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Stability and Ergodicity of Piecewise Deterministic Markov Processes

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***Abstract*— The main goal of this paper is to establish some equivalence results on stability, recurrence between a piecewise deterministic Markov process (PDMP for short) $\{X(t)\}$ and an embedded discrete-time Markov chain $\{\Theta_n\}$ generated by a Markov kernel G that can be explicitly characterized in terms of the three local characteristics of the PDMP contrary to the resolvent kernel. First we establish some important results characterizing $\{\Theta_n\}$ as a sampling of the PDMP $\{X(t)\}$ and deriving a connection between the probability of the first return time to a set for the discrete-time Markov chains generated by G and the resolvent kernel R of the PDMP. From these results we obtain equivalence results regarding recurrence and positive recurrence between $\{X(t)\}$ and $\{\Theta_n\}$**

I. Introduction

Piecewise-deterministic Markov processes (PDMP's for short) have been introduced in the literature by M.H.A. Davis [1] as a general class of stochastic models. PDMP's are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP $\{X(t)\}$ depends on three local characteristics, namely the flow Φ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from x the motion of the process follows the flow $\Phi(x,t)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate λ or when the flow $\Phi(x,t)$ hits the boundary of the state-space. In either case the location of the process at the jump time T_1 is selected by the transition measure $Q(\Phi(x,T_1),.)$ and the motion restarts from this new point as before.

Over the last decades a great deal of attention has been given to the stability properties and related ergodic theory of Markov processes. One of the main approaches to deal with these problems is to show that the recurrence properties of the Markov process under consideration are related to the recurrence properties of an associated discrete-time Markov chain obtained from a sampling of the original process, so that the well known discrete-time Markov chains results could be used (see for example the books [2], [3], [4] and the references therein).

In the continuous-time context, J. Azéma *et al* [5], [6] showed that a general Markov process and its associated resolvent admit the same recurrence properties. It was proved by P. Tuominen and R. Tweedie [7] that the recurrence structure of a Markov process $\{X(t)\}$ with transition semi-group $\{P^t\}$ and the Markov chain with kernel $K_F = \int P^t F(dt)$, where F is a distribution on $[0,\infty)$, are essentially equivalent, provided that a continuity assumption on $\{P^t\}$ is satisfied, an assumption later suppressed in a fundamental paper by S. Meyn and R. Tweedie [8]. It must be pointed out that these results are related to the sampling of a continuous-time process $\{X(t)\}$, sampled at random times defined by an independent undelayed renewal process. This idea of randomized sampling was generalized to state dependent sampling to provide some more powerful state dependent drift criterions in order to ensure stability of the original Markov process. Within this context, V. Malyšev and M. Men'šikov [9] derived a modified Foster-Lyapunov criterion to establish recurrence properties for discrete-time Markov chains with countable state space. S. Meyn and R. Tweedie [10] generalized this work to discrete-time Markov chains with a general state space and furthermore obtained state-dependent drift conditions to get geometric ergodic properties. The generalization to continuous-time models has been established by J. Dai and S. Meyn [11] in the context of general state space Markovian queueing models. In particular, they provided sufficient conditions for the existence of bounds on the long-run average moments and rates of convergence of the p^{th} moments to their steady-state values. Another paper related to this subject is [12].

The main goal of this paper is to establish equivalence results on stability such as (Harris) recurrence and positive (Harris) recurrence between a PDMP and a discrete-time Markov chain generated by a kernel G (see equations (2)-(4) for its definition) that can be explicitly characterized in terms of the three local characteristics of the PDMP. It should be noticed that the results developed in [6], [8], [7] would be hard to be applied for the PDMP's from the practical point of view because the transition semigroup of the PDMP as well as its associated resolvent kernel cannot be explicitly calculated from its local characteristics, as opposite to the kernel G. As shown in Theorem 3.1 below, G generates a Markov chain that corresponds to a state dependent sampling of the PDMP $\{X(t)\}$ providing an interesting parallel between our work in the continuous-time context and the results obtained in [10] in the discrete-time setting. However, it must be stressed that [10] provides general sufficient conditions to ensure that stability of the sampled chain implies stability of the Markov process, but not the converse. One of the main goals of our paper is

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to show the converse for PDMP's and, in fact, that the PDMP and the discrete-time Markov chain generated by this tractable kernel G have an equivalent recurrence structure. We show that the following equivalence results hold:

(i) The PDMP $\{X(t)\}$ is irreducible if and only if the Markov chain $\{\Theta_n\}$ associated to G is irreducible, see Proposition 4.1.

(ii) There is a one to one correspondence between the set of invariant measures for the PDMP $\{X(t)\}$ and for the Markov chain $\{\Theta_n\}$ associated to G, see Theorem 4.2.

