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Stabilization of nonlinear systems with delay in the input through backstepping

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Abstract— **We propose a new solution to the problem of globally asymptotically stabilizing a nonlinear system in feedback form with a known pointwise delay in the input. The result covers a family of systems wider than those studied in the literature and endows with control laws with a single delay, in contrast to the existing one, which include two distinct pointwise delays or distributed delays. The design strategy is based on the construction of an appropriate Lyapunov-Krasovskii functional.**

I. INTRODUCTION

Time-delay systems represent an important family of systems spanning a wide range of application including network control, population dynamics, biological systems to cite only a few. Most of the literature in the field is devoted to linear systems (see, for instance [5], [26], [24] and the references therein). Nevertheless, especially in the last two decades, some important results for nonlinear systems with delay have appeared. In particular, extensions to systems with delays of the two techniques of recursive design of control laws called *backstepping* and *forwarding* have been obtained. Forwarding [11] and backstepping [4], [25] approaches have been adapted to important families of systems with pointwise delays and delay-free inputs. It is worth mentioning that the problem of stabilizing nonlinear systems with time delayed inputs is also of interest due to the transport and measurement delays that naturally arise in control applications (see, e.g., [24]). Although such a problem appears as being difficult, a few papers present extensions of the forwarding approach to the case of retarded inputs (see, e.g., [15], [2], [29] and [21]). For *backstepping*, the situation is different: although backstepping is one of the most popular techniques of design of stabilizing control laws for nonlinear systems, which has been largely developed in the literature (see, for instance, [17], [14], [18], [1], [16] and the references therein), to the best of our knowledge, only three contributions [3], [19], [12] are devoted to the problem of extending the backstepping approach to the case where there are delays in the inputs. More precisely, stabilization is achieved in [3] and [12], via a control law with distributed terms over some time interval and in [19], stabilization is achieved via a control law with two pointwise delays.

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Motivated by the precedent considerations, the present work continues the efforts to extend the *backstepping approach* to nonlinear systems with a pointwise delay in the input. Since it owes a great deal to [19], we briefly describe its main result. Systems of the form:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h(x(t-\tau), z(t-\tau)) \ , \end{array} \right. \tag{1}$$

with $x \in \Re^n$, $z \in \Re$, $u \in \Re$ the input and $\tau \geq 0$ the delay under appropriate initial conditions, are considered. The existence of a control law $z_s(x)$ that globally asymptotically stabilizes the system $\dot{x} = f(x) + g(x)z_s(x)$ is assumed. Technical assumptions are also introduced to guarantee that the system

$$\dot{x}(t) = f(x(t)) + g(x(t))z_s(x(t-\tau)) \tag{2}$$

admits the origin as a globally asymptotically stable equilibrium point. Basically, these assumptions are growth conditions on the functions f and g that prevent the finite escape time phenomenon and make it possible an explicit construction of a strict Lyapunov-Krasovskii functional for the system (2). With these considerations in mind, the introduction of the operator $Z : C_{in} \mapsto \Re$ defined on the dynamics of (1) by

$$Z(t) = z(t) - z_s(x(t-\tau)) \ , \tag{3}$$

which plays the role of a change of coordinates, leads to a Lyapunov functional U and a control law

$$\begin{array}{rcl} u(t-\tau) & = & -\varepsilon[z(t-\tau) - z_s(x(t-2\tau))] \\ & & -h(x(t-\tau), z(t-\tau)) \\ & & +R(x(t-\tau), z(t-\tau)) \ , \end{array} \tag{4}$$

