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The value function of a finite fuel problem for a new class of singular stochastic controls

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Abstract— **We investigate, via the dynamic programming approach, a finite fuel nonlinear singular stochastic control problem of Bolza type. We prove that the associated value function is continuous and that its continuous extension to the closure of the domain coincides with the value function of a non singular control problem, for which we prove the existence of an optimal control. Moreover such a continuous extension is characterized as the unique viscosity solution of a quasi variational inequality with suitable boundary conditions of mixed type.**

I. INTRODUCTION

We study a finite fuel stochastic control problem with finite horizon via the dynamic programming approach. For any initial condition $(\bar{t}, \bar{k}, \bar{x}) \in [0, T[\times[0, K] \times \mathbf{R}^n$ we consider the nonlinear stochastic differential equation

$$x_t = \bar{x} + \int_{\bar{t}}^t A(r, x_r)\, dr + \int_{\bar{t}}^t B(r, x_r) u_r\, dr + \int_{\bar{t}}^t D(r, x_r)\, d\mathcal{W}_r, \tag{1}$$

where the functions A, B, and D are deterministic, $\{\mathcal{W}_t\}$ is a Brownian motion (not necessarily n–dimensional), and $\{u_t\}$ is a control. All the processes are assumed to be defined on a probability space $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$. Given a closed convex cone $\mathcal{K} \subset \mathbf{R}^m$, the class of admissible controls, denoted by $\mathcal{C}(\bar{t}, \bar{k}, \bar{x})$, is given by the set of $\mathcal{K}$–valued, $\{\mathcal{G}_t\}$–predictable processes verifying the constraint

$$k_T \doteq \bar{k} + \int_{\bar{t}}^T |u_t|\, dt \leq K, \quad Q-\text{a.s.}. \tag{2}$$

For any admissible control u we consider a cost of the form

$$\mathcal{J}(\bar{t}, \bar{k}, \bar{x}, u) = E_Q \left[\int_{\bar{t}}^T \left(l_0(r, x_r) + \langle l_1(r, x_r), u_r\rangle\right)\, dr + g(x_T)\right], \tag{3}$$

where l_0, l_1, and g are deterministic functions. The value function is defined as

$$\mathcal{V}(\bar{t}, \bar{k}, \bar{x}) = \inf_{u \in \mathcal{C}(\bar{t}, \bar{k}, \bar{x})} \mathcal{J}(\bar{t}, \bar{k}, \bar{x}, u). \tag{4}$$

In the paper we prove the continuity of the value function, and, via a dynamic programming principle, we show that the function V, which is the continuous extension of $\mathcal{V}$ to $[0, T] \times [0, K] \times \mathbf{R}^n$, is a viscosity solution of the following generalized Cauchy problem

$$\max\left\{-\frac{\partial v}{\partial t} + F(t, x, Dv, D^2 v), -\frac{\partial v}{\partial k} + H(t, x, Dv)\right\} = 0 \quad \text{in }]0, T[\times]0, K[\times\mathbf{R}^n, \tag{5}$$

$$\max\left\{-\frac{\partial v}{\partial t} + F(t, x, Dv, D^2 v), -\frac{\partial v}{\partial k} + H(t, x, Dv)\right\} \geq 0 \quad \text{on }]0, T[\times\{K\} \times \mathbf{R}^n, \tag{6}$$

$$v \leq g \text{ and if } v < g \text{ then } \max\left\{-\frac{\partial v}{\partial t} + F(t, x, Dv, D^2 v), -\frac{\partial v}{\partial k} + H(t, x, Dv)\right\} \geq 0 \quad \text{on } \{T\}\times]0, K] \times \mathbf{R}^n, \tag{7}$$

where Dv and $D^2 v$ denote the gradient and the matrix of the second derivatives of the function $v = v(t, k, x)$ with respect to the x variable,

$$F(t, x, p, S) \doteq -\langle A(t, x), p\rangle - l_0(t, x) - \frac{1}{2}\mathrm{Tr}\{\tilde{D}(t, x) S\},$$

where $\tilde{D}(t, x) \doteq D(t, x) D(t, x)^T$, and

$$H(t, x, p) \doteq \max_{w \in \mathcal{K},\ |w|=1} \{-\langle B(t, x) w, p\rangle - \langle l_1(t, x), w\rangle\}$$

for any $(t, x, p, S) \in \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{M}(n, n)$, where $\mathbf{M}(n, n)$ denotes the set of $n \times n$ real matrices. A uniqueness theorem proven in [MS2] allows us to characterize V as the unique viscosity solution to the above boundary value problem. V is in fact the value function of a non singular problem for which we can also prove the existence of an optimal control. Justified by the observation that optimal controls for the problem (1)-(4) may not exist and in fact quasi-optimal controls may be as close as desired to a control of impulsive type (see e.g. [FS]), we introduce an extension of our problem by considering a new set of controls, called auxiliary controls, which are bounded valued. We can show that our problem is equivalent to a problem of optimal stopping time within the class of auxiliary controls. Using a similar auxiliary control problem Dufour and Miller in [DM] proved the existence of an auxiliary optimal control for a Mayer problem with a dynamics like (1). From our point of view, that is of the dynamic programming approach, the equivalent problem is not standard and can be studied only applying an abstract version of the dynamic programming principle as introduced by Haussmann and Lepeltier [HL], which is based on the compactification method due to El Karoui et al. [EKNP].

We observe that the boundary conditions (6) and (7) are original in the setting of singular stochastic control problems. First of all, since we deal with problems of impulsive type, even considering a finite horizon problem, the limit

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$\lim_{\bar{t}\to T^-} \mathcal{V}(\bar{t},\bar{k},\bar{x})$ does not coincide in general with the final cost $g(\bar{x})$ and therefore, at time $\bar{t} = T$, we impose (7) which is an alternative between the quasi variational inequality (5) and $v = g$. At the boundary $\bar{k} = K$, instead, we introduce the supersolution condition (6) which replaces the Dirichlet condition $v(\bar{t},K,\bar{x}) = J(\bar{t},K,\bar{x},0)$, usually assumed in finite fuel control problems (see e.g. [FS]). It has the advantage that it does not require the computation of $J(\bar{t},K,\bar{x},0)$. Supersolution type conditions have been first considered by [S] for problems with state constraints, and in fact by considering k_t, the fuel consumed at time t, as a new variable, in view of (2) such a variable turns out to be constrained in $[0,K]$. Boundary value problems similar to (5)–(7) for first order Hamiltonians were already investigated by the authors in the context of impulsive deterministic control problems and in such a context, it is worth mentioning that our approach leads to approximation schemes for the numerical evaluation of the value functions which for the second order case has not yet been done.

