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Global Optimization of Linear Hybrid Systems with Varying Transition Times

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***Abstract*— Open loop optimal control problems with linear hybrid (discrete/continuous) systems embedded are often approximated as dynamic optimization problems. These problems are inherently nonconvex. A deterministic global optimization algorithm for linear hybrid systems with varying transition times is developed. First, the control parametrization enhancing transform is used to transform the problem from a linear hybrid system with scaled discontinuities and varying transition times into a nonlinear one with stationary discontinuities and fixed transition times. Next, a convexity theory is applied to construct a convex relaxation of the original nonconvex problem. This allows the problem to be solved in a branch-and-bound framework that can guarantee the solution to ε global optimality within a finite number of iterations.**

I. INTRODUCTION

Hybrid systems exhibit both discrete state and continuous state dynamics, and have become indispensable for modeling systems exhibiting discontinuities in their dynamics [1]. This article summarizes the work presented in [2] to locate the global solution of a specific class of dynamic optimization problems with linear time varying (LTV) hybrid systems embedded: problems in which the temporal sequence of modes is fixed, but the transition times between modes are allowed to vary. This is motivated by the fact that many practical problems can be expressed as open loop optimal control problems with hybrid systems embedded, which in turn can be approximated by dynamic optimization problems with hybrid systems embedded [1]. The latter transformation is carried out via control parametrization [3], a partial discretization method where the controls are approximated by a finite series of piecewise continuous basis functions over the time horizon.

The practicality of such a method hinges on the existence and uniqueness of the parametric sensitivities of the hybrid system (or the related adjoints), which are employed to calculate the gradients of the objective and constraint functionals used by the Master problem. Sufficient conditions for the existence and uniqueness of these sensitivities have been developed in [4], and these results indicate that the sensitivity trajectories of a hybrid system will usually exist a.e. in the parameter space. Subject to the key restriction that the temporal sequence of modes visited by executions of the hybrid system is unchanged throughout the parameter space (the timing of switches and jumps may still vary), the resulting Master NLP is smooth under mild assumptions and existing gradient based methods may be used to find local solutions. On the other hand, if the temporal sequence of modes varies as a function of the optimization parameters, then most resulting Master NLPs will exhibit some degree of nonsmoothness [5]. These critical observations explicate the requirement of fixing the sequence of modes, a key issue. The restriction to a fixed sequence of modes reduces the embedded hybrid system to a *multi-stage* dynamic system. For such a problem, obtaining the optimal switching times for a fixed sequence of modes is difficult because it is inherently nonconvex (see e.g., [6]).

In [2], the control parametrization enhancing transform (CPET) [7] is used to transform the problem into one with fixed transition times, at the expense of introducing nonlinearity into the embedded hybrid system. A convex relaxation theory is then developed for the global optimization of the resulting nonlinear hybrid system with a fixed sequence of modes and fixed transition times. Note that the results in the following sections will be presented without proof due to space constraints; the proofs can be found in [2].

II. HYBRID SYSTEMS: NOTATION

The modeling framework of [1] is used as a basis to define the hybrid system of interest.

Definition 1: The hybrid system considered is the 10-tuple $\mathcal{H} = (\mathcal{M}, \mathcal{E}, T_\mu, \sigma_1, \boldsymbol{\delta}, \mathbf{p}, \mathbf{x}, \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{T}}^0, \boldsymbol{\mathcal{T}})$, where

- $\mathcal{M} = \{1, \ldots, n_m\}$, $1 \leq n_m < +\infty$,
- $\mathcal{E} = \{1, \ldots, n_e\}$, $1 \leq n_e < +\infty$,
- $T_\mu = \{m_i\}_{i \in \mathcal{E}}$, $m_i \in \mathcal{M}, \ \forall i \in \mathcal{E}$,
- $\sigma_1 \in \mathbb{R}$,
- $\boldsymbol{\delta} \in \Delta \subset \mathbb{R}^{n_e}_+$,
- $\mathbf{p} \in P \subset \mathbb{R}^{n_p}$,
- $\mathbf{x} : \mathcal{E} \times P \times \Delta \times \mathbb{R} \to \mathbb{R}^{n_x}$,
- $\boldsymbol{\mathcal{F}} : \mathcal{M} \times \mathbb{R}^{n_x} \times P \times \Delta \times \mathbb{R} \to \mathbb{R}^{n_x}$,
- $\boldsymbol{\mathcal{T}}^0 : P \times \Delta \to \mathbb{R}^{n_x}$, and
- $\boldsymbol{\mathcal{T}} : \mathcal{E} \setminus \{n_e\} \times \mathbb{R}^{n_x} \times P \times \Delta \to \mathbb{R}^{n_x}$.

The elements of $\mathcal{M}$ are called the *modes* of $\mathcal{H}$. T_μ is called the *hybrid mode trajectory* and is the discrete state variable of $\mathcal{H}$. σ_1 is the initial time. $\boldsymbol{\delta}$ is the vector of nonnegative durations and $\mathbf{p}$ is the vector of parameters. $\mathbf{x}$ is the vector of continuous state variables, and $\boldsymbol{\mathcal{F}}$ is the vector field for $\mathbf{x}$. $\boldsymbol{\mathcal{T}}^0$ are the initial conditions, and $\boldsymbol{\mathcal{T}}$ are the *transition functions*. $\mathcal{E}$ is the index set for the *epochs*, which are defined in the following

Definition 2: The hybrid time trajectory of $\mathcal{H}$ is a finite sequence of intervals $T_\tau = \{I_i\}_{i \in \mathcal{E}}$ where $I_i = [\sigma_i, \tau_i]$, $\tau_i =$

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$\sigma_i + \delta_i$ for $i \in \mathcal{E}$ and $\sigma_i = \tau_{i-1}$ for $i = 2, \ldots, n_e$. The I_i are called epochs.