with

$$R(x,z) = \frac{\partial z_s}{\partial x}(x)[f(x) + g(x)z] \ ,$$

with $\varepsilon > 0$, which ensures that, for all $t \geq \tau$, the derivative of U along the trajectories of the closed-loop system is smaller than a negative definite function depending on $x(t)$, $z(t)$. To the best of the authors' knowledge, this pioneer technique made it possible for the first time to globally stabilize nonlinear systems in feedback form by retarded feedbacks of class C^1 without using distributed terms. However, such an approach has three limitations: (i) in the formula of the control law (4), both $x(t)$ and $x(t-\tau)$ are present, although in (1) only the delay τ is present in $u(t-\tau)$ and $h(x(t-\tau), z(t-\tau))$, (ii) the derivative of the Lyapunov-Krasovskii functional along the trajectories of the closed-loop system is negative definite only for values of time larger than τ. Although, as proved in [13], Lyapunov functionals of this type can usually be used to establish robustness results similar to those which can be derived from the so-called "classical" Lyapunov-Krasovskii functionals, they are not as familiar and convenient as classical Lyapunov-Krasovskii functionals (see for instance [27] and [8] for more information on the usefulness of Lyapunov-Krasovskii functionals), especially when one aims to determine ISS estimates. (iii) Finally, the function h in (1), which depends on the retarded values $x(t-\tau)$, $z(t-\tau)$, cannot depend on $x(t)$, $z(t)$. If it does, such a technique of construction *does not apply*.

In the present work, we overcome these limitations. In particular, we wish to point out that the main result of stabilization we are deriving applies to systems of the family:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) \\ & & +h_2(x(t-\tau), z(t-\tau)) \ , \end{array}\right. \tag{5}$$

with $x \in \Re^n$, $z \in \Re$, $u \in \Re$, under appropriate initial conditions. It is worth mentioning that such a class of systems is important due to the presence of the term without delay $h_1(x(t), z(t))$ and since this family encompasses the family of the systems of the form:

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \ , \\ & \vdots & \\ \dot{x}_{n-1}(t) & = & x_n(t) \ , \\ \dot{x}_n(t) & = & z(t), \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) \ , \end{array}\right. \tag{6}$$

with $x = (x_1, ..., x_n)$, which may result from the attempt to linearize a single-input single-output system including a single delay only in the input, which cannot be completed when $\tau > 0$ because the term $h_1(x(t), z(t))$ cannot be removed through a change of feedback. Interestingly, the contribution [6] presents results for the systems (6), under various assumptions on the growth properties of the term h_1. In contrast to the feedbacks of the present work, the control laws of [6] depend on the past values of the controls.

The Lyapunov based technique we are proposing relies on the use of an *operator* of a *new type*. It is reminiscent of the one introduced for the first time in [22] and [23] and also shares some features with the one used in [9] and [10] to reduce a system with delay to another one without delay. However neither the approach of [22] nor the one of [9] and [10] can be applied to stabilize systems of the form (5): the assumptions imposed in [22] are not satisfied by (5) (notice in particular that the main result of [22] is a result of exponential stabilization whereas (5) is not necessarily locally exponentially stabilizable) and the contributions [9] and [10] are not concerned with retarded inputs. The operator leads to the construction of a Lyapunov-Krasovskii functional whose derivative along the trajectories of (5) can be made negative definite by an appropriate choice of state feedback of class C^1. Using this strict Lyapunov-Krasovskii functional, we shall prove for a family of systems with additive disturbances that the control laws we propose give to the systems the desirable ISS property (see for instance [28], [18] for a detailed presentation of the notion of ISS) with respect to additive disturbances.

The remaining of the paper is organized as follows. In Section II, we introduce the general family of systems that will be studied and the assumptions that will be used throughout the paper. Next, the main result is stated and proved in Section III. A result of ISS robustness is established in Section IV. A second-order example in Section V illustrates the control design of the previous section. Some concluding remarks in Section VI complete the work.

Notations and definitions: • Denote $|\cdot|$ the Euclidean norm of matrices and vectors of any dimension. • Given $\phi : I \to \Re^p$ defined on an interval I, denote its (essential) supremum over I by $|\phi|_I$. • Let p be any positive integer. We denote $C_{\text{in}} = C([-\tau, 0], \Re^p)$ the set of all continuous $\Re^p$-valued functions defined on a given interval $[-\tau, 0]$. • For a continuous function $\varphi : [-\tau, +\infty) \to \Re^k$, for all $t \geq 0$, the function φ_t defined by $\varphi_t(\theta) = \varphi(t+\theta)$ for all $\theta \in [-\tau, 0]$ is sometimes called translation operator. • A function $\kappa : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}_\infty$ if $\kappa(0) = 0$, κ is continuous, increasing and unbounded. • The notations will be simplified whenever no confusion can arise from the context.

