)h^2 + 2\rho_q\alpha_{2,pq}^2)/\underline{\lambda}(P_p), \\
&(2n\overline{\lambda}(P_q)h^2/\rho_p)((N-1)/N)^2 + 2\rho_q\beta_{2,pq}^2/\rho_p\big\}.
\end{aligned}$$

C. Combined bound for sampling times

We now invoke Assumption 1, whose item 2 (the average dwell-time property) implies that $N_\sigma(k_0\tau_s, k\tau_s) \leq N_0 + (k-k_0)/m$ for every m such that $\tau_a \geq m\tau_s$. We want to derive a lower bound on m that guarantees convergence. We know from (2) that $N_\sigma(k_0\tau_s, k\tau_s)$ equals the number of intervals of the form $(\ell\tau_s, (\ell+1)\tau_s]$, $k_0 \leq \ell \leq k-1$ which contain a switch (among the total number $k - k_0$ of such intervals). Combining the conclusions of Sections V-A and V-B, we have the following bound for all $k \geq k_0$:

$$V_{\sigma(k\tau_s)}(x^*_k, E_k) \leq \left(\frac{\mu}{\nu}\right)^{N_0}\big(\mu^{1/m}\nu^{(m-1)/m}\big)^{k-k_0}\rho_{\sigma(k_0\tau_s)}E_{k_0}^2$$

(since $x^*_{k_0} = 0$). We want to ensure that $\mu^{1/m}\nu^{(m-1)/m} < 1$, which is equivalent to $m > 1 + \log\mu/\log(1/\nu)$. Thus if

$$\tau_a > (1 + \log\mu/\log(1/\nu))\tau_s \tag{25}$$

then there exists a $\theta \in (0,1)$ such that $V_{\sigma(k\tau_s)}(x^*_k, E_k) \leq (\mu/\nu)^{N_0}\theta^{k-k_0}(\max_{p\in\mathcal{P}}\rho_p)E_{k_0}^2$. This leads to

$$\|x^*_k\| \leq \left(\frac{\mu}{\nu}\right)^{N_0/2}\theta^{(k-k_0)/2}\sqrt{\frac{\max_{p\in\mathcal{P}}\rho_p}{\min_{p\in\mathcal{P}}\underline{\lambda}(P_p)}}E_{k_0}, \tag{26}$$

$$E_k \le \left(\frac{\mu}{\nu}\right)^{N_0/2} \theta^{(k-k_0)/2} \sqrt{\frac{\max_{p\in\mathcal{P}} \rho_p}{\min_{p\in\mathcal{P}} \rho_p}} E_{k_0}. \tag{27}$$

Recalling that, by (6), $\|x(k\tau_s)\| \le \|x_k^*\| + E_k$ for all k, we obtain an exponential decay bound for $\|x(k\tau_s)\|$ given by the sum of the right-hand sides of (26) and (27).

D. Intersample bound

Here we modify relevant calculations from Section IV to derive bounds that are simpler (in particular, we work with $t' = t'' = 0$) and more conservative, but apply to the whole sampling intervals and not just to the sampling times. Consider an interval $[k\tau_s, (k+1)\tau_s]$ with a possible switch at a time $k\tau_s + \bar{t}$ in its interior. On this interval we obtain $\|x(t)\| \le \|c_k\| + \|x(t) - c_k\| \le \|x_k^*\| + E_k(N-1)/N + \bar{E}_{k+1} = \alpha_{3,pq}\|x_k^*\| + \beta_{3,pq}E_k$, where $\alpha_{3,pq} := 1 + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s} - I\| + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s}\| \max_{0\le s\le\tau_s} \|e^{(A_p+B_pK_p)s} - I\|$ and $\beta_{3,pq} := \big(\alpha_{3,pq}(N-1) + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s}\| \times \max_{0\le s\le\tau_s} \|e^{A_p s}\|\big)/N$. As before, we can use the inequalities (17) to derive more conservative but more computationally friendly upper bounds. Invoking the earlier bounds (26) and (27), we conclude that for all $t \in [k\tau_s, (k+1)\tau_s)$, $k \ge k_0$ we have $\|x(t)\| \le \bar{c}\theta^{(k-k_0)/2} E_{k_0}$ where

$$\bar{c} := \left(\frac{\mu}{\nu}\right)^{N_0/2} \Big(\max_{p,q\in\mathcal{P}} \alpha_{3,pq} \sqrt{\max_{p\in\mathcal{P}} \rho_p / \min_{p\in\mathcal{P}} \underline{\lambda}(P_p)} + \max_{p,q\in\mathcal{P}} \beta_{3,pq} \sqrt{\max_{p\in\mathcal{P}} \rho_p / \min_{p\in\mathcal{P}} \rho_p} \Big).$$

We can now establish a continuous-time exponential decay bound: rewriting the previous bound as

$$\|x(t)\| \le \bar{c}\theta^{\frac{1}{2}\left(\lfloor \frac{t}{\tau_s} \rfloor - k_0\right)} E_{k_0} \le \bar{c}\big(1/\sqrt{\theta}\big)^{1+k_0} \theta^{\frac{t}{2\tau_s}} E_{k_0}$$

and recalling (21) and (22), we finally arrive at the desired exponential convergence property (4) with $\lambda := (1/(2\tau_s)) \log(1/\theta)$ and

$$g(r) := \bar{c}\big(1/\sqrt{\theta}\big)^{1+\eta(r)} \gamma(r). \tag{28}$$

E. Lyapunov stability

The proof of Lyapunov stability proceeds along the lines of [8], [15], and we omit it.