Definition 3: Consider the epoch $I_i = [\sigma_i, \tau_i]$ and its corresponding scaled time interval $\hat{I}_i = [\hat{\sigma}_i, \hat{\tau}_i] = [i-1, i]$. A scaled simple discontinuity occurring at time $t \in I_i$ is one that occurs at a fixed (stationary) point $s \in \hat{I}_i$ such that

$$\frac{s - \hat{\sigma}_i}{\hat{\tau}_i - \hat{\sigma}_i} = s - i + 1 = \frac{t - \sigma_i}{\tau_i - \sigma_i}.$$

It is clear from Definition 3 that there is a stationary simple discontinuity [8, Definition 2.1] at s^* in $\hat{I}_i$ iff there is a scaled simple discontinuity at t^* in I_i.

We will impose the following assumptions to make the optimization problem (introduced later) well posed:

A1. $\Delta = [\boldsymbol{\delta}^L, \boldsymbol{\delta}^U]$ and $P = [\mathbf{p}^L, \mathbf{p}^U]$ are nondegenerate interval vectors. The vector field $\boldsymbol{\mathcal{F}}(m, \cdot)$ for each $m \in \mathcal{M}$ is affine in the continuous state variables $\mathbf{x}$ and the parameters $\mathbf{p}$ so that for each epoch $I_{i \in \mathcal{E}}$ the continuous state variables evolve according to the LTV ODE system:

$$\dot{\mathbf{x}}(i, \mathbf{p}, \boldsymbol{\delta}, t) \equiv \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{i, \mathbf{p}, \boldsymbol{\delta}, t} = \mathbf{A}(m, \boldsymbol{\delta}, t)\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t) + \mathbf{B}(m, \boldsymbol{\delta}, t)\mathbf{p} + \mathbf{q}(m, \boldsymbol{\delta}, t), \ \forall t \in (\sigma_i, \tau_i].$$

Moreover, $\forall (m, i, \boldsymbol{\delta}) \in \mathcal{M} \times \mathcal{E} \times \Delta$, $\mathbf{A}(m, \boldsymbol{\delta}, \cdot)$, $\mathbf{B}(m, \boldsymbol{\delta}, \cdot)$ and $\mathbf{q}(m, \boldsymbol{\delta}, \cdot)$ are piecewise continuous on epoch I_i with a finite number of scaled simple discontinuities and are defined at any point of discontinuity.

A2. The initial conditions $\boldsymbol{\mathcal{T}}^0$ are affine functions so that the initial conditions for epoch I_1 are given by:

$$\mathbf{x}(1, \mathbf{p}, \boldsymbol{\delta}, \sigma_1) = \mathbf{E}(0)\mathbf{p} + \mathbf{J}(0)\boldsymbol{\delta} + \mathbf{k}(0).$$

A3. The transition functions $\boldsymbol{\mathcal{T}}(i, \cdot)$ for each $i \in \mathcal{E} \setminus \{n_e\}$ are affine functions so that the initial conditions for epochs $I_{i \in \mathcal{E} \setminus \{1\}}$ are given by:

$$\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, \sigma_i) = \mathbf{D}(i-1)\mathbf{x}(i-1, \mathbf{p}, \boldsymbol{\delta}, \tau_{i-1}) + \mathbf{E}(i-1)\mathbf{p} + \mathbf{J}(i-1)\boldsymbol{\delta} + \mathbf{k}(i-1). \qquad (1)$$

Definition 4: Given values for $\mathbf{p}$ and $\boldsymbol{\delta}$, the solution, or execution, of a hybrid system $\mathcal{H}$ subject to assumptions A1-A3 is $\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t)$, $t \in I_i$, $i \in \mathcal{E}$ where $\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t)$ is the solution of the ODE system in A1 with initial conditions A2 if $i = 1$ and A3 otherwise.

We shall now describe in words an execution of the hybrid system in time. The finite time horizon is partitioned into contiguous intervals called epochs. Starting from the initial conditions given by $\boldsymbol{\mathcal{T}}^0$, the continuous state variables $\mathbf{x}(1, \mathbf{p}, \boldsymbol{\delta}, \cdot)$ evolve in time, t, according to the differential equations defined by the vector field $\boldsymbol{\mathcal{F}}(m_1, \cdot)$ for a (possibly trivially zero) duration of δ_1. At time τ_1, a *transition* is made from mode m_1 to mode m_2. The transition functions in (1) map the value of the continuous state at τ_1 in epoch I_1 to an initial condition for epoch I_2 at time σ_2. The hybrid system then evolves according to the differential equations defined by the vector field $\boldsymbol{\mathcal{F}}(m_2, \cdot)$ for a duration of δ_2, and so on and so forth. Note that for epoch I_i, the system evolves continuously in time if $\delta_i > 0$, and it evolves discretely by making an instantaneous transition if $\delta_i = 0$.

Definition 5 (Implied State Bounds): Define the following convex sets for all $i \in \mathcal{E}$ where $\mathcal{S}_i \equiv [\sigma_1 + \sum_{j=1}^{i} \delta_j^L, \sigma_1 + \sum_{j=1}^{i} \delta_j^U]$. For any fixed $\underline{t} \in \mathcal{S}_i$,

$$X(i, \underline{t}; P, \Delta) \equiv [\mathbf{x}^L(\underline{t}), \mathbf{x}^U(\underline{t})] \mid \mathbf{x}^L(\underline{t}) \leq \mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, \underline{t}) \leq \mathbf{x}^U(\underline{t}), \forall (\mathbf{p}, \boldsymbol{\delta}) \in P \times \Delta.$$

In addition, $X(i, P, \Delta) \equiv [\mathbf{x}^L, \mathbf{x}^U] \mid X(i, \underline{t}; P, \Delta) \subset [\mathbf{x}^L, \mathbf{x}^U]\ \forall \underline{t} \in \mathcal{S}_i$.