II. Particular family of systems in feedback form

We consider the nonlinear systems:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) \\ & & +h_2(x(t-\tau), z(t-\tau)) \ , \end{array}\right. \tag{7}$$

with $x \in \Re^n$, $z \in \Re$, where $u \in \Re$ is the input, where $\tau > 0$ is a constant, where f, g, h_1, h_2 are functions of class C^1, under appropriate initial conditions. The main goal of the section is to develop a method for deriving stabilizing feedbacks for systems (7). To achieve it, we introduce a set of assumptions:

Assumption H1. *There exist a positive definite, radially unbounded function V of class C^1 and a function $z_s(x)$ of class C^2 such that $z_s(0) = 0$ and the function*

$$W : \Re^n \to \Re \ , \ W(x) = -\frac{\partial V}{\partial x}(x) F(x) \ , \tag{8}$$

where

$$F : \Re^n \to \Re^n \ , \ F(x) = f(x) + g(x) z_s(x) \ , \tag{9}$$

is positive definite.

Assumption H2. *There exist a constant $c_1 \geq 0$ and a nonnegative locally Lipschitz function G_1 such that the function*

$$\begin{array}{rcl} H & : & \Re^n \times \Re \to \Re \ , \\ H(x, Z) & = & -\frac{\partial z_s}{\partial x}(x)[F(x) + g(x)Z] \\ & & +h_1(x, Z + z_s(x)) \end{array} \tag{10}$$

satisfies, for all $x \in \Re^n, Z \in \Re$, the inequality

$$|H(x, Z)| \leq c_1 |Z| + G_1(x) \ . \tag{11}$$

There exist constants $c_2 > 0$ and $c_3 > 0$ such that, for all $x \in \Re^n$, the inequalities

$$\left| \frac{\partial V}{\partial x}(x) g(x) \right|^2 \leq c_2 W(x) \ , \tag{12}$$

$$G_1(x)^2 \leq c_3 W(x) \ , \tag{13}$$

are satisfied.

Assumption H3. *The delay τ and the constants c_i, $i = 1, 2, 3$, in Assumption H2 satisfy*

$$\tau < \left\{ \begin{array}{c} \min\left\{ \frac{1}{2\sqrt{2}c_1}, \frac{1}{\sqrt{6c_2c_3}} \right\} \quad if \quad c_1 > 0 \ , \\ \frac{1}{\sqrt{6c_2c_3}} \quad if \quad c_1 = 0 \ . \end{array} \right. \tag{14}$$

III. Main result

We are ready to state and prove the following result:

Theorem 1: Consider the system (7). Assume that it satisfies Assumptions H1 to H3. Then for all $L \in \Re$ such that

$$L \in \left[c_1 - \frac{1}{2\sqrt{2}\tau}, 0 \right) \tag{15}$$

the system (7) in closed-loop with

$$\begin{array}{l} u(x,z) = -h_2(x,z) + Le^{\tau L}(z - z_s(x)) \\ +e^{\tau L}\left[\frac{\partial z_s}{\partial x}(x)(f(x) + g(x)z) - h_1(x,z) \right] \end{array} \tag{16}$$

admits the origin as a globally asymptotically stable equilibrium point.

Discussion of Theorem 1.

– Assumption H1 is one of the fundamental assumptions of the backstepping method: it ensures the stabilizability of the x-subsystem of (7) with z as virtual input.

– Implicit growth restrictions on the functions f, g, h_1 are imposed in Assumption H2. They allow us performing our Lyapunov design. However, they are not simple technical assumptions which might be removed without changing the result. It is worth pointing out that in [19], it is proved that there are some systems (1), which do not satisfy Assumption H2 and for which there exists no feedback $u(x(t-\tau), z(t-\tau))$ so that the finite escape time phenomenon does not occur.