(iii) The PDMP $\{X(t)\}$ is recurrent if and only if the Markov chain $\{\Theta_n\}$ associated to G is recurrent, see Theorem 4.7.

(iv) The PDMP is Harris recurrent if and only if the Markov chain associated to G is Harris recurrent, see Theorem 4.9.

(v) The PDMP is positive recurrent (respectively, positive Harris recurrent) if and only if the Markov chain associated to G is recurrent (respectively Harris recurrent) with invariant measure satisfying a boundedness condition, see Corollary 4.8 (respectively Corollary 4.11).

The paper is organized as follows. In section II we present some basic definitions related to the motion of a PDMP, introduce the Markov kernel G, and recall some classical definitions related with Markov processes both in the discrete-time and continuous-time context. Some preliminary results are derived in section III that will be important to obtain the equivalence properties for the stability of the PDMP's and the Markov kernel G. In section IV, it will be established that the stability and recurrence properties are equivalent for the PDMP's and the kernel G.

II. Definition of the PDMP and the Markov kernel G

In this section we first present some standard notation and some basic definitions related to the motion of a PDMP $\{X(t)\}$. For further details the reader is referred to [1]. Afterwards we introduce the Markov kernel G, which we will use for characterizing the recurrence and the Harris recurrence structure of the PDMP $\{X(t)\}$. At the end of this section, we recall some classical definitions related with Markov processes both in the discrete-time and continuous-time context. For a complete exposition on the subject, the reader is referred to the works of Meyn and Tweedie [2], [13], [14], [15]. We follow closely the notation in Meyn and Tweedie [2].

Let $\mathbb{R}_+$ be the set of nonnegative real numbers. The set of natural numbers is denoted by $\mathbb{N}$, and $\mathbb{N}^* \doteq \mathbb{N} - \{0\}$. For any metric space H, the borel σ-field of H is denoted by $\mathcal{B}(H)$. The indicator of a set A is denoted by $\mathbf{1}_A$ ($\mathbf{1}_A(x) = 1$ if $x \in A$, $\mathbf{1}_A(x) = 0$ if $x \notin A$). Let E and F be two metric spaces. A kernel K on $E \times \mathcal{B}(F)$ is a map $K : E \times \mathcal{B}(F) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ such that for $x \in E$, $K(x, .)$ is a nonnegative σ-finite measure on $(F, \mathcal{B}(F))$ and for any $A \in \mathcal{B}(F)$, $K(., A)$ is a measurable function on E. The kernel I_A on $(E, \mathcal{B}(E))$ is defined for any set $A \in \mathcal{B}(E)$ by $I_A(x, B) = \mathbf{1}_{A \cap B}(x)$.

We present next the definition of the motion of a PDMP. Let E^0 be an open subset of $\mathbb{R}^n$ and ∂E^0 its boundary. A PDMP is determined by its local characteristics $(\mathfrak{X}, \lambda, Q)$ where:

• $\mathfrak{X}$ is a Lipschitz continuous vector field, $\mathfrak{X} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which determines a flow $\Phi(x, t)$ such that $\frac{\partial}{\partial t}\Phi(x, t) = \mathfrak{X}(\Phi(x, t))$ and $\Phi(x, 0) = x$ for all $x \in \mathbb{R}^n$.

Define $\Gamma^+ \doteq \{x \in \partial E^0 : x = \Phi(y, t)$ for some $y \in E^0, t > 0$, and $\Phi(y, s) \in E^0, \forall s \in [0, t[\}$, and $\Gamma^- \doteq \{x \in \partial E^0 : x = \Phi(y, -t)$ for some $y \in E^0, t > 0$, and $\Phi(y, -s) \in E^0, \forall s \in [0, t[\}$. $\Gamma^+ \subset \partial E^0$ represents the boundary points at which the flow exits from E^0. $\Gamma^- \subset \partial E^0$ is characterized by the fact that the flow starting from a point in Γ^- will not leave E^0 immediately. Therefore it is natural to define the state space for the PDMP by $E \doteq E^0 \cup \Gamma^- - \Gamma^- \cap \Gamma^+$. For all x in E, let us denote by $t_*(x) \doteq \inf\{t > 0 : \Phi(x, t) \in \partial E^0\}$, with the convention $\inf \emptyset = \infty$.

• The jump rate $\lambda : E \rightarrow \mathbb{R}_+$ is assumed to be a measurable function satisfying: $(\forall x \in E)$ $(\exists \varepsilon > 0)$ such that $\displaystyle\int_0^{\varepsilon} \lambda(\Phi(x, s))ds < \infty$.

• $Q : E \cup \Gamma^+ \times \mathcal{B}(E) \rightarrow [0, 1]$ is a transition measure satisfying the following property: $(\forall x \in E \cup \Gamma^+)$ $Q(x, E - \{x\}) = 1$.