Now consider the following

Problem 1:

$$\min_{\mathbf{p} \in P, \boldsymbol{\delta} \in \Delta} F(\mathbf{p}, \boldsymbol{\delta}) = \sum_{i=1}^{n_e} \left\{ \sum_{j=1}^{n_{\phi i}} \phi_{ij}\Big(\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, \alpha_{ij}(\boldsymbol{\delta})), \mathbf{p}, \boldsymbol{\delta}\Big) + \int_{\sigma_i(\boldsymbol{\delta})}^{\tau_i(\boldsymbol{\delta})} f_i\big(\mathbf{x}, \mathbf{p}, \boldsymbol{\delta}, t\big)\, \mathrm{d}t \right\},$$

subject to the following point and isoperimetric constraints,

$$\mathbf{G}(\mathbf{p}, \boldsymbol{\delta}) = \sum_{i=1}^{n_e} \left\{ \sum_{j=1}^{n_{\eta i}} \boldsymbol{\eta}_{ij}\Big(\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, \beta_{ij}(\boldsymbol{\delta})), \mathbf{p}, \boldsymbol{\delta}\Big) + \int_{\sigma_i(\boldsymbol{\delta})}^{\tau_i(\boldsymbol{\delta})} \mathbf{g}_i\big(\mathbf{x}, \mathbf{p}, \boldsymbol{\delta}, t\big)\, \mathrm{d}t \right\} \leq \mathbf{0},$$

where $\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t)$ is given by the solution of the embedded hybrid system in Definition 1 subject to assumptions A1-A3; f_i and $\mathbf{g}_i$ are piecewise continuous mappings $f_i : X(i, P, \Delta) \times P \times \Delta \times \mathcal{S}_i \to \mathbb{R}$ and $\mathbf{g}_i : X(i, P, \Delta) \times P \times \Delta \times \mathcal{S}_i \to \mathbb{R}^{n_c}$ for all $i \in \mathcal{E}$, where only a finite number of scaled simple discontinuities are allowed; $n_{\phi i}$ is an arbitrary number of scaled point objectives in epoch I_i, $\alpha_{ij}(\boldsymbol{\delta}) \in I_i$ such that $\alpha_{ij}(\boldsymbol{\delta}) = \sigma_i + \delta_i(\hat{\alpha}_{ij} - i + 1)$ for some fixed $\hat{\alpha}_{ij} \in \hat{I}_i$, and ϕ_{ij} is a continuous mapping $\phi_{ij} : X(i, P, \Delta) \times P \times \Delta \to \mathbb{R}$ for all $j = 1, \ldots, n_{\phi i}$ and $i \in \mathcal{E}$; and $n_{\eta i}$ is an arbitrary number of scaled point constraints in epoch I_i, $\beta_{ij} \in I_i$ such that $\beta_{ij}(\boldsymbol{\delta}) = \sigma_i + \delta_i(\hat{\beta}_{ij} - i + 1)$ for some fixed $\hat{\beta}_{ij} \in \hat{I}_i$, and $\boldsymbol{\eta}_{ij}$ is a continuous mapping $\boldsymbol{\eta}_{ij} : X(i, P, \Delta) \times P \times \Delta \to \mathbb{R}^{n_c}$ for all $j = 1, \ldots, n_{\eta i}$ and $i \in \mathcal{E}$. Additionally, we require that the set $G = \{(\mathbf{p}, \boldsymbol{\delta}) \in P \times \Delta \mid \mathbf{G}(\mathbf{p}, \boldsymbol{\delta}) \leq \mathbf{0}\}$, is nonempty.

III. The Transform

The CPET [7] is implemented as follows. Consider the original independent variable time (t) in Problem 1. We now wish to construct a new time scale in which the varying epoch durations (transition times) are fixed, $s \in [0, n_e]$. The transformation (CPET) from $t \in [\sigma_1, \sigma_1 + \sum_{i=1}^{n_e} \delta_i^U]$ to $s \in [0, n_e]$ is defined by

$$\frac{\mathrm{d}t}{\mathrm{d}s} = v(\boldsymbol{\delta}, s), \quad t(\boldsymbol{\delta}, 0) = \sigma_1, \qquad (2)$$

where the function $v : \Delta \times [0, n_e] \to \mathbb{R}$ is called the *enhancing control*. It is a piecewise constant function with possible simple discontinuities at the prefixed knots $1, \ldots, n_e - 1$,

$$v(\boldsymbol{\delta}, s) = \sum_{i=1}^{n_e} \delta_i \chi_i(s),$$

where $\chi_i(s)$ is the indicator function defined by

$$\chi_i(s) = \begin{cases} 1 & \text{if } s \in [i-1, i], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\begin{aligned} t(\boldsymbol{\delta}, s) &= \sigma_1 + \int_0^s v(\boldsymbol{\delta}, z)\, \mathrm{d}z \\ &= \sigma_1 + \delta_i(s - (i-1)) + \sum_{j=1}^{i-1} \delta_j = (s - i + 1)\delta_i + \sigma_i \end{aligned} \quad (3)$$

for $s \in [i-1, i],\ i \in \mathcal{E}$, where the value of the enhancing control in the transformed time interval $(i-1, i)$ corresponds to the value of the duration of epoch I_i in the original time scale. In addition, the scaled simple discontinuities, point objectives and point constraints in Problem 1 become stationary simple discontinuities, point objectives and point constraints in the new time scale, according to Definition 3. Finally, let $\mathbf{x}' \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$. It follows from the CPET that

$$\begin{aligned} \frac{\mathbf{x}'(i, \mathbf{p}, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))}{v(\boldsymbol{\delta}, s)} = \Big(&\mathbf{A}(m, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))\mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s)) \\ &+\mathbf{B}(m, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))\mathbf{p} + \mathbf{q}(m, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))\Big), \end{aligned}$$

where t is an additional differential state variable that has to satisfy (2). We can substitute for the explicit form of $t(\boldsymbol{\delta}, s)$ to obtain

$$\begin{aligned} \hat{\mathbf{x}}'(i, \mathbf{p}, \boldsymbol{\delta}, s) = v(\boldsymbol{\delta}, s)\Big(&\hat{\mathbf{A}}(m, \boldsymbol{\delta}, s)\hat{\mathbf{x}}(\mathbf{p}, \boldsymbol{\delta}, s) \\ &+\hat{\mathbf{B}}(m, \boldsymbol{\delta}, s)\mathbf{p} + \hat{\mathbf{q}}(m, \boldsymbol{\delta}, s)\Big), \end{aligned} \quad (4)$$

where $\hat{\mathbf{x}}(i, \mathbf{p}, \boldsymbol{\delta}, s) \equiv \mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))$, $\hat{\mathbf{x}}' \equiv \frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}s}$, $\hat{\mathbf{A}}(m, \boldsymbol{\delta}, s) \equiv \mathbf{A}(m, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))$, $\hat{\mathbf{B}}(m, \boldsymbol{\delta}, s) \equiv \mathbf{B}(m, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))$, $\hat{\mathbf{q}}(m, \boldsymbol{\delta}, s) \equiv \mathbf{q}(m, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))$, and $t(\boldsymbol{\delta}, s)$ is given by (3).