Finally we refer to [MS1] for the proofs of the theorems that are not proved here and for a more complete bibliography.

II. The data, the auxiliary control problem

Throughout the paper we will use the notation $\mathbf{R}_+ = [0,+\infty[$ and the following hypotheses

(A0) There are some constants L_1, L_2 such that the deterministic functions $A : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$, $B : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{M}(n,m)$, and $D : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{M}(n,p)$ verify for all $t,s \in \mathbf{R}_+$ and $x,y \in \mathbf{R}^n$

$$\begin{aligned}&|A(t,x)| + |B(t,x)| + |D(t,x)| \le L_1(1+|x|),\\ &|A(t,x)-A(s,y)| + |B(t,x)-B(s,y)| +\\ &|D(t,x)-D(s,y)| \le L_2(|t-s|+|x-y|).\end{aligned}$$

(A1) There are some constants L, L_3 such that the functions $l_0 : \mathbf{R}_+\times\mathbf{R}^n \to \mathbf{R}$, $l_1 : \mathbf{R}_+\times\mathbf{R}^n \to \mathbf{R}^m$, and $g : \mathbf{R}^n \to \mathbf{R}$ verify for all $t,s \in \mathbf{R}_+$ and $x,y \in \mathbf{R}^n$

$$\begin{aligned}&|l_0(t,x)-l_0(s,y)| + |l_1(t,x)-l_1(s,y)| \le\\ &L(|t-s|+|x-y|),\\ &|g(x)-g(y)| \le L|x-y|;\\ &|l_0(t,x)| + |l_1(t,x)| + |g(x)| \le L_3. \qquad (8)\end{aligned}$$

Remark 1: If we replace the boundedness hypothesis (8) with

$$|l_0(t,x)|+|l_1(t,x)|+|g(x)| \le L_3(1+|x|)\ \forall t \in \mathbf{R}_+,\ x \in \mathbf{R}^n, \qquad (9)$$

the main results of the paper remain true, except that, of course, the value function $\mathcal{V}$ is no more bounded but it turns out to verify $|\mathcal{V}(t,k,x)| \le \bar{C}(1+|x|)$ for some $\bar{C}$ and $\forall(t,k,x) \in [0,T[\times[0,K]\times\mathbf{R}^n$.

A. The auxiliary control problem and the equivalence of the two problems

Definition 1 (Auxiliary control problem): For any $(\bar{t},\bar{k},\bar{x}) \in [0,T]\times[0,K]\times\mathbf{R}^n$ an *auxiliary control* is a term

$$\beta = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{w_s\},\{(t_s,k_s,\xi_s)\},\theta),$$

where the following **(B1)** and **(B2)** are assumed.

(B1) (Ω,F,P) is a complete probability space, with a right continuous complete filtration $\{F_s\}$,
$\{w_s\}$ is a $\overline{B}_m(1)\cap\mathcal{K}$–valued control defined on $[0,T+K]\times\Omega$ which is $\{F_s\}$–predictable,
θ is an $\{F_s\}$–stopping time such that $\theta \le T+K$,

and

(B2) $\{(t_s,k_s,\xi_s)\}$ is an $\mathbf{R}^{2+n}$–valued $\{F_s\}$–progressively measurable process with continuous paths, such that, for $0 \le s \le T+K$,

$$\begin{cases} t_s = \bar{t} + \int_0^s w_{0_\sigma}\,d\sigma\\ k_s = \bar{k} + \int_0^s |w_\sigma|\,d\sigma\\ \xi_s = \bar{x} + \int_0^s (A(t_\sigma,\xi_\sigma)w_{0_\sigma} + B(t_\sigma,\xi_\sigma)w_\sigma)\,d\sigma\\ \quad + \int_0^s D(t_\sigma,\xi_\sigma)\sqrt{w_{0_\sigma}}\,dW_\sigma,\end{cases}$$

where $\{W_s\}$ is a standard p–dimensional $\{F_s\}$–Brownian motion defined on $[0,T+K]\times\Omega$ and where we set $w_{0_s}(\omega) \doteq 1-|w_s(\omega)|\ \ \forall(s,\omega)$ just for the sake of notation.

The cost corresponding to an auxiliary control β is of the form

$$\begin{aligned}J(\bar{t},\bar{k},\bar{x},\beta) \doteq E_P\Big[&\int_0^\theta (l_0(t_\sigma,\xi_\sigma)w_{0_\sigma} + \langle l_1(t_\sigma,\xi_\sigma),w_\sigma\rangle)\,d\sigma\\ &+g(\xi_\theta) + G(t_\theta,k_\theta)\Big],\end{aligned}$$

where $G(T,k) = 0$ for all $k \le K$ and $G(t,k) = +\infty$ otherwise. We use $\Gamma(\bar{t},\bar{k},\bar{x})$ to denote the set of auxiliary controls, while

$$\Gamma^a(\bar{t},\bar{k},\bar{x}) \doteq \left\{\beta \in \Gamma(\bar{t},\bar{k},\bar{x}) : \quad J(\bar{t},\bar{k},\bar{x},\beta) < +\infty\right\} \qquad (10)$$

denotes the subset of admissible auxiliary controls. We define for every $(\bar{t},\bar{k},\bar{x}) \in [0,T]\times[0,K]\times\mathbf{R}^n$ the auxiliary value function as

$$V(\bar{t},\bar{k},\bar{x}) \doteq \inf_{\beta\in\Gamma^a(\bar{t},\bar{k},\bar{x})} J(\bar{t},\bar{k},\bar{x},\beta). \qquad (11)$$

The original problem and the one formulated in the above definition are equivalent in the following sense

Theorem 1 (Equivalence): Assume (A0), (A1). Then for any initial condition $(\bar{t},\bar{k},\bar{x}) \in [0,T[\times[0,K]\times\mathbf{R}^n$ one has

i) $\mathcal{C}(\bar{t},\bar{k},\bar{x}) \hookrightarrow \Gamma^a(\bar{t},\bar{k},\bar{x})$, that is, for every control $c \in \mathcal{C}(\bar{t},\bar{k},\bar{x})$ there exists an admissible auxiliary control $\beta \in \Gamma^a(\bar{t},\bar{k},\bar{x})$ such that $J(\bar{t},\bar{k},\bar{x},\beta) = \mathcal{J}(\bar{t},\bar{k},\bar{x},c)$;

ii) for any admissible auxiliary control $\beta \in \Gamma^a(\bar{t},\bar{k},\bar{x})$ there is a sequence of controls $c^n \in \mathcal{C}(\bar{t},\bar{k},\bar{x})$ such that $\lim_n \mathcal{J}(\bar{t},\bar{k},\bar{x},c^n) = J(\bar{t},\bar{k},\bar{x},\beta)$;

iii)
$$V(\bar{t},\bar{k},\bar{x}) = \mathcal{V}(\bar{t},\bar{k},\bar{x}). \qquad (12)$$