From these characteristics, it can be shown [1, p. 62-66] that there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{P_x\}_{x \in E})$ such that the motion of the process $\{X(t)\}$ starting from a point $x \in E$ may be constructed as follows. Take a random variable T_1 such that

$$P_x(T_1 > t) \doteq \begin{cases} e^{-\Lambda(x,t)} & \text{for } t < t_*(x) \\ 0 & \text{for } t \geq t_*(x) \end{cases}$$

where for $x \in E$ and $t \in [0, t_*(x)[$

$$\Lambda(x, t) \doteq \int_0^t \lambda(\Phi(x, s))ds. \tag{1}$$

If T_1 generated according to the above probability is equal to infinity, then for $t \in \mathbb{R}_+$, $X(t) = \Phi(x, t)$. Otherwise select independently an E-valued random variable (labelled X_1) having distribution $Q(\Phi(x, T_1), .)$. The trajectory of $\{X(t)\}$ starting at x, for $t \leq T_1$, is given by

$$X(t) \doteq \begin{cases} \Phi(x, t) & \text{for } t < T_1, \\ X_1 & \text{for } t = T_1. \end{cases}$$

Starting from $X(T_1) = X_1$, we now select the next inter-jump time $T_2 - T_1$ and post-jump location $X(T_2) = X_2$ is a similar way.

This gives a strong Markov process $\{X(t)\}$ with jump times $\{T_k\}_{k \in \mathbb{N}}$ (where $T_0 \doteq 0$). The transition semigroup of the process $\{X(t)\}$ is denoted by $\{P^t\}_{t \in \mathbb{R}_+}$. We denote by $\{\mathcal{F}_t^X\}_{t \in \mathbb{R}_+}$ the filtration generated by the process $\{X(t)\}$.

It is assumed in all the paper that for all $(t, x) \in \mathbb{R}_+ \times E$, $E_x\Big[\displaystyle\sum_k \mathbf{1}_{\{T_k \leq t\}}\Big] < \infty$ implying in particular that $T_k \rightarrow \infty$

as $k \to \infty$. This is a standard assumption, see for example equations (24.4) or (24.8) in [1].

Now let us introduce the sub-stochastic kernels H and J and the Markov kernel G:

$$H(x, A) \doteq \int_0^{t_*(x)} e^{-\{s+\Lambda(x,s)\}} \mathbf{1}_A(\Phi(x, s))ds, \tag{2}$$

$$J(x, A) \doteq \int_0^{t_*(x)} \lambda(\Phi(x, s))e^{-\{s+\Lambda(x,s)\}} Q(\Phi(x, s), A)ds + e^{-\{t_*(x)+\Lambda(x,t_*(x))\}} Q(\Phi(x, t_*(x)), A), \tag{3}$$

$$G(x, A) \doteq J(x, A) + H(x, A). \tag{4}$$

In [16], it was shown that G as defined in (4) is a Markov kernel.
The resolvent kernel associated to the process $\{X(t)\}_{t\in\mathbb{R}_+}$ is denoted by

$$R(x, A) \doteq \int_0^{\infty} P^t(x, A)e^{-t}dt. \tag{5}$$

As shown in [16], R can be written in terms of H and J as follows:

$$R = \sum_{j=0}^{\infty} J^j H. \tag{6}$$

Let $\{\Theta_n\}$ (respectively $\{\Upsilon_n\}$) be the Markov chain associated to the Markov kernel G (respectively R). In Theorem 3.1 below it will be shown how the Markov chain $\{\Theta_n\}$ can be generated from the sample paths of the PDMP $\{X(t)\}$.

In what follows we will present some definitions considering a discrete-time Markov chain $\{\chi_n\}$ with Markov kernel S that could be either $\{\Theta_n\}$ (with $S = G$) or $\{\Upsilon_n\}$ (with $S = R$). The first return time of a set $A \in \mathcal{B}(E)$ for the PDMP $\{X(t)\}$ and for the Markov chain $\{\chi_n\}$ are defined respectively as follows:

$$\tau_A^X \doteq \inf\{t > 0 : X(t) \in A\}, \ \ \tau_A^S \doteq \inf\{n \geq 1 : \chi_n \in A\}.$$

Associated to these first return times, we have the return time probability of a set $A \in \mathcal{B}(E)$ for the PDMP $\{X(t)\}$ and for the Markov chain $\{\chi_n\}$, given respectively by

$$L^X(x, A) \doteq P_x(\tau_A^X < \infty), \ \ L^S(x, A) \doteq P_x(\tau_A^S < \infty).$$

The number of visits to a set A is defined for the PMDP $\{X(t)\}$ and for the Markov chain $\{\chi_n\}$ respectively as

$$\eta_A^X \doteq \int_0^{\infty} \mathbf{1}_A(X(t))dt, \qquad \eta_A^S \doteq \sum_{n=1}^{\infty} \mathbf{1}_A(\chi_n).$$