The objective function and constraints after the CPET are then given by

$$\begin{aligned} \hat{F}(\mathbf{p}, \boldsymbol{\delta}) = \sum_{i=1}^{n_e} \Bigg\{ &\sum_{j=1}^{n_{\phi i}} \phi_{ij}\Big(\hat{\mathbf{x}}(i, \mathbf{p}, \boldsymbol{\delta}, \hat{\alpha}_{ij}), \mathbf{p}, \boldsymbol{\delta}\Big) \\ &+\int_{i-1}^{i} f_i\Big(\hat{\mathbf{x}}, \mathbf{p}, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s)\Big) v(\boldsymbol{\delta}, s)\, \mathrm{d}s \Bigg\}, \end{aligned} \quad (5)$$

$$\begin{aligned} \hat{\mathbf{G}}(\mathbf{p}, \boldsymbol{\delta}) = \sum_{i=1}^{n_e} \Bigg\{ &\sum_{j=1}^{n_{\eta i}} \boldsymbol{\eta}_{ij}\Big(\hat{\mathbf{x}}(i, \mathbf{p}, \boldsymbol{\delta}, \hat{\beta}_{ij}), \mathbf{p}, \boldsymbol{\delta}\Big) \\ &+\int_{i-1}^{i} \mathbf{g}_i\Big(\hat{\mathbf{x}}, \mathbf{p}, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s)\Big) v(\boldsymbol{\delta}, s)\, \mathrm{d}s \Bigg\}. \end{aligned} \quad (6)$$

Note that $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ are no longer a function of $\boldsymbol{\delta}$. Henceforth, we shall use the superscript prime notation to denote the transformed time derivative, i.e., $' \equiv \frac{\mathrm{d}}{\mathrm{d}s}$. We are now able to formally state the transformed hybrid system and problem.

Definition 6: A CPET hybrid system is the 8-tuple $\hat{\mathcal{H}} = (\mathcal{M}, \mathcal{E}, T_\mu, \hat{\mathbf{p}}, \hat{\mathbf{x}}, \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{T}}^0, \boldsymbol{\mathcal{T}})$, where $\mathcal{M}$, $\mathcal{E}$ and T_μ are as defined in Definition 1, and

- $\hat{\mathbf{p}} = (\mathbf{p}, \boldsymbol{\delta}) \in \hat{P} = P \times \Delta \subset \mathbb{R}^{n_p + n_e}$,
- $\hat{\mathbf{x}} : \mathcal{E} \times \hat{P} \times \mathbb{R} \to \mathbb{R}^{n_x}$,
- $\boldsymbol{\mathcal{F}} : \mathcal{M} \times \mathbb{R}^{n_x} \times \hat{P} \times \mathbb{R} \to \mathbb{R}^{n_x}$,
- $\boldsymbol{\mathcal{T}}^0 : \hat{P} \to \mathbb{R}^{n_x}$, and
- $\boldsymbol{\mathcal{T}} : \mathcal{E} \setminus \{n_e\} \times \mathbb{R}^{n_x} \times \hat{P} \to \mathbb{R}^{n_x}$.

As before, the elements of $\mathcal{M}$ are called the modes of $\hat{\mathcal{H}}$. T_μ is called the hybrid mode trajectory and is the discrete state variable of $\hat{\mathcal{H}}$. $\hat{\mathbf{p}}$ is the vector of parameters. $\hat{\mathbf{x}}$ is the vector of continuous state variables, and $\boldsymbol{\mathcal{F}}$ is the vector field for $\mathbf{x}$. $\boldsymbol{\mathcal{T}}^0$ are the initial conditions, and $\boldsymbol{\mathcal{T}}$ are the transition functions. $\mathcal{E}$ remains the index set for the epochs, which are defined in the following

Definition 7: The hybrid time trajectory of $\hat{\mathcal{H}}$ is a finite sequence of intervals $T_\tau = \{\hat{I}_i\}_{i \in \mathcal{E}}$ where $\hat{I}_i = [\hat{\sigma}_i, \hat{\tau}_i] = [i-1, i]$. The $\hat{I}_i$ are called epochs.

From the previous analysis, the CPET transform of $\mathcal{H}$ subject to assumptions A1-A3 will result in a CPET hybrid system $\hat{\mathcal{H}}$ subject to the following assumptions:

B1. $\hat{P} = [\hat{\mathbf{p}}^L, \hat{\mathbf{p}}^U] = [(\mathbf{p}^L, \boldsymbol{\delta}^L), (\mathbf{p}^U, \boldsymbol{\delta}^U)]$ is a nondegenerate interval vector. The vector field $\boldsymbol{\mathcal{F}}(m, \cdot)$ for each $m \in \mathcal{M}$ is nonlinear in the continuous state variables $\hat{\mathbf{x}}$ and the parameters $\hat{\mathbf{p}}$ so that for each epoch $\hat{I}_{i \in \mathcal{E}}$ the continuous state variables evolve according to the nonlinear system in (4), written as the following:

$$\hat{\mathbf{x}}'(i, \hat{\mathbf{p}}, s) = \boldsymbol{\mathcal{F}}(m, \hat{\mathbf{x}}, \hat{\mathbf{p}}, s), \quad \forall s \in (i-1, i]. \quad (7)$$

Moreover, $\forall (m, i, \hat{\mathbf{p}}, \hat{x}) \in \mathcal{M} \times \mathcal{E} \times \hat{P} \times \mathbb{R}^{n_x}$, $\boldsymbol{\mathcal{F}}(m, \hat{\mathbf{x}}, \hat{\mathbf{p}}, \cdot)$ is piecewise continuous on epoch $\hat{I}_i$ with a finite number of stationary simple discontinuities and is defined at any point of discontinuity.