– Since in the absence of delay, globally asymptotically stabilizing feedbacks can be determined through the backstepping approach, the result of [20], which is a result of robustness with respect to the presence of a delay in the input, applies in some cases when extra assumptions, which pertain in particular to the growth of $\frac{\partial f}{\partial x}$, $\frac{\partial g}{\partial x}$ $\frac{\partial h_1}{\partial x}$, $\frac{\partial h_1}{\partial z}$, $\frac{\partial^2 z_s}{\partial^2 x}$, are satisfied. In this particular case these assumptions are complicated. For the sake of brevity we do not give give them.

– One can easily check that if the system (7) is linear and asymptotically stabilizable, then there always exists a quadratic Lyapunov function V and a linear function z_s such that Assumptions H1 and H2 are satisfied.

– Assumptions H1 to H3 do not imply that the system (7) with $\tau = 0$ admits an exponentially stabilizable linear approximation at the origin: they are satisfied for instance by the two-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & x(t)z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) \ , \end{array} \right. \tag{17}$$

studied in [19] with $V(x) = \ln(1 + x^2)$, $z_s(x) = -\frac{\omega x^2}{1+x^2}$, $\omega > 0$, where τ is sufficiently small relative to ω. This is a remarkable feature of Theorem 1 because most of the stabilization results for systems with delays apply only to systems that can be locally exponentially stabilized by a feedback of class C^1.

Proof. To begin with, we observe that Assumption H3 ensures that there exist negative constants L such that the inequality (15) is satisfied. Now, to simplify the control design, we perform the change of coordinate $Z = z - z_s(x)$ and take u under the form $u(t-\tau) = v(x(t-\tau), z(t-\tau) - z_s(x(t-\tau))) - h_2(x(t-\tau), z(t-\tau))$, where v is a function to be determined later. With these transformations, the system (7) becomes

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & F(x(t)) + g(x(t))Z(t) \ , \\ \dot{Z}(t) & = & v(x(t-\tau), Z(t-\tau)) \\ & & +H(x(t), Z(t)) \ , \end{array} \right. \tag{18}$$

with F defined in (9) and H defined in (10). Then, we start the construction of a Lyapunov functional candidate by introducing the operator $\alpha : C_{in} \mapsto \Re$ defined by

$$\alpha(\phi_x, \phi_Z) = \int_{-\tau}^{0} e^{-L(m+\tau)} v(\phi_x(m), \phi_Z(m)) dm \ , \tag{19}$$

where L is the constant given in Theorem 1. Along the trajectories of (18), it satisfies

$$\alpha(x_t, Z_t) = \int_{t-\tau}^{t} e^{L(t-m-\tau)} v(x(m), Z(m)) dm \ , \tag{20}$$

and its time-derivative, denoted simply $\dot{\alpha}(t)$, satisfies

$$\begin{array}{rcl} \dot{\alpha}(t) & = & L\alpha(x_t, Z_t) + e^{-\tau L} v(x(t), Z(t)) \\ & & -v(x(t-\tau), Z(t-\tau)) \ . \end{array} \tag{21}$$

It follows that the operator $\beta : C_{in} \mapsto \Re$ defined by

$$\beta(\phi_x, \phi_Z) = \phi_Z(0) + \alpha(\phi_x, \phi_Z) \tag{22}$$

is such that, for all $t \geq 0$,

$$\begin{array}{rcl} \dot{\beta}(t) & = & L\alpha(x_t, Z_t) + e^{-\tau L} v(x(t), Z(t)) \\ & & +H(x(t), Z(t)) \ . \end{array} \tag{23}$$