B. Existence of an auxiliary optimal control

We recall that in a relexed control the $\overline{B}_m(1)\cap\mathcal{K}$–valued process $\{w_s\}$ is replaced by an $\mathbf{M}_1(\overline{B}_m(1)\cap\mathcal{K})$–valued process $\{\mu_s\}$, where $\mathbf{M}_1(\overline{B}_m(1)\cap\mathcal{K})$ is the space of probability measures on $\overline{B}_m(1)\cap\mathcal{K}$. We will extend any bounded measurable map $\psi : \overline{B}_m(1)\cap\mathcal{K} \to \mathbf{R}$ to $\mathbf{M}_1(\overline{B}_m(1)\cap\mathcal{K})$ by setting

$$\psi(\mu) = \int_{\overline{B}_m(1)\cap\mathcal{K}} \psi(w)\mu(dw).$$

For a detailed definition we refer to the Appendix. For each $(\bar t,\bar k,\bar x)$ the set of relaxed controls will be denoted by $\Gamma(\bar t,\bar k,\bar x)$ and $\forall\tilde\alpha\in\tilde\Gamma(\bar t,\bar k,\bar x)$ we define the cost

$$J(\bar t,\bar k,\bar x,\tilde\alpha)=E_P\Big[\int_0^\theta (l_0(t_\sigma,\xi_\sigma)(1-|\mu_\sigma|)+ \langle l_1(t_\sigma,\xi_\sigma),\mu_\sigma\rangle)\,d\sigma+g(\xi_\theta)+G(t_\theta,k_\theta)\Big]. \tag{13}$$

We use $\tilde\Gamma^a(\bar t,\bar k,\bar x)$ to denote the subset of admissible relaxed controls, that is, with finite cost.

The set $\Gamma^a(\bar t,\bar k,\bar x)$ can be naturally embedded in $\tilde\Gamma^a(\bar t,\bar k,\bar x)$, therefore the inequality

$$\inf_{\tilde\alpha\in\tilde\Gamma^a(\bar t,\bar k,\bar x)} J(\bar t,\bar k,\bar x,\tilde\alpha)\le \inf_{\alpha\in\Gamma^a(\bar t,\bar k,\bar x)} J(\bar t,\bar k,\bar x,\alpha),$$

is trivially verified. In fact, also the converse inequality holds true.

Theorem 2 (Existence): Assume (A0), (A1). Then for any $(\bar t,\bar k,\bar x)\in[0,T]\times[0,K]\times\mathbf{R}^n$,

$$V(\bar t,\bar k,\bar x)=\inf_{\alpha\in\Gamma^a(\bar t,\bar k,\bar x)} J(\bar t,\bar k,\bar x,\alpha)=\inf_{\tilde\alpha\in\tilde\Gamma^a(\bar t,\bar k,\bar x)} J(\bar t,\bar k,\bar x,\tilde\alpha).$$

Moreover, the infimum over relaxed controls is attained and so is the infimum over auxiliary controls.

C. Dynamic Programming Principle DPP

In order to state a dynamic programming principle for our problem we need for each $(\bar t,\bar k,\bar x)\in[0,T]\times[0,K]\times\mathbf{R}^n$ the notion of control rule R which will be defined in the Appendix. For each $(\bar t,\bar k,\bar x)$ we will denote by $\mathcal{R}(\bar t,\bar k,\bar x)$ the space of control rules and by $\mathcal{R}^a(\bar t,\bar k,\bar x)$ the subset for which the cost is finite.

Let us notice that the auxiliary control problem is in fact an *unconstrained* stopping time control problem. Indeed, from Definition 1 it follows that for all $(\bar t,\bar k,\bar x)$ such that either $\bar t>T$ or $\bar k>K$, the set of admissible auxiliary controls $\Gamma^a(\bar t,\bar k,\bar x)$ is empty. Hence the auxiliary value function V might be extended to the whole set $[0,+\infty[\times[0,+\infty[\times\mathbf{R}^n$ in a natural way by setting $V=+\infty$ outside $[0,T]\times[0,K]\times\mathbf{R}^n$.

Proposition 1 (DPP): Assume (A0), (A1). For any $(\bar t,\bar k,\bar x)\in[0,T]\times[0,K]\times\mathbf{R}^n$, one has

$$V(\bar t,\bar k,\bar x)=\inf\Big\{E_R\Big[\int_0^{\rho'}(l_0(t_\sigma,\xi_\sigma)(1-|\mu_\sigma|)+ \langle l_1(t_\sigma,\xi_\sigma),\mu_\sigma\rangle)\,d\sigma+V(t_{\rho'},k_{\rho'},\xi_{\rho'})\Big]\Big\}, \tag{14}$$

where the infimum is taken over the set $\mathcal{R}^a(\bar t,\bar k,\bar x)$ and $\rho'=\rho\wedge\theta$, ρ being any finite stopping time such that $0\le\rho\le\theta$.

III. Continuity of the value function

Theorem 3: Let (A0), (A1) hold. Then the value function V is bounded and continuous. More precisely, there exists some $\bar C>0$ such that V satisfies the following:

$$|V(\bar t,\bar k,\bar x)|\le\bar C \qquad \forall(\bar t,\bar k,\bar x)\in[0,T]\times[0,K]\times\mathbf{R}^n;$$

$$|V(\bar t_1,\bar k_1,\bar x_1)-V(\bar t_2,\bar k_2,\bar x_2)|\le \bar C\left[|\bar x_1-\bar x_2|+(1+|\bar x_1|\vee|\bar x_2|)\left(|\bar t_1-\bar t_2|^{1/2}+|\bar k_1-\bar k_2|\right)\right]$$

for all $(\bar t_1,\bar k_1,\bar x_1)$, $(\bar t_2,\bar k_2,\bar x_2)\in[0,T]\times[0,K]\times\mathbf{R}^n$.

Proof: Boundedness. It is very easy to see that for any initial condition $(\bar t,\bar k,\bar x)\in[0,T]\times[0,K]\times\mathbf{R}^n$ the set of admissible control rules is non empty. Since the stopping time θ is bounded from above by $T+K$, the boundedness of V follows therefore straightforwardly from the boundedness of both the process $\{w_s\}$ and the data l_0, l_1, and g.