B2. The initial conditions $\boldsymbol{\mathcal{T}}^0$ are functions so that the initial conditions for epoch $\hat{I}_1$ are given by:

$$\hat{\mathbf{x}}(1, \hat{\mathbf{p}}, 0) = \mathbf{E}(0)\mathbf{p} + \mathbf{J}(0)\boldsymbol{\delta} + \mathbf{k}(0).$$

B3. The transition functions $\boldsymbol{\mathcal{T}}(i, \cdot)$ for each $i \in \mathcal{E} \setminus \{n_e\}$ are functions so that the initial conditions for epochs $\hat{I}_{i \in \mathcal{E} \setminus \{1\}}$ are given by:

$$\begin{aligned} \hat{\mathbf{x}}(i, \hat{\mathbf{p}}, i-1) = &\mathbf{D}(i-1)\hat{\mathbf{x}}(i-1, \hat{\mathbf{p}}, i-1) \\ &+\mathbf{E}(i-1)\mathbf{p} + \mathbf{J}(i-1)\boldsymbol{\delta} + \mathbf{k}(i-1). \end{aligned} \quad (8)$$

Definition 8: Given a value for $\hat{\mathbf{p}}$, the solution, or execution, of a CPET hybrid system $\hat{\mathcal{H}}$ subject to assumptions B1-B3 is $\hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s),\ s \in \hat{I}_i,\ i \in \mathcal{E}$ where $\hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s)$ is the solution of the ODE system in B1 with initial conditions B2 if $i = 1$ and B3 otherwise.

The major differences between a CPET hybrid system $\hat{\mathcal{H}}$ subject to assumptions B1-B3 and a hybrid system $\mathcal{H}$ subject to assumptions A1-A3 are (a) the initial time for $\hat{\mathcal{H}}$ is fixed at $s = 0$, and the durations of all epochs are 1, and (b) the form of the underlying differential equations for each mode.

The corresponding relaxations for the implied state bounds are given by the following:

Definition 9 (Implied State Bounds): Define the following convex sets for all $i \in \mathcal{E}$. For any fixed $\underline{s} \in \hat{I}_i$:

$$\hat{X}(i, \underline{s}; \hat{P}) \equiv [\hat{\mathbf{x}}^L(\underline{s}), \hat{\mathbf{x}}^U(\underline{s})] \mid \hat{\mathbf{x}}^L(\underline{s}) \leq \hat{\mathbf{x}}(\hat{\mathbf{p}}, \underline{s}) \leq \hat{\mathbf{x}}^U(\underline{s}), \forall \hat{\mathbf{p}} \in \hat{P}.$$

In addition, $\hat{X}(i, \hat{P}) \equiv [\hat{\mathbf{x}}^L, \hat{\mathbf{x}}^U] \mid \hat{X}(i, \underline{s}; \hat{P}) \subset [\hat{\mathbf{x}}^L, \hat{\mathbf{x}}^U], \forall\ \underline{s} \in \hat{I}_i$.

The transformed problem is then given by the following

Problem 2:

$$\min_{\hat{\mathbf{p}} \in \hat{P}} \ \hat{F}(\hat{\mathbf{p}}) \quad \text{s.t. } \hat{\mathbf{G}}(\hat{\mathbf{p}}) \leq \mathbf{0},$$

where $\hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s)$ is given by the solution of the embedded nonlinear hybrid system in Definition 6 subject to assumptions B1-B3; $\hat{F}(\hat{\mathbf{p}})$ and $\hat{\mathbf{G}}(\hat{\mathbf{p}})$ are given by (5) and (6) respectively; $\hat{f}_i$ and $\hat{\mathbf{g}}_i$ are piecewise continuous mappings $\hat{f}_i : \hat{X}(i, \hat{P}) \times \hat{P} \times \hat{I}_i \to \mathbb{R}$ and $\hat{\mathbf{g}}_i : \hat{X}(i, \hat{P}) \times \hat{P} \times \hat{I}_i \to \mathbb{R}^{n_c}$, for all $i \in \mathcal{E}$, with a finite number of stationary simple discontinuities; $n_{\phi i}$ is the number of fixed point objectives in epoch $\hat{I}_i$, $\hat{\alpha}_{ij} \in \hat{I}_i$ and $\hat{\phi}_{ij}$ is a continuous mapping $\hat{\phi}_{ij} : \hat{X}(i, \hat{\alpha}_{ij}; \hat{P}) \times \hat{P} \to \mathbb{R}$ for all $j = 1, \ldots, n_{\phi i}$ and $i \in \mathcal{E}$; and $n_{\eta i}$ is the number of fixed point constraints in epoch I_i, $\hat{\beta}_{ij} \in \hat{I}_i$ and $\hat{\boldsymbol{\eta}}_{ij}$ is a continuous mapping $\hat{\boldsymbol{\eta}}_{ij} : \hat{X}(i, \hat{\beta}_{ij}; \hat{P}) \times \hat{P} \to \mathbb{R}^{n_c}$ for all $j = 1, \ldots, n_{\eta i}$ and $i \in \mathcal{E}$. Additionally, we require that the set $\hat{G} = \{\hat{\mathbf{p}} \in \hat{P} \mid \hat{\mathbf{G}}(\hat{\mathbf{p}}) \leq \mathbf{0}\}$ is nonempty.

Lemma 1: Consider $\mathcal{H}$ subject to A1-A3 and $\hat{\mathcal{H}}$ subject to B1-B3. Then, for any $(\hat{\mathbf{p}}, s) \in \hat{P} \times \hat{I}_i$, $\hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s) = \mathbf{x}(i, \mathbf{p}, \boldsymbol{\delta}, t(\boldsymbol{\delta}, s))$ for all $i \in \mathcal{E}$, where $t(\boldsymbol{\delta}, s)$ is given by (3).

IV. CONVEX RELAXATION

To solve Problem 2, a convex relaxation has to be constructed for the objective function (5) and constraints (6), subject to the transformed nonlinear hybrid system. This will enable a convex relaxation of the problem to be solved. It is shown later that the constructed convex relaxations possess the same consistent bounding properties of the convex relaxation techniques used in their construction. This implies that their incorporation into a branch-and-bound framework [9] leads to an infinitely convergent algorithm [10], which implies ε global optimality within a finite number of iterations.