Selecting the function

$$v(x, Z) = -e^{\tau L} H(x, Z) + Le^{\tau L} Z \ , \tag{24}$$

which corresponds to the control (16), we obtain

$$\dot{\beta}(t) = L\beta(x_t, Z_t) \ . \tag{25}$$

We note that $\beta(x_t, Z_t)$ is a solution of an exponentially stable system since $L < 0$. We shall use this property implicitly later. For the time being, we observe that the equality

$$\begin{array}{rcl} Z(t) & = & \beta(x_t, Z_t) \\ & - & \displaystyle\int_{t-\tau}^{t} e^{L(t-m-\tau)} v(x(m), Z(m)) dm \ , \end{array} \tag{26}$$

with v defined in (24) and the negativity of L imply that, for all $t \geq 0$, the inequality

$$|Z(t)| \leq \int_{t-\tau}^{t} |H(x(m), Z(m)) - LZ(m)|dm + |\beta(x_t, Z_t)| \quad (27)$$

holds. From (11) in Assumption H2 and the definition of H in (10), we deduce that, for all $t \geq 0$,

$$|Z(t)| \leq \int_{t-\tau}^{t} [c_4|Z(m)| + G_1(x(m))]dm + |\beta(x_t, Z_t)| \ , \quad (28)$$

with $c_4 = c_1 - L$. This inequality together with the inequality (57), Cauchy-Schwartz inequality imply that, for all $t \geq 0$,

$$\begin{aligned} Z(t)^2 &\leq \tfrac{5}{4}\tau \int_{t-\tau}^{t} [c_4|Z(m)| + G_1(x(m))]^2 dm + 5\beta(x_t, Z_t)^2 \\ &\leq \tfrac{5}{2}\tau \int_{t-\tau}^{t} [c_4^2 Z(m)^2 + G_1(x(m))^2]dm + 5\beta(x_t, Z_t)^2 \ . \end{aligned} \quad (29)$$

Then, we introduce a new operator $\gamma : C_{in} \mapsto \Re$ defined by

$$\gamma(\phi_Z) = \int_{-\tau}^{0} \int_{\ell}^{0} \phi_Z(m)^2 dm d\ell \ , \quad (30)$$

which satisfies, along the trajectories of (18), for all $t \geq 0$, $\gamma(Z_t) = \int_{t-\tau}^{t} \int_{\ell}^{t} Z(m)^2 dm d\ell$ and

$$\begin{aligned} \dot{\gamma}(t) &= \tau Z(t)^2 - \int_{t-\tau}^{t} Z(m)^2 dm \\ &= -\tau Z(t)^2 - \int_{t-\tau}^{t} Z(m)^2 dm + 2\tau Z(t)^2 \ . \end{aligned} \quad (31)$$

Combining this equality with (29), after some straightforward algebraic manipulations, we obtain, for all $t \geq 0$, the inequality

$$\begin{aligned} \dot{\gamma}(t) \leq\ & -\tau Z(t)^2 + (5\tau^2 c_4^2 - 1)\int_{t-\tau}^{t} Z(m)^2 dm \\ & + 5\tau^2 \int_{t-\tau}^{t} G_1(x(m))^2 dm \\ & + 10\tau\beta(x_t, Z_t)^2 \ . \end{aligned} \quad (32)$$

The inequality (15) implies that $5\tau^2 c_4^2 - 1 \leq -3\tau^2 c_4^2$. It follows that, for all $t \geq 0$,

$$\begin{aligned} \dot{\gamma}(t) \leq\ & -\tau Z(t)^2 - 3c_4^2\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & + 5\tau^2 \int_{t-\tau}^{t} G_1(x(m))^2 dm \\ & + 10\tau\beta(x_t, Z_t)^2 \ . \end{aligned} \quad (33)$$