If F is a probability distribution on $\mathbb{R}_+$ (respectively b is a probability on $\mathbb{N}^*$), then the stochastic kernel K_F^X (respectively K_b^S) associated to $\{X(t)\}$ (respectively $\{\chi_n\}$) is defined on $E \times \mathcal{B}(E)$, $\forall x \in E$, $\forall A \in \mathcal{B}(E)$, by:

$$K_F^X(x, A) \doteq \int_0^{\infty} P^t(x, A)F(dt), \tag{7}$$

$$K_b^S(x, A) \doteq \sum_{k=0}^{\infty} b(k)S^k(x, A). \tag{8}$$

A set $C \in \mathcal{B}(E)$ is called a petite set for $\{X(t)\}$ (respectively for $\{\chi_n\}$) if there exist a probability distribution F on $\mathbb{R}_+$ (respectively a probability b on $\mathbb{N}^*$), and a non-trivial measure ν on $(E, \mathcal{B}(E))$ such that $(\forall A \in \mathcal{B}(E))$, $(\forall x \in C)$, $K_F^X(x, A) \geq \nu(A)$ (respectively $(\forall A \in \mathcal{B}(E))$, $(\forall x \in C)$, $K_b^S(x, A) \geq \nu(A)$).

A positive measure μ (respectively π) is called invariant for the PMDP $\{X(t)\}$ (respectively for the Markov chain $\{\chi_n\}$) if it is a σ-finite measure satisfying $\mu = \mu P^t$ for all $t \geq 0$ (respectively $\pi = \pi S$).

The next definitions apply for both the continuous-time as well as the discrete-time process, and therefore we suppress the superscript X or S. A Markov process is called φ-irreducible (and φ an irreducibility measure) if for some σ-finite measure φ, we have that $E_x(\eta_A) > 0$ for all $x \in E$ whenever $\varphi(A) > 0$. A set $A \in \mathcal{B}(E)$ is said to be full if $\varphi(A^c) = 0$. An irreducibility measure ψ is called maximal irreducible if for any other φ irreducibility measure, we have that $\psi \gg \varphi$. A Markov process is called recurrent if for some σ-finite measure φ, we have that $E_x(\eta_A) = \infty$ for all $x \in E$ whenever $\varphi(A) > 0$, and Harris recurrent if $E_x(\eta_A) = \infty$ is replaced by $P_x(\eta_A = \infty) = 1$. If the Markov process is Harris recurrent then there exists a unique (up to constant multiples) invariant measure. The Markov process is said to be positive Harris recurrent if it is Harris recurrent and the invariant measure is finite.

III. PRELIMINARY RESULTS

In this section we present some preliminary results that will be very important to characterize the recurrence and Harris recurrence structure of the PDMP $\{X(t)\}$. First in Theorem 3.1 the Markov chain $\{\Theta_n\}$ generated by the kernel G is shown to be related to the sample path of the PDMP. It is interesting to remark that $\{\Theta_n\}$ corresponds to a sampling of the continuous-time process $\{X(t)\}$ at random times that depends on a combination of a sequence of independent and identically distributed exponential times with the sequence $\{T_k\}$ of jump times of the PDMP $\{X(t)\}$. Moreover it must be pointed out that the Markov kernel G does not correspond to a generalized resolvent, as studied in the fundamental paper of Meyn and Tweedie [8]. An easy consequence of Theorem 3.1 presented in Corollary 3.2 is that if the first return time of the Markov chain $\{\Theta_n\}$ to a set A is finite then the first return time of the PDMP $\{X(t)\}$ to the same set A is finite. Consequently, it will be easy to deduce from the this result that if the Markov chain $\{\Theta_n\}$ is Harris recurrent then so is the process $\{X(t)\}$. The last two theorems of this section show that:

- the probability of the first return time of $\{\Theta_n\}$ to a set A to be finite is bounded below by the probability of the first return time of $\{\Upsilon_n\}$ to the same set A to be finite (see Theorem 3.3).
- the average number of visits of $\{\Theta_n\}$ to a set A is bounded below by the average number of visits of the

Markov chain $\{\Upsilon_n\}$ (generated by the resolvent) to the same set A (see Theorem 3.4).

Theorem 3.4 will be used in the next section to show that if the process $\{X(t)\}$ is recurrent then so is the Markov chain $\{\Theta_n\}$. An important consequence of Theorem 3.3 is that if the process $\{X(t)\}$ is Harris recurrent then so is the Markov chain $\{\Theta_n\}$. Theorem 3.3 is surprising and far from trivial to show.

We have the following result, proved in [17], which shows how the Markov chain $\{\Theta_n\}$ could be generated from the sample path realizations of $\{X(t)\}$.