The steps for constructing the convex relaxation are outlined below:

1) Estimating the implied state bounds, $\hat{X}(i, \underline{s}; \hat{P})$ in Definition 9;
2) Constructing convex and concave relaxations for the states;
3) Applying convex relaxation techniques on subsets of Euclidean spaces to construct the required convex relaxation.

Definition 10: Let $\hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s)$ be the solution of $\hat{\mathcal{H}}$ subject to assumptions B1-B3, and let $\hat{x}_j(i, \hat{\mathbf{p}}, s) \in \hat{\mathcal{X}}_j(i, \hat{\mathbf{p}}, s)$ for each $\hat{\mathbf{p}} \in \hat{P}$, $i \in \mathcal{E}$, $j = 1, \ldots, n_x$ where $\hat{\mathcal{X}}_j(i, \hat{\mathbf{p}}, s) \subset \mathbb{R}$ is a bounding set that is known independently. For each fixed $\underline{s} \in \hat{I}_{i \in \mathcal{E}}$, let $\varrho_j(i, \mathbf{r}, \underline{s}) = \inf \hat{\mathcal{X}}_j(i, \mathbf{r}, \underline{s})$ and $\varsigma_j(i, \mathbf{r}, \underline{s}) = \sup \hat{\mathcal{X}}_j(i, \mathbf{r}, \underline{s})$ for each $\mathbf{r} \in \hat{P}, j = 1, \ldots, n_x$. Furthermore, let $\hat{\mathcal{X}}(i, \underline{s})$ be defined pointwise in (transformed) time for each $i \in \mathcal{E}$ by $\hat{\mathcal{X}}(i, \underline{s}) = [\mathbf{z}^L, \mathbf{z}^U]$ such that

$$z_j^L = \inf_{\mathbf{r} \in \hat{P}} \varrho_j(i, \mathbf{r}, \underline{s}), \ z_j^U = \sup_{\mathbf{r} \in \hat{P}} \varsigma_j(i, \mathbf{r}, \underline{s}), \ \forall j = 1, \ldots, n_x$$

where z_j^L and z_j^U are in the extended real number system.

Theorem 1: Consider $\hat{\mathcal{H}}$ subject to assumptions B1-B3. If the following conditions are satisfied for all $i \in \mathcal{E}$ and $j = 1, \ldots, n_x$,

D1. $v_j(\hat{\sigma}_i) < \min_{\mathbf{r} \in \hat{P}} \hat{x}_j(i, \mathbf{r}, \hat{\sigma}_i)$

D2. $w_j(\hat{\sigma}_i) > \max_{\mathbf{r} \in \hat{P}} \hat{x}_j(i, \mathbf{r}, \hat{\sigma}_i)$

and additionally for all $\mathbf{v}(s), \mathbf{w}(s) \in H(s)$, $s \in [i-1, i]$,

D3.

$$v'_j = \underline{h}_j(m_i, \mathbf{v}, \mathbf{w}, s; \hat{P}) < \inf_{\substack{\mathbf{z} \in \hat{\mathcal{X}}(i,s) \cap H(s), \mathbf{r} \in \hat{P} \\ z_j = v_j(s)}} \mathcal{F}_j(m_i, \mathbf{z}, \mathbf{r}, s)$$

D4.

$$w'_j = \overline{h}_j(m_i, \mathbf{v}, \mathbf{w}, s; \hat{P}) > \sup_{\substack{\mathbf{z} \in \hat{\mathcal{X}}(i,s) \cap H(s), \mathbf{r} \in \hat{P} \\ z_j = w_j(s)}} \mathcal{F}_j(m_i, \mathbf{z}, \mathbf{r}, s)$$

where $H(s) \equiv \{\mathbf{z} \mid \mathbf{v}(s) \leq \mathbf{z} \leq \mathbf{w}(s)\}$, then

$$\mathbf{v}(s) < \hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s) < \mathbf{w}(s), \qquad \forall (\hat{\mathbf{p}}, s) \in \hat{P} \times \hat{I}_i, \ i \in \mathcal{E}.$$

It is also assumed that the solutions, in the sense of Carathéodory, to the differential systems in $\mathbf{v}$ and $\mathbf{w}$ exist and are unique, for all $i \in \mathcal{E}$.

By asserting uniqueness of the solution of the bounding differential equations, the conditions of the above theorem may be relaxed to

D1. $v_j(\hat{\sigma}_i) \leq \min_{\mathbf{r} \in \hat{P}} \hat{x}_j(\mathbf{r}, \hat{\sigma}_i)$

D2. $w_j(\hat{\sigma}_i) \geq \max_{\mathbf{r} \in \hat{P}} \hat{x}_j(\mathbf{r}, \hat{\sigma}_i)$

D3.

$$v'_j = \underline{h}_j(m_i, \mathbf{v}, \mathbf{w}, s; \hat{P}) \leq \inf_{\substack{\mathbf{z} \in \hat{\mathcal{X}}(i,s) \cap H(s), \mathbf{r} \in \hat{P} \\ z_j = v_j(s)}} \mathcal{F}_j(m_i, \mathbf{z}, \mathbf{r}, s)$$

D4.

$$w'_j = \overline{h}_j(m_i, \mathbf{v}, \mathbf{w}, s; \hat{P}) \geq \sup_{\substack{\mathbf{z} \in \hat{\mathcal{X}}(i,s) \cap H(s), \mathbf{r} \in \hat{P} \\ z_j = w_j(s)}} \mathcal{F}_j(m_i, \mathbf{z}, \mathbf{r}, s)$$

i.e., replacing the strict inequalities with regular inequalities (see [11, Remark 12.X]). Furthermore, by asserting regular inequalities, the result of the theorem also permits

$$\mathbf{v}(s) \leq \hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s) \leq \mathbf{w}(s), \qquad \forall(\hat{\mathbf{p}}, s) \in \hat{P} \times \hat{I}_i,\ i \in \mathcal{E}.$$

We will assume that the uniqueness of the constructed bounding differential equations hold $\forall \hat{\mathbf{p}} \in \hat{P}, s \in [0, n_e]$, and so it is understood that reference to Theorem 1 also refers to the regular inequalities just described.