Lipschitz continuity in x. Fix $(\bar t,\bar k,\bar x_1),(\bar t,\bar k,\bar x_2)\in[0,T]\times[0,K]\times\mathbf{R}^n$ and assume that $V(\bar t,\bar k,\bar x_1)\ge V(\bar t,\bar k,\bar x_2)$. One has

$$0\le V(\bar t,\bar k,\bar x_1)-V(\bar t,\bar k,\bar x_2)\le \sup_{P\in\mathcal{R}^a(\bar t,\bar k,\bar x_2)}\left(J(\bar t,\bar k,\bar x_1,Q)-J(\bar t,\bar k,\bar x_2,P)\right)$$

for every $Q\in\mathcal{R}^a(\bar t,\bar k,\bar x_1)$. Take $P\in\mathcal{R}^a(\bar t,\bar k,\bar x_2)$ arbitrary and let $(\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{\mu_s\},\{(t_s,k_s,\xi_{2_s})\},\theta)$ be the associated relaxed control. By the definition of control rules, there exists an extension $(\tilde\Omega,\tilde{\mathcal{F}},\tilde P,\{\tilde{\mathcal{F}}_s\})$ of $(\Omega,\mathcal{F},P,\{\mathcal{F}_s\})$, i.e. there exists another probability space $(\Omega',\mathcal{F}',\mathcal{F}'_s,P')$ such that $\tilde\Omega=\Omega\times\Omega'$, $\tilde{\mathcal{F}}=\mathcal{F}\times\mathcal{F}'$, $\tilde{\mathcal{F}}_s=\mathcal{F}_s\times\mathcal{F}'_s$ and $\tilde P=P\times P'$. We can extend the process $\{((t_\cdot,k_\cdot,\xi_\cdot),\mu_\cdot,\theta)\}$ to $\tilde\Omega$ by the following: for $\tilde\omega=(\omega,\omega')\in\tilde\Omega$,

$$((t_\cdot,k_\cdot,\xi_\cdot),\mu_\cdot,\theta)(\tilde\omega)=((t_\cdot,k_\cdot,\xi_\cdot),\mu_\cdot,\theta)(\omega).$$

On $(\tilde\Omega,\tilde{\mathcal{F}},\tilde P,\tilde{\mathcal{F}}_s)$ there exists a standard p–dimensional Brownian motion $\{W_s\}$ such that for $s\in[0,T+K]$,

$$\begin{aligned}(t_s,k_s,\xi_{2s})=\big(&\bar t+\textstyle\int_0^s(1-|\mu|_\sigma)\,d\sigma,\bar k+\int_0^s|\mu_\sigma|\,d\sigma,\\ &\bar x_2+\textstyle\int_0^s(A(t_\sigma,\xi_{2\sigma})(1-|\mu|_\sigma)+B(t_\sigma,\xi_{2\sigma})\mu_\sigma)\,d\sigma\\ &+\textstyle\int_0^s D(t_\sigma,\xi_{2\sigma})\sqrt{1-|\mu|_\sigma}\,dW_\sigma\big),\end{aligned}$$

the control $\tilde\beta=(\tilde\Omega,\tilde{\mathcal{F}},\tilde P,\{\tilde{\mathcal{F}}_s\},\{\mu_s\},\{(t_s,k_s,\xi_{2s})\},\theta)\in\tilde\Gamma^a(\bar t,\bar k,\bar x_2)$, where, by the definition of the set $\mathcal{Z}$ in (25), θ is the first time in which $\chi_{s\ge\theta}$ jumps from 0 to 1 and $J(\bar t,\bar k,\bar x_2,\tilde\beta)=J(\bar t,\bar k,\bar x_2,\tilde P)=J(\bar t,\bar k,\bar x_2,P)$.

Consider the equations with the initial condition $(\bar t,\bar k,\bar x_1)$, for $s\in[0,T+K]$,

$$\begin{aligned}(t_s,k_s,\xi_{1s})=\big(&\bar t+\textstyle\int_0^s(1-|\mu|_\sigma)\,d\sigma,\bar k+\int_0^s|\mu_\sigma|\,d\sigma,\\ \xi_{1s}&=\bar x_1+\textstyle\int_0^s(A(t_\sigma,\xi_{1\sigma})(1-|\mu|_\sigma)+B(t_\sigma,\xi_{1\sigma})\mu_\sigma)\,d\sigma+\\ &\textstyle\int_0^s D(t_\sigma,\xi_{1\sigma})\sqrt{1-|\mu|_\sigma}\,dW_\sigma\big)\end{aligned} \tag{15}$$

on the stochastic basis $(\tilde\Omega,\tilde{\mathcal{F}},\tilde P,\{\tilde{\mathcal{F}}_s\})$. Under assumptions (A0), (A1), the strong solution to (15) exists and one can see that $\tilde\alpha=(\tilde\Omega,\tilde{\mathcal{F}},\tilde P,\{\tilde{\mathcal{F}}_s\},\{\mu_s\},\{(t_s,k_s,\xi_{1s})\},\theta)\in\tilde\Gamma^a(\bar t,\bar k,\bar x_1)$. Therefore there exists a control rule $Q\in\mathcal{R}^a(\bar t,\bar k,\bar x_1)$ such that

$$J(\bar t,\bar k,\bar x_1,\tilde\alpha)=J(\bar t,\bar k,\bar x_1,Q).$$

We have

$$\begin{aligned}&J(\bar t,\bar k,\bar x_1,Q)-J(\bar t,\bar k,\bar x_2,P)\le\\ &E_{\tilde P}\Big[\int_0^\theta|l_0(t_\sigma,\xi_{1\sigma})-l_0(t_\sigma,\xi_{2\sigma})|\,|1-|\mu_\sigma||\,d\sigma+\\ &\int_0^\theta|l_1(t_\sigma,\xi_{1\sigma})-l_1(t_\sigma,\xi_{2\sigma})||\mu_\sigma|\,d\sigma+g(\xi_{1\theta})-g(\xi_{2\theta})|\Big]\le\\ &L\,E_{\tilde P}\Big[\int_0^\theta|\xi_{1\sigma}-\xi_{2\sigma}|\,d\sigma\Big]+L\,E_{\tilde P}[|\xi_{1\theta}-\xi_{2\theta}|]\end{aligned}$$

where we have used the Lipschitz continuity of l_0, l_1, and g and L is the same as in (A1). Let us define $\hat\xi_{i_s}\doteq\xi_{i_{s\wedge\theta}}$ for all $s\ge0$ and $i=1,2$. By the Burkholder–Gundy's and Gronwall's inequalities we obtain that there exists a constant

C, depending on the Lipschitz constant L_2 in (A0) and on $T+K$, such that, for all $0 \le \sigma \le T+K$,

$$E_{\tilde P}\left[\sup_{s\le\sigma}(|\hat\xi_{1_s}-\hat\xi_{2_s}|^2)\right] \le C|\bar x_1-\bar x_2|^2.$$

Since from the definitions of $\{\hat\xi_{1_s}\}$ and $\{\hat\xi_{2_s}\}$ it follows that

$$E_{\tilde P}\left[\int_0^\theta |\xi_{1\sigma}-\xi_{2\sigma}|\,d\sigma\right] \le E_{\tilde P}\left[\int_0^{T+K} |\hat\xi_{1_\sigma}-\hat\xi_{2\sigma}|\,d\sigma\right] \le \left(\int_0^{T+K} E_{\tilde P}\left[\sup_{s\le\sigma}(|\hat\xi_{1_s}-\hat\xi_{2_s}|^2)\right]\,d\sigma\right)^{1/2}, \tag{16}$$

in view of the arbitrariness of $P\in\mathcal R^a(\bar t,\bar k,\bar x_2)$, the previous estimates yield that

$$0 \le V(\bar t,\bar k,\bar x_1)-V(\bar t,\bar k,\bar x_2) \le \bar C|\bar x_1-\bar x_2|$$

for a suitable constant $\bar C$, depending just on L, L_2, and $T+K$.