Theorem 3.1: On the probability space $(\Omega, \mathcal{F}_t^X, \mathcal{F}, P_x)$ let $\{s_n\}_{n\geq 0}$ be a sequence of independent and identically distributed $\mathbb{R}_+$-valued random variables with exponential distribution with parameter equal to 1 such that $\vee_{t\geq 0}\mathcal{F}_t^X$ and $\sigma\{s_k : k \geq 0\}$ are independent. Let the sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ be defined as follows: $\tau_0 = 0$, and

$$\tau_{n+1} \doteq \sum_{k=0}^{n} \mathbf{1}_{\{T_k\leq\tau_n<T_{k+1}\}}\Big[(\tau_n + s_{n+1}) \wedge T_{k+1}\Big]. \quad (9)$$

Then $\{X(\tau_n)\}$ is Markov chain with transition probability given by G.

Proof: The proof of this result is presented in [17]. ■

Without loss of generality, it can be considered that $\Theta_n = X(\tau_n)$ since $\{\Theta_n\}$ was defined in section II as a Markov chain generated by G and from the previous theorem, $\{\Theta_n\}$ and $\{X(\tau_n)\}$ have the same probability distribution. An important corollary of the previous theorem is the following inclusion for the first return times of the Markov chain $\{\Theta_n\}$ and the process $\{X(t)\}$:

Corollary 3.2: For any set $A \in \mathcal{B}(E)$,

$$\{\tau_A^G < \infty\} \subset \{\tau_A^X < \infty\}. \quad (10)$$

Proof: This is a straightforward consequence from the fact that we can consider $\Theta_n = X(\tau_n)$, as shown in Theorem 3.1. ■

We have the following important theorem establishing a link between the probability of the first return time to a set for the Markov chains $\{\Theta_n\}$ and $\{\Upsilon_n\}$.

Theorem 3.3: For every $x \in E$, and $A \in \mathcal{B}(E)$,

$$L^G(x, A) \geq L^R(x, A). \quad (11)$$

Proof: The proof of this result is presented in [17]. ■

Combining (10) and (11) we have, for every $x \in E$ and $A \in \mathcal{B}(E)$, the following important inequalities:

$$L^R(x, A) \leq L^G(x, A) \leq L^X(x, A).$$

We conclude this section with the following theorem, providing a link between the average numbers of visits for the Markov chains generated by the kernel G and R.

Theorem 3.4: For every $x \in E$, and $A \in \mathcal{B}(E)$,

$$U^G(x, A) \geq U^R(x, A). \quad (12)$$

Proof: The proof of this result is presented in [17]. ■

IV. Characterization of the recurrence and Harris recurrence structure of the PDMP in terms of the Markov kernel G

The aim of this section is to characterize the (Harris) recurrence properties between the PMDP $\{X(t)\}$ and the Markov chain $\{\Theta_n\}$ generated by the kernel G. First, it is proved in Proposition 4.1 that $\{X(t)\}$ is irreducible if and only if $\{\Theta_n\}$ is irreducible. Then a generalization of Theorem 3.5 in [16] is presented in Theorem 4.2 giving a one to one correspondence between the invariant (positive and σ-finite) measures for the PDMP $\{X(t)\}$ and the Markov chain $\{\Theta_n\}$. Using the preliminary results derived in the previous section, it is shown in Theorem 4.7 and 4.9 that the PDMP $\{X(t)\}$ is recurrent (respectively Harris recurrent) if and only if the Markov chain $\{\Theta_n\}$ is recurrent (respectively Harris recurrent). One would expect a natural generalization of such equivalence results for positivity between the processes $\{X(t)\}$ and $\{\Theta_n\}$. In fact, this result does not hold. Indeed, it is shown in Corollary 4.8 that the positive recurrence of the process $\{X(t)\}$ is equivalent to a weaker form of stability for the Markov chain $\{\Theta_n\}$. Namely, $\{X(t)\}$ is positive recurrent if and only if $\{\Theta_n\}$ is recurrent and its unique invariant measure π satisfies the condition given by $\pi H(E) < \infty$ which is far less demanding than positive recurrence for $\{\Theta_n\}$. A similar result will be proved for the positive Harris recurrence of $\{X(t)\}$ (see Corollary 4.11).

We have the following proposition characterizing the irreducibility of the process $\{X(t)\}$ and the Markov chain $\{\Theta_n\}$.

Proposition 4.1: The PDMP $\{X(t)\}$ is irreducible if and only if the Markov chain $\{\Theta_n\}$ is irreducible.