The bounding set $\hat{\mathcal{X}}(i, \hat{\mathbf{p}}, s)$ makes it possible to tighten the implied state bounds obtained when physical insight from the problem in the form of invariants (e.g., conservation laws) and bounds is available, see [12], [13] for examples.

Theorem 1 enables a hybrid system of bounding differential equations to be constructed to obtain the following set,

$$\hat{X}(i, \underline{s}; \hat{P}) \equiv \{\mathbf{z} \mid \mathbf{v}(\underline{s}) \leq \mathbf{z} \leq \mathbf{w}(\underline{s})\}. \tag{9}$$

The most difficult aspect of applying the theorem lies in obtaining the extrema in conditions D1 – D4. As stated in [13], while computing the exact solution to the optimization problem would yield the tightest bounds possible from the theorem, actually solving the optimization problems at each integration step in a numerical integration would be a prohibitively expensive task. Hence, in practice, the solutions to the optimization problems are estimated by interval arithmetic [14] pointwise in time.

Corollary 1: Consider $\hat{\mathcal{H}}$ subject to assumptions B1-B3. Define the following interval valued functions,

$$Y(\hat{\sigma}_1) = [\mathbf{y}^L(\hat{\sigma}_1), \mathbf{y}^U(\hat{\sigma}_1)] = \mathbf{E}(0)P + \mathbf{J}(0)\Delta + \mathbf{k}(0), \tag{10}$$

$$Y(\hat{\sigma}_{l+1}) = [\mathbf{y}^L(\hat{\sigma}_{l+1}), \mathbf{y}^U(\hat{\sigma}_{l+1})] = \mathbf{D}(l)[\mathbf{v}(\hat{\tau}_l), \mathbf{w}(\hat{\tau}_l)] + \mathbf{E}(l)P + \mathbf{J}(l)\Delta + \mathbf{k}(l),\ \forall l = 1, \ldots, n_e - 1, \tag{11}$$

and let $\Gamma_j(m_i, v_j, Z(j, i, s), \hat{P}, s) = [\gamma_j^L(m_i), \gamma_j^U(m_i)]$ and $\Lambda_j(m_i, w_j, Z(j, i, s), \hat{P}, s) = [\lambda_j^L(m_i), \lambda_j^U(m_i)]$ be inclusion monotonic interval extensions of $\mathcal{F}_j(m_i, \hat{x}_j, \hat{\mathbf{x}}_{k\neq j}, \hat{\mathbf{p}}, s)$ for all $i \in \mathcal{E}$, $j = 1, \ldots, n_x$, where

$$Z(j, i, s) = \{\mathbf{z}_{k\neq j} \mid cmax(\mathbf{v}_{k\neq j}(s), \boldsymbol{\varphi}_{k\neq j}(i, s)) \leq \mathbf{z}_{k\neq j} \leq cmin(\mathbf{w}_{k\neq j}(s), \boldsymbol{\psi}_{k\neq j}(i, s))\},$$

and $\hat{X}(i, s; \hat{P}) = [\boldsymbol{\varphi}(i, s), \boldsymbol{\psi}(i, s)]$ is defined in (9) and obtained from Theorem 1. Then, for all $j = 1, \ldots, n_x$, $s \in [i-1, i]$ and $i \in \mathcal{E}$, the following system of differential equations and initial conditions

$$v_j' = \gamma_j^L(m_i, \mathbf{v}, \mathbf{w}, \hat{\mathbf{p}}^L, \hat{\mathbf{p}}^U, s), \qquad v_j(\hat{\sigma}_i) = y_j^L(\hat{\sigma}_i), \tag{12}$$

$$w_j' = \lambda_j^U(m_i, \mathbf{v}, \mathbf{w}, \hat{\mathbf{p}}^L, \hat{\mathbf{p}}^U, s), \qquad w_j(\hat{\sigma}_i) = y_j^U(\hat{\sigma}_i), \tag{13}$$

bounds the transformed hybrid system,

$$\mathbf{v}(s) \leq \hat{\mathbf{x}}(i, \hat{\mathbf{p}}, s) \leq \mathbf{w}(s), \qquad \forall(\hat{\mathbf{p}}, s) \in \hat{P} \times \hat{I}_i,\ i \in \mathcal{E}.$$

Next, we will show how convex and concave relaxations for the states of the transformed hybrid system can be constructed.

Definition 11: Consider the following functions, $f : Z \times \hat{P} \times S \to \mathbb{R}$ and $\mathbf{z} : S \to Z$ where $Z \subset \mathbb{R}^{n_x}$, $\hat{P} \subset \mathbb{R}^{n_p+n_e}$, $S \subset \mathbb{R}$ and $f(\cdot, \underline{s})$ is differentiable on some suitable open set containing $Z \times \hat{P}$ for each $\underline{s} \in S$. Define the funtion $\mathcal{L}_f|_{\boldsymbol{\zeta}^*(s)} : Z \times \hat{P} \times S \to \mathbb{R}$ to be a linearization of f at the point $\boldsymbol{\zeta}^*(s) = (\mathbf{z}^*(s), \hat{\mathbf{p}}^*)$ where $(\mathbf{z}^*(s), \hat{\mathbf{p}}^*) \in Z \times \hat{P}$, and given by the following:

$$\mathcal{L}_f|_{\boldsymbol{\zeta}^*(s)}(\mathbf{z}, \hat{\mathbf{p}}, s) = f(\mathbf{z}^*, \hat{\mathbf{p}}^*, s) + \sum_{k=1}^{n_x} \left.\frac{\partial f}{\partial z_k}\right|_{(\boldsymbol{\zeta}^*(s), s)} \Big(z_k(s) - z_k^*(s)\Big) + \sum_{k=1}^{n_p} \left.\frac{\partial f}{\partial \hat{p}_k}\right|_{(\boldsymbol{\zeta}^*(s), s)} \big(\hat{p}_k - \hat{p}_k^*\big).$$