Hölder continuity in t. Fix $(\bar t_1,\bar k,\bar x)$, $(\bar t_2,\bar k,\bar x)\in[0,T]\times[0,K]\times\mathbf R^n$ and assume that $V(\bar t_1,\bar k,\bar x)\ge V(\bar t_2,\bar k,\bar x)$.

First case, $\bar t_1<\bar t_2$. One has

$$0\le V(\bar t_1,\bar k,\bar x)-V(\bar t_2,\bar k,\bar x)\le \sup_{P\in\mathcal R^a(\bar t_2,\bar k,\bar x)}\left(J(\bar t_1,\bar k,\bar x,Q)-J(\bar t_2,\bar k,\bar x,P)\right)$$

for every $Q\in\mathcal R^a(\bar t_1,\bar k,\bar x)$. Take $P\in\mathcal R^a(\bar t_2,\bar k,\bar x)$ arbitrary and let $(\Omega,\mathcal F,P,\{\mathcal F_s\},\{\mu_s\},\{(t_{2_s},k_s,\xi_{2_s})\},\theta_2)$ be the associated relaxed control. Now, as in the previous step, there exist an extension $(\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\})$ of $(\Omega,\mathcal F,P,\{\mathcal F_s\})$, and a standard Brownian motion $\{W_s\}$ on $(\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\})$ such that, for $s\in[0,T+K]$,

$$(t_{2s},k_s,\xi_{2s})=\Big(\bar t_2+\int_0^s(1-|\mu_\sigma|)\,d\sigma,\ \bar k+\int_0^s|\mu_\sigma|\,d\sigma,\ \bar x+\int_0^s\left(A(t_{2_\sigma},\xi_{2_\sigma})(1-|\mu_\sigma|)+B(t_{2_\sigma},\xi_{2_\sigma})\mu_\sigma\right)d\sigma+\int_0^s D(t_{2_\sigma},\xi_{2_\sigma})\sqrt{1-|\mu_\sigma|}\,dW_\sigma\Big),$$

the control $\tilde\beta=(\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\},\{\mu_s\},\{(t_{2_s},k_s,\xi_{2_s})\},\theta_2)\in\tilde\Gamma^a(\bar t_2,\bar k,\bar x)$, and $J(\bar t_2,\bar k,\bar x,\tilde\beta)=J(\bar t_2,\bar k,\bar x,P)$. Let us now consider the relaxed control that one obtains from the definition of $\tilde\beta$ when μ_s is replaced by $\mu_s\chi_{\{s\le\theta_2\}}$ for $s\ge0$. It is easy to see that this control belongs to $\tilde\Gamma^a(\bar t_2,\bar k,\bar x)$, and that the corresponding cost coincides with $J(\bar t_2,\bar k,\bar x,P)$. With a small abuse of notation, from now on let us use $\tilde\beta$ to denote such control.

Let us introduce the stopping time $\theta_1\doteq\theta_2+(\bar t_2-\bar t_1)$ and let $\{(t_{1_s},k_s,\xi_{1_s})\}$ be the strong solution to

$$(t_{1_s},k_s,\xi_{1_s})=\Big(\bar t_1+\int_0^s(1-|\mu_\sigma|)\,d\sigma,\ \bar k+\int_0^s|\mu_\sigma|\,d\sigma,\ \bar x+\int_0^s\left(A(t_{1_\sigma},\xi_{1_\sigma})(1-|\mu_\sigma|)+B(t_{1_\sigma},\xi_{1_\sigma})\mu_\sigma\right)d\sigma+\int_0^s D(t_{1_\sigma},\xi_{1_\sigma})\sqrt{1-|\mu_\sigma|}\,dW_\sigma\Big)$$

on the stochastic basis $(\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\})$ for $s\in[0,T+K]$. $\tilde\beta$ admissible implies that $\theta_2\le(T-\bar t_2)+(K-\bar k)$, $t_{2_{\theta_2}}=T$, and $k_{\theta_2}\le K$ (see Remark 2.2 in [MS1]). Hence, one deduces

$$\theta_1\le(T-\bar t_2)+(K-\bar k)+(\bar t_2-\bar t_1)\le T+K.$$

Moreover, since we identified μ_s with $\mu_s\chi_{\{s\le\theta_2\}}$ one has

$$t_{1_{\theta_1}}=t_{2_{\theta_2}}+(\theta_1-\theta_2)-(\bar t_2-\bar t_1),\qquad k_{\theta_1}=k_{\theta_2}\le K.$$

Therefore $t_{1_{\theta_1}}=T$, $k_{\theta_1}\le K$, and the control $\tilde\alpha=(\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\},\{\mu_s\},\{(t_{1_s},k_s,\xi_{1_s})\},\theta_1)$ is in $\tilde\Gamma^a(\bar t_1,\bar k,\bar x)$. Thus there exists a control rule $Q\in\mathcal R^a(\bar t_1,\bar k,\bar x)$ such that $J(\bar t_1,\bar k,\bar x,\tilde\alpha)=J(\bar t_1,\bar k,\bar x,Q)$. By some calculation we can deduce that

$$J(\bar t_1,\bar k,\bar x,Q)-J(\bar t_2,\bar k,\bar x,P)\le L\left[E_{\tilde P}[|\xi_{1_{\theta_1}}-\xi_{2_{\theta_2}}|^2]\right]^{\frac12}+ LE_{\tilde P}\left[\int_0^{\theta_2}\left(|t_{1_\sigma}-t_{2_\sigma}|+|\xi_{1_\sigma}-\xi_{2_\sigma}|\right)d\sigma\right]+L_3(\bar t_2-\bar t_1).$$