Proof: Suppose that the Markov chain $\{\Theta_n\}$ is φ-irreducible. From Proposition 4.2.1 in [2], page 87, whenever $\varphi(A) > 0$ for any $A \in \mathcal{B}(E)$ we have that $L^G(x, A) > 0$ for all $x \in E$. From (10) we have that $L^X(x, A) > 0$ for all $x \in E$ whenever $\varphi(A) > 0$ for $A \in \mathcal{B}(E)$. By using Proposition 2.1 in [15], it follows that the PDMP $\{X(t)\}$ is μ-irreducible with $\mu = \varphi R$, where we recall that R is the resolvent defined in (5).
Now suppose that $\{X(t)\}$ is Ψ-irreducible. Then for $A \in \mathcal{B}(E)$ with $\Psi(A) > 0$, we have for all $x \in E$, $E_x\big[\eta_A^X\big] > 0$. Since $E_x\big[\eta_A^X\big] = U^R(x, A)$ (see, for instance, [7]), it implies that $L^R(x, A) > 0$. From Theorem 3.3, we get the result. ■

Recall that by definition an invariant measure is always σ-finite and positive. The next result shows that there exists a one to one correspondence between the set of invariant measures for the PDMP $\{X(t)\}$ and the set of invariant measures for the Markov chain $\{\Theta_n\}$ generated by G. It extends Theorem 3.5 in [16] that was restricted to the set of invariant probability measures for the PDMP's.

Theorem 4.2: $i)$ If μ is an invariant measure for $\{X(t)\}$ then $\mu \sum_{j=0}^{\infty} J^j$ is invariant for $\{\Theta_n\}$ and $\mu \sum_{j=0}^{\infty} J^j H = \mu$.

$ii)$ If π is an invariant measure for $\{\Theta_n\}$ then πH is invariant for $\{X(t)\}$ and $\pi H \sum_{j=0}^{\infty} J^j = \pi$.

Proof: Let us show i). Let μ be an invariant measure for $\{X(t)\}$ and set $\pi = \sum_{j=0}^{\infty} \mu J^j$. Let us show that π is σ-finite. Since μ is σ-finite, there exists a partition $\{A_i\}$ of E such that $\mu(A_i) < \infty$. Define $C_n = \bigcup_{i=1}^{n} A_i$, and $B_{n,m} = \{y \in E : H(y, C_n) > \frac{1}{m}\}$ for $m \in \mathbb{N}^*$. Notice now that for every $x \in E$, we have that $0 < H(x, E) < 1$, and so $\bigcup_{n,m} B_{n,m} = E$. From (6) we have that $\mu = \mu R = \sum_{j=0}^{\infty} \mu J^j H = \pi H$, so that $\infty > \mu(C_n) = \int_E H(y, C_n)\pi(dy) \geq \int_{B_{n,m}} H(y, C_n)\pi(dy) \geq \frac{1}{m}\pi(B_{n,m})$ showing that π is σ-finite. Finally notice from (4) and (6) that $\pi G = \pi J + \pi H = \sum_{j=1}^{\infty} \mu J^j + \mu R = \sum_{j=1}^{\infty} \mu J^j + \mu = \sum_{j=0}^{\infty} \mu J^j = \pi$ showing that π is invariant for $\{\Theta_n\}$. Moreover, $\mu \sum_{j=0}^{\infty} J^j H = \mu R = \mu$ completing the proof of $i)$.

Let us show now ii). Let π be an invariant measure for $\{\Theta_n\}$. For any $n \in \mathbb{N}^*$, we have $\pi \sum_{j=1}^{n} J^j H + \pi H = \pi G \sum_{j=0}^{n} J^j H = \pi \sum_{j=1}^{n} J^j H + \pi J^{n+1} H + \pi H \sum_{j=0}^{n} J^j H$. In order to cancel out the identical term $\pi \sum_{j=1}^{n} J^j H$ on both sides of the previous equation one first need to check that all the measures under consideration are σ-finite. Since $\pi = \pi G = \pi H + \pi J$, it can be shown easily by induction that $\pi J^j H \leq \pi$ and so $\pi J^j H$ is σ-finite for all $j \in \mathbb{N}$. Consequently, for all $j \in \mathbb{N}^*$, the measures $\pi \sum_{j=1}^{n} J^j H$, πH, $\pi J^{n+1} H$, and $\pi H \sum_{j=0}^{n} J^j H$ are σ-finite, implying that $\pi H = \pi J^{n+1} H + \pi H \sum_{j=0}^{n} J^j H$. Moreover, it can be shown that $J^n(x, A) = E_x\big[e^{-T_n} \mathbf{1}_A[X(T_n)]\big]$ for all $n \in \mathbb{N}^*$, $x \in E$ and $A \in \mathcal{B}(E)$. By using the dominated convergence theorem and the fact that $\lim_{n\to\infty} T_n = +\infty$, it follows that for all $A \in \mathcal{B}(E)$ $\lim_{n\to\infty} \pi J^n(A) = 0$. Combining these equations, we have that $\mu = \pi H = \pi H \sum_{j=0}^{\infty} J^j H = \mu R$, and from Lemma 1 in [5] it follows that μ is an invariant measure for $\{X(t)\}$. Now we have that $\pi H \sum_{j=0}^{n} J^j + \pi J^{n+1} + \pi \sum_{j=1}^{n} J^j = \pi + \pi \sum_{j=1}^{n} J^j$. It follows that $\pi H \sum_{j=0}^{n} J^j + \pi J^{n+1} = \pi$ by using the same arguments as above. Thus $\lim_{n\to\infty} \pi H \sum_{j=0}^{n} J^j = \pi$, showing $ii)$. ∎