Theorem 2: For $i \in \mathcal{E}$ and $j = 1, \ldots, n_x$, define the funtions $u_j(m_i, \cdot, \underline{s}) : \hat{X}(i, s; \hat{P}) \times \hat{P} \to \mathbb{R}$ and $o_j(m_i, \cdot, \underline{s}) : \hat{X}(i, s; \hat{P}) \times \hat{P} \to \mathbb{R}$ for each fixed $\underline{s} \in \hat{I}_i$. Let the following conditions be satisfied for all $i \in \mathcal{E}$, $j = 1, \ldots, n_x$ and each fixed $\underline{s} \in \hat{I}_i$,

E1. $u_j(m_i, \cdot, \underline{s})$ is a convex underestimator and $o_j(m_i, \cdot, \underline{s})$ is a concave overestimator for $\mathcal{F}_j(m_i, \cdot, \underline{s})$ on $\hat{X}(i, \underline{s}; \hat{P}) \times \hat{P}$;

E2. $u_j(m_i, \cdot, \underline{s})$ and $o_j(m_i, \cdot, \underline{s})$ are differentiable on some suitable open set containing $\hat{X}(i, s; \hat{P}) \times \hat{P}$ along some reference trajectory $\boldsymbol{\zeta}^*(s) = (\mathbf{z}^*(s), \hat{\mathbf{p}}^*) \in \hat{X}(i, s; \hat{P}) \times \hat{P}$;

and the following ODE system be constructed,

$$c_j' = h_{c,j}(m_i, \mathbf{c}, \mathbf{C}, \hat{\mathbf{p}}, s) = \inf_{\substack{\mathbf{z}\in\mathcal{C}(\hat{\mathbf{p}},s) \\ z_j = c_j(s)}} \mathcal{L}_{u_j(m_i,\cdot)}(\mathbf{z}, \hat{\mathbf{p}}, s)\big|_{(\boldsymbol{\zeta}^*(s), s)},\ s \in (i-1, i],$$

$$C_j' = h_{C,j}(m_i, \mathbf{c}, \mathbf{C}, \hat{\mathbf{p}}, s) = \sup_{\substack{\mathbf{z}\in\mathcal{C}(\hat{\mathbf{p}},s) \\ z_j = C_j(s)}} \mathcal{L}_{o_j(m_i,\cdot)}(\mathbf{z}, \hat{\mathbf{p}}, s)\big|_{(\boldsymbol{\zeta}^*(s), s)},\ s \in (i-1, i],$$

with initial conditions for each epoch $\hat{I}_i$ given by

$$\mathbf{c}(\hat{\mathbf{p}}, 0) = \mathbf{C}(\hat{\mathbf{p}}, 0) = \mathbf{E}(0)\mathbf{p} + \mathbf{J}(0)\boldsymbol{\delta} + \mathbf{k}(0), \tag{14}$$

$$[\mathbf{c}(\hat{\mathbf{p}}, \hat{\sigma}_{l+1}), \mathbf{C}(\hat{\mathbf{p}}, \hat{\sigma}_{l+1})] = \mathbf{D}(l)[\mathbf{c}(\hat{\mathbf{p}}, \hat{\tau}_l), \mathbf{C}(\hat{\mathbf{p}}, \hat{\tau}_l)] + \mathbf{E}(l)\mathbf{p} + \mathbf{J}(l)\boldsymbol{\delta} + \mathbf{k}(l), \tag{15}$$

for $l = 1, \ldots, n_e - 1$, where $\mathcal{C}(\hat{\mathbf{p}}, s) = \{\mathbf{z} \mid \mathbf{c}(\hat{\mathbf{p}}, s) \leq \mathbf{z} \leq \mathbf{C}(\hat{\mathbf{p}}, s)\}$. Then, for each fixed $\underline{s} \in \hat{I}_i$, $\mathbf{c}(\cdot, \underline{s})$ is a convex underestimator and $\mathbf{C}(\cdot, \underline{s})$ is a concave overestimator for $\hat{\mathbf{x}}(i, \cdot, \underline{s})$ on $\hat{P}$, for all $i \in \mathcal{E}$.

Note that the infima and suprema in Theorem 2 are attained at the vertices of the set $\mathcal{C}(\hat{\mathbf{p}}, s)$ due to the properties of the linearizations, and are easily computed, see [12, Theorem 6.16]. The next theorem demonstrates the convergence properties of the convex relaxations constructed using the relaxation techniques presented in this section.

Theorem 3: Consider the following convex relaxation of

(5),

$$\hat{U}(\hat{\mathbf{p}};\hat{P}) = \sum_{i=1}^{n_e} \left\{ \sum_{j=1}^{n_{\phi i}} \hat{\psi}_{ij}\Big(\mathbf{c}(\hat{\mathbf{p}},\hat{\alpha}_{ij}),\mathbf{C}(\hat{\mathbf{p}},\hat{\alpha}_{ij}),\hat{\mathbf{p}};,\hat{X}(i,\hat{\alpha}_{ij};\hat{P}),\hat{P}\Big) + \int_{i-1}^{i} \hat{u}_i\Big(\mathbf{c},\mathbf{C},\hat{\mathbf{p}},s;\hat{X}(i,s;\hat{P}),\hat{P}\Big)v(\boldsymbol{\delta},s)\ \mathrm{d}s \right\}, \quad (16)$$

where $\hat{\psi}_{ij}$ and $\hat{u}_i$ are constructed using any relaxation technique that possesses a consistent bounding operation [10, Definition IV.4, pg. 128], the convex and concave relaxations for the state and derivatives are constructed using Theorem 2, and the estimation of the state bounds constructed using Corollary 1. If the interval vector $\hat{P}_k$ in any partition on $\hat{P}$ approaches degeneracy $\hat{P}^*$, then the lower bound on this partition $\hat{U}(\hat{\mathbf{p}};\hat{P}_k)$ converges pointwise to the objective function value $\hat{F}(\hat{\mathbf{p}})$ in this same partition.

V. CONCLUSION

The global optimization problem with continuous time linear hybrid systems embedded has been considered where the embedded systems have varying time transitions. The CPET has been utilized to transform the problem into a global optimization problem with nonlinear hybrid systems embedded where the transitions are now fixed in time. A method of constructing convex relaxations for the transformed problem has been developed that is shown to be convergent within a branch-and-bound framework.

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