We continue our Lyapunov construction by considering the functional $U_1 : C_{in} \mapsto \Re$ defined by

$$U_1(\phi_x, \phi_Z) = V(\phi_x(0)) + c_5\gamma(\phi_Z) \ , \quad (34)$$

where V is the function given by Assumption H1 and where $c_5 > 0$ is a constant to be chosen later. Then, from Assumption H1 and (33), we deduce that, for all $t \geq 0$,

$$\begin{aligned} \dot{U}_1(t) \leq\ & -W(x(t)) + \tfrac{\partial V}{\partial x}(x(t))g(x(t))Z(t) \\ & - c_5\tau Z(t)^2 - c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & + 5c_5\tau^2 \int_{t-\tau}^{t} G_1(x(m))^2 dm \\ & + 10c_5\tau\beta(x_t, Z_t)^2 \ , \end{aligned} \quad (35)$$

with $c_6 = 3c_5c_4^2$. From the inequality (12) in Assumption H2, we deduce that, for all $t \geq 0$,

$$\begin{aligned} \dot{U}_1(t) \leq\ & -W(x(t)) + \sqrt{c_2 W(x(t))}|Z(t)| \\ & - c_5\tau Z(t)^2 - c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & + 5c_5c_3\tau^2 \int_{t-\tau}^{t} W(x(m))dm \\ & + 10c_5\tau\beta(x_t, Z_t)^2 \ . \end{aligned} \quad (36)$$

This inequality leads us to consider the functional $U_2 : C_{in} \mapsto \Re$ defined by

$$\begin{aligned} & U_2(\phi_x, \phi_Z) = k\beta(\phi_x, \phi_Z)^2 + U_1(\phi_x, \phi_Z) \\ & + 6c_5c_3\tau^2 \int_{-\tau}^{0} \int_{\ell}^{0} W(\phi_x(m))dm d\ell \ , \end{aligned} \quad (37)$$

with $k = \frac{1+10c_5\tau}{-2L}$ (since $L < 0$, $k > 0$). Bearing in mind (25), through elementary calculations, we obtain that for all $t \geq 0$, the following inequality holds:

$$\begin{aligned} \dot{U}_2(t) \leq\ & (6c_5c_3\tau^3 - 1)W(x(t)) \\ & + \sqrt{c_2 W(x(t))}|Z(t)| - c_5\tau Z(t)^2 \\ & - \beta(x_t, Z_t)^2 - c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & - c_5c_3\tau^2 \int_{t-\tau}^{t} W(x(m))dm \ . \end{aligned} \quad (38)$$

Therefore there exists a constant $c_7 \in (0, c_5\tau)$ such that, for all $t \geq 0$,

$$\begin{aligned} \dot{U}_2(t) \leq\ & -c_7 W(x(t)) - c_7 Z(t)^2 - \beta(x_t, Z_t)^2 \\ & - c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & - c_5c_3\tau^2 \int_{t-\tau}^{t} W(x(m))dm \ , \end{aligned} \quad (39)$$

if there exists $c_5 \in \left(0 \frac{1}{6c_3\tau^3}\right)$ such that the inequality:

$$c_2 + 4c_5\tau(6c_5c_3\tau^3 - 1) < 0 \quad (40)$$

holds (this fact can be proved by using the inequality $\sqrt{c_2 W(x(t))}|Z(t)| \leq \frac{c_2}{4(c_5\tau - c_7)} W(x(t)) + (c_5\tau - c_7)|Z(t)|^2$). Since Assumption H3 ensures that $\tau < \frac{1}{\sqrt{6c_2c_3}}$, the constant $c_5 = \frac{1}{12c_3\tau^3}$ is such that (40) is satisfied because this value for c_5 leads to $c_2 + 4c_5\tau(6c_5c_3\tau^3 - 1) = c_2 - \frac{1}{6c_3\tau^2} < 0$. However, although the right hand side of (39) is nonpositive, we cannot apply the Lyapunov-Krasovskii Theorem yet (see for instance [7], [18]) because we do not know if there is a function λ of class $\mathcal{K}_\infty$ such that, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$ the inequality:

$$\lambda(|\phi_x(0)|^2 + \phi_Z(0)^2) \leq U_2(\phi_x, \phi_Z) \quad (41)$$

is satisfied. To overcome this obstacle, we replace U_2 by the functional

$$\begin{array}{l} U_3(\phi_x, \phi_Z) = U_2(\phi_x, \phi_Z) \\ +\frac{c_7}{2} \int_{-\tau}^{0} [W(\phi_x(m)) + \phi_Z(m)^2] dm \ . \end{array} \quad (42)$$

The rest of the proof is omitted.