Remark 4.3: A straightforward consequence of Theorem 4.2 is the following result: There exists a finite invariant measure for $\{X(t)\}$ if and only if there exists an invariant measure π for $\{\Theta_n\}$ satisfying $\pi H(E) < \infty$. Note that this result was already proved in Theorem 3.5 in [16].

The next two results show that if the PDMP is recurrent then so is the Markov chain generated by G and *vice versa*.

Proposition 4.4: If $H \in \mathcal{B}(E)$ is recurrent for the process $\{X(t)\}$ then H is recurrent for the Markov chain $\{\Theta_n\}$.

Proof: If $H \in \mathcal{B}(E)$ is recurrent for $\{X(t)\}$ then there exists a measure ν on $(H, \mathcal{B}(H))$ such that for all $A \in \mathcal{B}(H)$ with $\nu(A) > 0$, $E_x\big[\eta_A^X\big] = U^R(x, A) = \infty$ for every $x \in H$. From Theorem 3.4, it follows that $U^G(x, A) = \infty$, showing the result. ∎

Proposition 4.5: If A is an absorbing set for G then

i) for all $n \in \mathbb{N}_*$ $I_A J^n f(x) = I_A (J I_A)^n f(x)$ for every bounded positive measurable function f.

ii) A is an absorbing set for R.

Proof: Since A is an absorbing set for G, then for all $x \in E$, $I_A G(x, A^c) = 0$, consequently $I_A H \mathbf{1}_{A^c}(x) = 0$ and $I_A J \mathbf{1}_{A^c}(x) = 0$. Let us show now i) by induction on n. Consider a positive measurable function f bounded by a constant c. For $n = 1$ we have that $I_A J f(x) = I_A J I_A f(x) + I_A J I_{A^c} f(x) = I_A J I_A f(x)$ since $I_A J I_{A^c} f(x) \leq c I_A J \mathbf{1}_{A^c}(x) = 0$. Suppose now that $I_A J^n f(x) = I_A (J I_A)^n f(x)$. We have that $I_A J^{n+1} f(x) = I_A J^n J f(x) = I_A (J I_A)^n J f(x) = (I_A J)^n I_A J f(x) = (I_A J)^n I_A J I_A f(x) = I_A (J I_A)^{n+1} f(x)$, showing item $i)$.

Let us show now that $R(x, A) = 1$ for every $x \in A$. For all $x \in E$, we have

$$I_A R \mathbf{1}_{A^c}(x) = I_A \sum_{k=0}^{\infty} J^k H \mathbf{1}_{A^c}(x) = 0,$$

showing the last part of the result. ∎

Let A be an absorbing set for a discrete-time Markov chain $\{\chi_n\}$ with Markov kernel S. Then define

$$A_S^{\infty} \doteq \{x \in E : L^S(x, A) = 1\}.$$

An absorbing set A is called maximal absorbing if $A = A_S^{\infty}$.

Corollary 4.6: If H is maximal absorbing set for G then H is a maximal absorbing set for R

Proof: From Proposition 4.5, it follows that H is an absorbing set for R. Consequently, $H \subset H_R^{\infty} \doteq \{x \in E : L^R(x, H) = 1\}$. By definition, we have $H = H_G^{\infty} \doteq \{x \in E : L^G(x, H) = 1\}$. However, from Theorem 3.3 we have $H_R^{\infty} \subset H_G^{\infty}$, implying $H = H_G^{\infty} = H_R^{\infty}$ ∎

Theorem 4.7: The PDMP $\{X(t)\}$ is recurrent if and only if the Markov chain $\{\Theta_n\}$ associated to G is recurrent.

Proof: Suppose that $\{\Theta_n\}$ is recurrent. Let φ a maximal irreducibility measure for $\{\Theta_n\}$. Then from Proposition 9.0.1 in [2], $E = H \cup T$ where $T \in \mathcal{B}(E)$ is φ-null and transient for $\{\Theta_n\}$ and $H \in \mathcal{B}(E)$ is non-empty and maximal absorbing for $\{\Theta_n\}$ and every subset of H in $\mathcal{B}(E)^+ \doteq \{A \in \mathcal{B}(E) : \varphi(A) > 0\}$ is Harris recurrent. Combining Corollary 4.6, and a slight modification of Proposition 2.1 in [7], it follows that H is a closed set for the process $\{X(t)\}$: $(\forall x \in H)$, $P_x\big(X(t) \in H, \text{ for all } t \in \mathbb{R}_+\big) = 1$. Consequently, for all $A \in \mathcal{B}(H)^+$, $(\forall x \in H)$, $1 = L^G(x, A) \leq L^X(x, A)$ and by using Theorem 1.1 in [8] the process $\{X(t)\}$ restricted to H is Harris recurrent. Therefore, $\{X(t)\}$ is recurrent on E. The converse follows from Proposition 4.4, giving the result. ∎