VI. Simulation Example

We simulated the above control strategy with the following data: $\mathcal{P} = \{1,2\}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $K_1 = \begin{pmatrix} -2 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} 0 & -1 \end{pmatrix}$, $x_0 = (2,2)^T$, $E_0 = 0.5$, $\tau_s = 0.5$, $N = 5$ (Assumption 3 is satisfied), $\tau_d = 1.05$, $\tau_a = 7.55$, and $N_0 = 5$. Figure 1 plots a typical behavior of x_1 (in solid red) and $\hat{x}_1$ (in dashed green) versus time; switches are marked by blue circles. Observe the initial "zooming-out" phase and the nonsmooth behavior of x when $\hat{x}$ experiences a jump (causing a jump in u). For this example, the theoretical lower bound on the average dwell time from the formula (25) is about 85.5 which is, not surprisingly, quite conservative.

VII. Conclusions

We presented a result on sampled-data quantized state feedback stabilization of switched linear systems. We believe there is room for improving it by relaxing the slow-switching assumption, by refining the reachable set bounds, and by allowing state jumps. We also envision developing similar results for nonlinear dynamics, state-dependent switching, and output feedback.

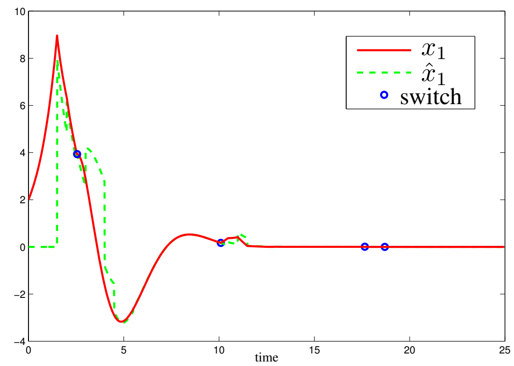

Fig. 1. Simulation example

Acknowledgment. The author thanks Aneel Tanwani for helpful comments on an earlier draft.

References

[1] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proc. IEEE*, vol. 95, pp. 108–137, 2007.
[2] D. Liberzon, *Switching in Systems and Control*. Boston: Birkhäuser, 2003.
[3] Z. Sun and S. S. Ge, *Switched Linear Systems: Control and Design*. London: Springer, 2005.
[4] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Review*, vol. 49, pp. 545–592, 2007.
[5] G. N. Nair, S. Dey, and R. J. Evans, "Infimum data rates for stabilising Markov jump linear systems," in *Proc. 42nd IEEE Conf. on Decision and Control*, 2003, pp. 1176–1181.
[6] C. Zhang, K. Chen, and G. E. Dullerud, "Stabilization of Markovian jump linear systems with limited information—a convex approach," in *Proc. 2009 American Control Conf.*, 2009, pp. 4013–4019.
[7] Q. Ling and H. Lin, "Necessary and sufficient bit rate conditions to stabilize quantized Markov jump linear systems," in *Proc. 2010 American Control Conf.*, 2010, pp. 236–240.
[8] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Trans. Automat. Control*, vol. 48, pp. 304–307, 2003.
[9] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th IEEE Conf. on Decision and Control*, 1999, pp. 2655–2660.
[10] L. Vu and D. Liberzon, "Supervisory control of uncertain linear time-varying plants," *IEEE Trans. Automat. Control*, vol. 56, pp. 27–42, 2011.
[11] S. Tatikonda and S. K. Mitter, "Control under communication constraints," *IEEE Trans. Automat. Control*, vol. 49, pp. 1056–1068, 2004.
[12] J. P. Hespanha, A. Ortega, and L. Vasudevan, "Towards the control of linear systems with minimum bit-rate," in *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*, 2002.
[13] Y. Sharon and D. Liberzon, "Input-to-state stabilizing controller for systems with coarse quantization," *IEEE Trans. Automat. Control*, to appear. Available online; preliminary version appeared in Proc. 46th CDC, 2007, pp. 20–25.
[14] Z. P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.
[15] D. Liberzon and J. P. Hespanha, "Stabilization of nonlinear systems with limited information feedback," *IEEE Trans. Automat. Control*, vol. 50, pp. 910–915, 2005.