IV. ISS closed-loop systems

In this section, we use the Lyapunov-Krasovskii functional constructed in the previous section to establish that, under supplementary assumptions, the control (16) gives an ISS property to the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))[z(t) + w_1(t)] \ , \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) \\ & & +h_2(x(t-\tau), z(t-\tau)) + w_2(t) \ , \end{array} \right. \quad (43)$$

with $x \in \Re^n$, $z \in \Re$, where $u \in \Re$ is the input, under appropriate initial conditions, where $\tau > 0$ is a constant, and f, g, h_1, h_2 are functions of class C^1 and where w_1 and w_2 are disturbances. More precisely, we establish the following result:

Theorem 2: Consider the system (43). Assume that it satisfies Assumptions H1 to H3 with a function W that is radially unbounded. Assume also that there exists a constant $c_8 > 0$ such that, for all $x \in \Re^n$,

$$\left| \frac{\partial z_s}{\partial x}(x) \right| \leq c_8 \ . \quad (44)$$

Then, for all $L \in \Re$ such that (15) holds, the system (43) in closed-loop with the control (16) is ISS with respect to (w_1, w_2).

Proof. For the sake of brevity, the proof is omitted.

V. Illustrative example

In this section, we illustrate Theorem 2. We consider the pendulum equations:

$$\left\{ \begin{array}{c} \dot{x}(t) = z(t) + w_1(t) \ , \\ \dot{z}(t) = u(t-\tau) + a\sin(x(t)) - bz(t) + w_2(t) \ , \end{array} \right. \quad (45)$$

where $a \neq 0$, b and τ are positive constants, $x \in \Re$, $z \in \Re$, where $u \in \Re$ is the input and where w_1, w_2 are disturbances. Due to the presence of the term $a\sin(x) - bz$, the technique of [19] does not apply to (45).

For this system, with the notations of Section IV, $f(x) = 0$, $g(x) = 1$, $h_1(x, z) = a\sin(x) - bz$, $h_2(x, z) = 0$. We apply Theorem 2 with the following functions $z_s(x) = -rx$, where $r > 0$ and $r \neq b$ and $V(x) = \frac{1}{2}x^2$. Then, with the notation of Section III,

$$W(x) = rx^2 \ , \ F(x) = -rx \ , \ \frac{\partial V}{\partial x}(x)g(x) = x \quad (46)$$

and $$H(x, Z) = r[-rx + Z] + a\sin(x) - b(Z - rx)$$
.

It follows that, for all $(x, Z) \in \Re^2$,

$$|H(x, Z)| \ \leq \ c_1|Z| + G_1(x) \ , \quad (47)$$

with $c_1 = |b - r|$, $G_1(x) = |(b - r)rx + a\sin(x)|$. Finally, noticing that W is positive definite and radially unbounded, $\frac{\partial z_s}{\partial x}(x) = -r$,

$$\left(\frac{\partial V}{\partial x}(x)g(x) \right)^2 = x^2 = c_2 W(x) \quad (48)$$

with $c_2 = \frac{1}{r}$ and

$$G_1(x)^2 \ \leq \ 2[r^2(b-r)^2 + a^2]x^2 = c_3 W(x) \ , \quad (49)$$

with $c_3 = 2\frac{r^2(b-r)^2+a^2}{r}$, we deduce from Theorem 2 that if

$$\tau < \min \left\{ \frac{1}{2\sqrt{2}|b-r|}, \frac{1}{\sqrt{12}} \frac{r}{\sqrt{r^2(b-r)^2 + a^2}} \right\},$$

then the system (45) in closed-loop with the control law

$$\begin{array}{rcl} u_r(x, z) & = & Le^{\tau L}(z + rx) \\ & & +e^{\tau L}\left[-rz - a\sin(x) + bz\right] \ , \end{array} \quad (50)$$