In order to conclude the proof, let us introduce for $s\ge0$ the processes $\hat\xi_{i_s}\doteq\xi_{i_{s\wedge\theta_i}}$, for $i=1,2$. Since

$$t_{1_{\theta_2}}=t_{2_{\theta_2}}-(\bar t_2-\bar t_1),$$

one can prove that

$$E_{\tilde P}\left[\sup_{s\le\sigma}|\hat\xi_{2_s}-\hat\xi_{1_s}|^2\right]\le C^2(1+|\bar x|)^2|\bar t_2-\bar t_1|,$$

for every $0\le\sigma\le T+K$, with C a suitable constant depending on L_1, L_2 in (A0) and $T+K$ which yields

$$E_{\tilde P}\left[\left|\xi_{2_{\theta_2}}-\xi_{1_{\theta_1}}\right|^2\right]= E_{\tilde P}\left[\left|\hat\xi_{2_{T+K}}-\hat\xi_{1_{T+K}}\right|^2\right]\le C^2(1+|\bar x|)^2|\bar t_2-\bar t_1|,$$

Therefore, by (16) we obtain $J(\bar t_1,\bar k,\bar x,Q)-J(\bar t_2,\bar k,\bar x,P)\le \bar C\left[(1+|\bar x|)|\bar t_1-\bar t_2|^{\frac12}+|\bar t_1-\bar t_2|\right]$, which, by the arbitrariness of P, yields

$$0\le V(\bar t_2,\bar k,\bar x)-V(\bar t_1,\bar k,\bar x)\le\bar C(1+|\bar x|)|\bar t_1-\bar t_2|^{\frac12},$$

for some constant $\bar C$ depending on the constants L, L_2, L_3, and $T+K$ in (A0), (A1).

Second case, $\bar t_1>\bar t_2$. Consider the Dynamic Programming Principle (14) for $V(\bar t_2,\bar k,\bar x)$,

$$V(\bar t_2,\bar k,\bar x)=\inf_{R\in\mathcal R^a(\bar t_2,\bar k,\bar x)}\left\{E_R\left[\int_0^{r\wedge\theta}\left(l_0(t_\sigma,\xi_\sigma)(1-|\mu_\sigma|)+\langle l_1(t_\sigma,\xi_\sigma),\mu_\sigma\rangle\right)d\sigma+V(t_{r\wedge\theta},k_{r\wedge\theta},\xi_{r\wedge\theta})\right]\right\},$$

where we choose the (deterministic) time $r=\bar t_1-\bar t_2$. It is easy to see that there exists an admissible control rule $P\in\mathcal R^a(\bar t_2,\bar k,\bar x)$ associated to a relaxed control $(\Omega,\mathcal F,P,\{\mathcal F_s\},\{\mu_s\},\{(t_s,k_s,\xi_s)\},\theta)$ and such that $P(\mu_s=\delta_{\{0\}}\quad 0\le s\le\theta,\quad \theta=T-\bar t_2)=1$. Then $P(\theta\ge r)=1$; by the boundedness of l_0 one has

$$V(\bar t_2,\bar k,\bar x)\le E_P\left[\int_0^r|l_0(t_\sigma,\xi_\sigma)|\,d\sigma+V(t_r,k_r,\xi_r)\right]\le L_3r+E_P[V(t_r,k_r,\xi_r)],$$

and by the Lipschitz continuity of the value function in x,

$$V(t_r,k_r,\xi_r)\le V(t_r,k_r,\bar x)+C|\xi_r-\bar x|.$$

Hence

$$V(\bar t_2,\bar k,\bar x)-E_P[V(t_r,k_r,\bar x)]\le L_3r+CE_P[|\xi_r-\bar x|]\le L_3r+C(E_P[|\xi_r-\bar x|^2])^{\frac12}. \tag{17}$$

From the definition of control rules, we know that under P,

$$(t_r,k_r,\xi_r)=(\bar t_2+r=\bar t_1,\ \bar k,\ \bar x+\int_0^r A(t_\sigma,\xi_\sigma)\,d\sigma+M_r) \tag{18}$$

where $\{M_r\}$ is a continuous square integrable martingale with $\langle M\rangle_r = \int_0^r \mathrm{Tr}\{\tilde D(t_\sigma,\xi_\sigma)\}\,d\sigma$. Therefore by the Burkholder-Davis-Gundy inequality there exists a constant C, depending on L_1 in (A0), such that

$$E_P[|\xi_r-\bar x|^2] \le C^2(1+|\bar x|)^2(r^2+r). \tag{19}$$

Therefore (17), (18) and (19) yield

$$0 \le V(\bar t_2,\bar k,\bar x) - V(\bar t_1,\bar k,\bar x) \le \bar C(1+|\bar x|)|\bar t_2-\bar t_1|^{\frac12}$$

for some constant $\bar C$ depending on the constants introduced in (A0), (A1).

Lipschitz continuity in k. Fix $(\bar t,\bar k_1,\bar x)$, $(\bar t,\bar k_2,\bar x) \in [0,T]\times[0,K]\times\mathbf{R}^n$, and assume that $V(\bar t,\bar k_1,\bar x) \ge V(\bar t,\bar k_2,\bar x)$.

First case, $\bar k_1 < \bar k_2$. One has

$$0 \le V(\bar t,\bar k_1,\bar x) - V(\bar t,\bar k_2,\bar x) \le \\ \sup_{P\in\mathcal{R}^a(\bar t,\bar k_2,\bar x)} \big(J(\bar t,\bar k_1,\bar x,Q) - J(\bar t,\bar k_2,\bar x,P)\big)$$

for every $Q\in\mathcal{R}^a(\bar t,\bar k_1,\bar x)$. As in the previous step, take $P\in\mathcal{R}^a(\bar t,\bar k_2,\bar x)$ arbitrary and let $\{W_s\}$ be a standard Brownian motion on a suitable $(\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\})$ such that

$$\tilde\beta = (\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\},\{\mu_s\},\{(t_s,k_{2_s},\xi_s)\},\theta_2) \in \tilde\Gamma^a(\bar t,\bar k_2,\bar x)$$

is a relaxed control, where, for $s\in[0,T+K]$,

$$(t_s,k_{2_s},\xi_s) = (\bar t+\textstyle\int_0^s(1-|\mu_\sigma|)\,d\sigma, \bar k_2+\int_0^s|\mu_\sigma|\,d\sigma, \\ \bar x+\int_0^s\big(A(t_\sigma,\xi_\sigma)(1-|\mu_\sigma|)+B(t_\sigma,\xi_\sigma)\mu_\sigma\big)\,d\sigma+ \\ \int_0^s D(t_\sigma,\xi_\sigma)\sqrt{1-|\mu_\sigma|}\,dW_\sigma,)$$

and $J(\bar t,\bar k_2,\bar x,\tilde\beta) = J(\bar t,\bar k_2,\bar x,P)$. Moreover, setting for $s\ge 0$,