The next corollary emphasizes a split with the previous equivalence results. Indeed it is shown that the process is positive recurrent if and only if $\{\Theta_n\}$ satisfies a weaker condition: recurrence and a technical condition for its unique (σ-finite) invariant measure.

Corollary 4.8: The PDMP $\{X(t)\}$ is positive recurrent if and only if the Markov chain $\{\Theta_n\}$ associated to G is recurrent with invariant measure π satisfying $\pi H(E) < \infty$.

Proof: The result easily follows from Remark 4.3 and Theorem 4.7. ■

We prove now that the Harris recurrent properties are equivalent for $\{X(t)\}$ and $\{\Theta_n\}$.

Theorem 4.9: The PDMP $\{X(t)\}$ is Harris recurrent if and only if the Markov chain $\{\Theta_n\}$ is Harris recurrent.

Proof: Suppose that the Markov chain $\{\Theta_n\}$ is Harris recurrent. Denote by Ψ^G a maximal irreducibility measure for the Markov chain $\{\Theta_n\}$. Then for any set $A \in \mathcal{B}(E)$ satisfying $\Psi^G(A) > 0$ it follows from Corollary 3.2 that $1 = L^G(x, A) \leq L^X(x, A)$ for all $x \in E$. Therefore, $\{X(t)\}$ is Harris recurrent by using Theorem 1.1 in [8].

Now assume that the PDMP $\{X(t)\}$ is Harris recurrent. From the equivalence results in [7], if the PDMP $\{X(t)\}$ is Harris recurrent then the Markov chain $\{\Upsilon_n\}$ associated to the resolvent R is Harris recurrent. Moreover, by using Proposition 4.1 $\{\Theta_n\}$ is irreducible. Let us denote by Ψ^G (respectively Ψ^R) a maximal irreducible measure for $\{\Theta_n\}$ (respectively $\{\Upsilon_n\}$). According to the definition of Harris recurrence (see [2, page 200]), we want to show that if $\Psi^G(A) > 0$ then $P_x\big(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \{\Theta_n \in A\}\big) = 1$ for all $x \in E$. From (ii) and (iii) of Proposition 5.5.5 in [2], it follows that there exists an increasing sequence of petite sets $\{C_k\}_{k\in\mathbb{N}}$ for $\{\Theta_n\}$ such that $E = \cup_{k\in\mathbb{N}} C_k$ with $\Psi^G(C_k) > 0$ and $\Psi^R(C_k) > 0$ for all $k \in \mathbb{N}$. Since $\{\Upsilon_n\}$ is Harris recurrent we have for all $k \in \mathbb{N}$ that $L^R(x, C_k) = 1$ for all $x \in C_k$. From Theorem 3.3 we have that $L^G(x, C_k) = 1$, for all $x \in C_k$, and from Proposition 9.1.1 in [2], it follows that C_k is Harris recurrent for $\{\Theta_n\}$. The remaining of the proof follows now the same steps as the end of the proof of Theorem 9.1.4 in [2], and it will be presented for the sake of completeness. From Lemma 5.5.1 in [2], we have that for all $A \in \mathcal{B}(E)$ with $\Psi^G(A) > 0$, there exists $\delta > 0$ such that $\inf_{x\in C_k} L^G(x, A) > \delta$. However C_k is Harris recurrent for $\{\Theta_n\}$ and from Theorem 9.1.3 (i) in [2], we have that for all $x \in C_k$, $P_x\big(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \{\Theta_n \in A\}\big) = 1$. The result follows after recalling that $E = \bigcup_{k\in\mathbb{N}} C_k$. ■

Remark 4.10: In the previous proof note that if A is a set such that $\psi^G(A) > 0$ then it does not necessarily imply that $\psi^R(A) > 0$. That is why we needed to proceed through the tool of petite sets.

As for the positive recurrence property, the following result points out the split with the previous theorem by showing that the positive Harris recurrence is equivalent to a weaker form of stability for the chain $\{\Theta_n\}$.

Corollary 4.11: The PDMP $\{X(t)\}$ is positive Harris recurrent if and only if the Markov chain $\{\Theta_n\}$ associated to G is Harris recurrent with invariant measure π satisfying $\pi H(E) < \infty$.

Proof: Combining Remark 4.3 and Theorem 4.9, we obtain the result. ■

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