with

$$L \in \left[|b - r| - \frac{1}{2\sqrt{2}\tau}, 0 \right) \quad (51)$$

is globally ISS with respect to (w_1, w_2). Observe that, rewriting u_r as

$$u_r(x, z) = e^{\tau L}\left[Lrx - a\sin(x) + (L + b - r)z\right] \ , \quad (52)$$

and choosing $r < b$ and $L = r - b$, we obtain the feedback

$$u_r(x, z) = e^{\tau(r-b)}\left[(r - b)rx - a\sin(x)\right] \quad (53)$$

which is independent from z. This feature may be of interest in the cases where the variable of velocity z cannot be measured. ISS for the system (45) can be also achieved by linear output feedbacks

$$u_l(x) = -sx \ , \ s \in \Re \ , \quad (54)$$

when the delay τ is sufficiently small. However, the families of stabilizing feedbacks (54) and (53) do not have the same features. Since, for all $(x, z) \in \Re^2$,

$$|u_r(x, z)| \leq e^{\tau(r-b)}\left[(b - r)r|x| + |a|\right] \ , \quad (55)$$

and since r can be chosen arbitrary close to b (independently from τ), for any $\epsilon > 0$, the family (53) contains elements which satisfy, for all $(x, z) \in \Re^2$, the inequality

$$|u_r(x, z)| \leq \epsilon|x| + |a| \ . \quad (56)$$

(For instance, a possible choice for obtaining this is by choosing $r = b - \min\left\{\frac{\varepsilon}{b}, \frac{b}{2}\right\}$). By contrast, one can prove that if a feedback $u_l(x) = -sx$ stabilizes the system (45), then there exists $x_l \in \Re$ such that $|u_l(x_l)| > \varepsilon_0|x_l| + 2|a|$ with $\varepsilon_0 = \frac{|a|}{9}$. We omit this proof.

We conclude that the family of stabilizing control laws u_r given in (53) has an advantage over the family of the stabilizing linear control laws u_l given in (54), relative to the size of their elements: roughly speaking, outside a compact set, no stabilizing linear feedback will be smaller than some of the feedbacks provided by Theorem 2.

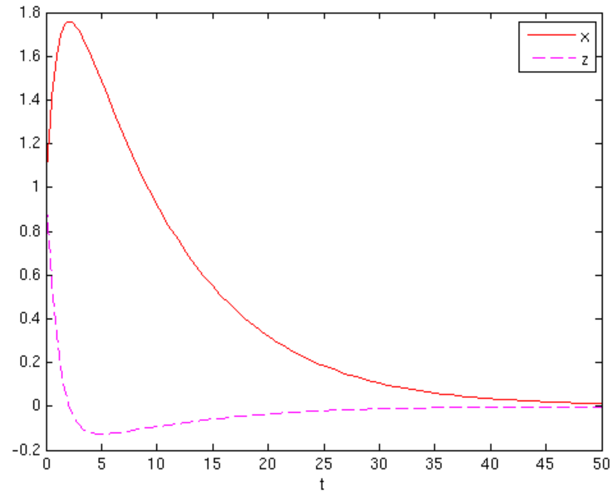

Fig. 1. Numerical simulation of a solution of the system (45) in closed-loop with the control (53) with $a = b = 1$, $r = 0.8879$, $\tau = 0.1806$.

VI. Conclusion

We have developed a new backstepping method for a new family of systems with delay in the input. We have obtained state feedbacks of class C^1 whose analytic expressions include delay information, and explicit expressions of strict Lyapunov-Krasovskii functionals for the closed-loop systems. Much remains to be done.

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VIII. Useful inequalities

For all real numbers A, B and $\rho \in (0, +\infty)$ the inequality

$$(A + B)^2 \leq (1 + \rho)A^2 + (1 + \frac{1}{4\rho})B^2 \ , \tag{57}$$

is satisfied. For all real numbers A, B and for all $\rho \in (0, 1)$, the inequality

$$(A + B)^2 \geq \rho A^2 - \frac{\rho}{1 - \rho} B^2 \tag{58}$$

is satisfied.