$$k_{1_s} \doteq \bar k_1 + \int_0^s |\mu_\sigma|\,d\sigma = k_{2_s} - (\bar k_2-\bar k_1),$$

one easily sees that $\tilde\alpha = (\tilde\Omega,\tilde{\mathcal F},\tilde P,\{\tilde{\mathcal F}_s\},\{\mu_s\},\{(t_s,k_{1_s},\xi_s)\},\theta_2) \in \tilde\Gamma^a(\bar t,\bar k_1,\bar x)$. As before, there exists a control rule $Q\in\mathcal{R}^a(\bar t,\bar k_1,\bar x)$ such that $J(\bar t,\bar k_1,\bar x,\tilde\alpha) = J(\bar t,\bar k_1,\bar x,Q)$. Since the cost functional J and the state process $\{\xi_s\}$ do not depend explicitly on the k variable, one has that

$$J(\bar t,\bar k_2,\bar x,P) = J(\bar t,\bar k_1,\bar x,Q).$$

As a consequence, in this case, $V(\bar t,\bar k_1,\bar x) = V(\bar t,\bar k_2,\bar x)$.

Second case, $\bar k_1 > \bar k_2$. Consider the Dynamic Programming Principle (14) for $V(\bar t,\bar k_2,\bar x)$,

$$V(\bar t,\bar k_2,\bar x) = \inf_{R\in\mathcal{R}^a(\bar t,\bar k_2,\bar x)}\Big\{E_R\Big[\int_0^{r\wedge\theta}\big(l_0(t_\sigma,\xi_\sigma) \\ (1-|\mu_\sigma|)+\langle l_1(t_\sigma,\xi_\sigma),\mu_\sigma\rangle\big)\,d\sigma + V(t_{r\wedge\theta},k_{r\wedge\theta},\xi_{r\wedge\theta})\Big]\Big\},$$

where we choose the (deterministic) time $r=\bar k_1-\bar k_2$. Let us fix an arbitrary $w\in\mathcal K$ with $|w|=1$. Then there exists a control rule $P\in\mathcal{R}^a(\bar t,\bar k_2,\bar x)$ associated to a relaxed control

$$\tilde\beta = (\Omega,\mathcal F,P,\{\mathcal F_s\},\{\mu_s\},\{(t_s,k_s,\xi_s)\},\theta)$$

such that $P(\mu_s=\delta_{\{w\}} \quad 0\le s\le\theta, \quad \theta=K-\bar k_2)=1$, and $J(\bar t,\bar k_2,\bar x,\tilde\beta) = J(\bar t,\bar k_2,\bar x,P)$. Then, arguing as in the case "$\bar t_1>\bar t_2$" of the proof of the continuity in t, we can deduce that an estimate analogous to (17) is still verified, that is

$$V(\bar t,\bar k_2,\bar x) - E_P[V(t_r,k_r,\bar x)] \le \\ L_3 r + CE_P[|\xi_r-\bar x|] \le L_3 r + C(E_P[|\xi_r-\bar x|^2])^{\frac12}. \tag{20}$$

Now, under P we have

$$(t_r,k_r,\xi_r) = (\bar t,\bar k_2+r=\bar k_1,\bar x+\int_0^r B(t_\sigma,\xi_\sigma)\,d\sigma) \tag{21}$$

Therefore, since $E_P[|B(t_s,\xi_s)|^2] \le E_P[[L_1(1+|\xi_s|)]^2] \le C^2(1+|\bar x|)^2$, we deduce that for $0\le r\le\theta$,

$$E_P[|\xi_r-\bar x|^2] \le C^2(1+|\bar x|)^2 r^2. \tag{22}$$

Then (20), (21) and (22) yield

$$0 \le V(\bar t,\bar k_2,\bar x) - V(\bar t,\bar k_1,\bar x) \le \bar C(1+|\bar x|)|\bar k_2-\bar k_1|.$$

The proof of Theorem 3 is so concluded. ∎

IV. Dynamic Programming Equation and Boundary Conditions

This section is devoted to show that the value function $\mathcal V$ is a viscosity solution of (5)–(7). To this aim, we refer to [MS1] for the definition of viscosity sub– and supersolution to (5)–(7) which is based on the definition in [CIL] . We also give a formal derivation of the boundary value problem described in the Introduction in the following subsection.

A. Heuristic derivation of the quasi variational inequality and of the boundary conditions.

It is quite easy to deduce heuristically the boundary value problem (5)–(7) once we consider the value function V of the auxiliary optimization control problem defined in Definition 1 to which the original control problem, described in the Introduction, is equivalent. The auxiliary control problem is indeed formulated as an *unconstrained stopping time problem*, with bounded controls and discontinuous final cost given by

$$\tilde G(t,k,x) \doteq g(x) - G(t,k) \qquad \forall(t,k,x)\in\mathbf{R}^{2+n}.$$

Therefore, assuming V of class $C^{1,2}$, using Ito's formula and arguing as usual (see e.g. [FS]), we can deduce from the dynamic programming principle (14) that V verifies the following equation

$$\tilde F\left(x,DV,\frac{\partial V}{\partial t},\frac{\partial V}{\partial k},D^2V\right) = 0 \qquad \text{in }]0,T[\times]0,K[\times\mathbf{R}^n,$$

$$\tilde F(x,p_x,p_t,p_k,S) \doteq \max_{\{(w_0,w):\,|w|\le1,\quad w\in\mathcal K,\ w_0+|w|=1\}} \\ \{-\tfrac12 w_0\mathrm{Tr}\{\tilde D(t,x)S\} - \langle A(t,x)w_0+B(t,x)w,p_x\rangle - \\ l_0(t,x)w_0 - \langle l_1(t,x),w\rangle - p_t\,w_0 - p_k|w|\},$$

which is in turn equivalent to the quasi variational inequality (5), as shown in [MS2].

More precisely one can show that the value function of an optimal stopping time problem verifies

$$\max\left\{\tilde F\left(x,DV,\frac{\partial V}{\partial t},\frac{\partial V}{\partial k},D^2V\right);V-\tilde G\right\} = 0 \text{ in } \mathbf{R}^{2+n}, \tag{23}$$

due to the fact that the controller can decide to stop as soon as it is convenient (for the derivation of (23) in a viscosity framework we refer e.g. to [BP]). Since $V(t,k,x) = +\infty$ outside $[0,T] \times [0,K] \times \mathbf{R}^n$ and the lower semicontinuous exit cost $\tilde{G}(t,k,x)$ is equal to $g(x)$ for $(t,x,k) \in \{T\} \times [0,K] \times \mathbf{R}^n$ and to $+\infty$ otherwise, by (23) it easily follows that (7) holds for every $(t,x,k) \in \{T\} \times [0,K] \times \mathbf{R}^n$ and that (6) holds for $(t,x,k) \in [0,T[\times[0,K] \times \mathbf{R}^n$.

We underline that, as far as we know, there is not in the literature a dynamic programming principle for the problem described in the Introduction, hence, even assuming the value function $\mathcal{V}$ regular enough, there is no way to deduce the equation and the boundary conditions (5)–(7) directly for the original control problem.

B. Existence and uniqueness

We have the following two theorems which characterize the function $\mathcal{V}$.

Theorem 4: Assume (A0), (A1). Then the value function $\mathcal{V} : [0,T] \times [0,K] \times \mathbf{R}^n \to \mathbf{R}$ solves the boundary value problem (5)–(7) in the viscosity sense.

Theorem 5: Assume (A0), (A1). Then the value function $\mathcal{V} : [0,T] \times [0,K] \times \mathbf{R}^n \to \mathbf{R}$ is the unique viscosity solution of (5)–(7) among the bounded functions defined on $[0,T] \times [0,K] \times \mathbf{R}^n$ which are continuous on $\partial(]0,T[\times]0,K[\times\mathbf{R}^n)$.

V. Appendix

Definition 2: [Relaxed controls] Given $(\bar{t},\bar{k},\bar{x}) \in [0,T] \times [0,K] \times \mathbf{R}^n$ we say that $\tilde{\alpha}$ is a relaxed control and we write $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t},\bar{k},\bar{x})$ if

$$\tilde{\alpha} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s,k_s,\xi_s)\}, \theta)$$

where the following **(B3′)**, **(B4′)** are assumed.

(B3′) $(\Omega,\mathcal{F},P)$ is a probability space with a filtration $\{\mathcal{F}_s\}$, $\{\mu_s\}$ is a $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K})$–valued process defined on $[0,T+K]\times\Omega$ which is $\{\mathcal{F}_s\}$–progressively measurable, θ is an $\{\mathcal{F}_s\}$–stopping time such that $\theta \le T+K$,

(B4′) $\{(t_s,k_s,\xi_s)\}$ is a $\mathbf{R}^{2+n}$–valued $\{\mathcal{F}_s\}$–progressively measurable process for $s \in [0,T+K]$, with continuous paths, such that $(t_s,k_s,\xi_s) = (\bar{t},\bar{k},\bar{x})$ for $s=0$, for any $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$, $\mathcal{M}_s(\varphi,\tilde{\alpha})$ is a $(P,\{\mathcal{F}_s\})$ square integrable martingale for $s \in [0,T+K]$, where

$$M_s(\varphi,\tilde{\alpha}) \doteq \varphi(t_s,k_s,\xi_s) - \int_0^s \mathcal{L}\varphi(t_\sigma,k_\sigma,\xi_\sigma,\mu_\sigma)\,d\sigma,$$

and where

$$\begin{aligned}\mathcal{L}\varphi(t,k,x,w) \doteq \Big[\tfrac{1}{2}\textstyle\sum_{ij}\tilde{D}_{ij}(t,x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(t,k,x)+\\ \textstyle\sum_i A_i(t,x)\frac{\partial\varphi}{\partial x_i}(t,k,x)+\frac{\partial\varphi}{\partial t}(t,k,x)\Big]\,w_0\\ +\textstyle\sum_i\langle B_i(t,x),w\rangle\frac{\partial\varphi}{\partial x_i}(t,k,x)+\frac{\partial\varphi}{\partial k}(t,k,x)|w|,\end{aligned} \tag{24}$$

Definition 3: [Control rules] In order to introduce a canonical space for the problem, let us define the following spaces

$$\mathcal{C}^{2+n} = \{f : [0,T+K] \to \mathbf{R}^{2+n},\ f \text{ continuous}\},$$

endowed with the topology of uniform convergence;

$$\mathcal{U} \doteq \{\nu : [0,T+K] \to \mathbf{M}_1(\overline{B}_m(1)\cap\mathcal{K}),\ \nu \text{ Borel measurable}\},$$

endowed with the stable topology;

$$\mathcal{Z} = \{\zeta : [0,T+K] \to \mathbf{R},\ \zeta = \chi_{s\ge\Delta},\ \Delta \in [0,+\infty]\} \tag{25}$$

endowed with the topology of weak convergence of the corresponding (point) probability measures. We denote the map $\zeta \to \Delta$ by $\Delta(\cdot)$. Let $\tilde{\mathcal{C}}$, $\tilde{\mathcal{U}}$, $\tilde{\mathcal{Z}}$ denote their Borel σ–fields, let $\tilde{\mathcal{C}}_s$, $\tilde{\mathcal{U}}_s$, $\tilde{\mathcal{Z}}_s$ denote the σ–fields up to time s (e.g., $\tilde{\mathcal{Z}}_s = \sigma\{\zeta(s') :\ 0 \le s' \le s\}$), and let us introduce the canonical setting

$$\begin{aligned}\Omega = \mathcal{C}^{2+n}\times\mathcal{U}\times\mathcal{Z}, \qquad \mathcal{F} \doteq \tilde{\mathcal{C}}\times\tilde{\mathcal{U}}\times\tilde{\mathcal{Z}},\\ \mathcal{F}_s \doteq \tilde{\mathcal{C}}_s\times\tilde{\mathcal{U}}_s\times\tilde{\mathcal{Z}}_s.\end{aligned} \tag{26}$$

Notice that Ω is metrizable and separable under the product topology. Fix $(\bar{t},\bar{k},\bar{x}) \in [0,T] \times [0,K] \times \mathbf{R}^n$, and let Ω, $\mathcal{F}$ and $\{\mathcal{F}_s\}$ be defined by (26). We say that R is a control rule and we write $R \in \mathcal{R}(\bar{t},\bar{k},\bar{x})$ if R is a probability measure on the canonical space $(\Omega,\mathcal{F})$, such that

$$\tilde{\alpha} = (\Omega,\mathcal{F},R,\{\mathcal{F}_s\},\{\mu_s\},\{(t_s,k_s,\xi_s)\},\theta)$$

is a relaxed control (i.e., $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t},\bar{k},\bar{x})$), where

$$(t_s,k_s,\xi_s)(\omega) = f_s, \quad \mu_s(\omega) = \nu_s, \quad \theta(\omega) = \Delta(\zeta)$$

for $\omega = (f,\nu,\zeta) \in \Omega$. Finally, we define the cost associated to R as $J(\bar{t},\bar{k},\bar{x},R) \doteq J(\bar{t},\bar{k},\bar{x},\tilde{\alpha})$ where $J(\bar{t},\bar{k},\bar{x},\tilde{\alpha})$ is given in (13).

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