Rendezvous under noisy measurements

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***Abstract*— We describe a distributed algorithm for solving the rendezvous problem based on consensus protocols. We extend our previous work by considering the case when the evolution of the system is affected by measurement noise. The consensus formulation allows us to derive conditions for convergence of the system towards a ball with a finite radius. We derive an upper bound on the radius of the ball and show how it depends on the magnitude of the noise. We also present examples showing that the bound is tight and can be in fact achieved, but that typically the convergence is much better than the bound suggests.**

I. INTRODUCTION

In robotic networks, *rendezvous* refers to the task of controlling agents in a formation towards a common location without active communication among them. Several distributed algorithms for solving the rendezvous problem are currently available in the literature. Originally presented in [1], the problem has been extended to both synchronous [2] and asynchronous [3], [4] cases. The proposed algorithms are all distributed in the sense that each robot takes the decision based only on the information it can gather from a certain subset of the rest of the agents in the formation.

Parallel to this work, there has been much research in the control community on consensus algorithms. Originally described in [5] and applied to parallel computing [6], [7], consensus protocols were brought to the control community by [8]–[10] and have been since extensively studied. Variations of the consensus protocol have also been studied. For instance the so called gossip algorithms by the computer network community [11], also known as *aggregation protocols* [12], are examples of such alternative formulations.

Observe that if the agents move freely in $\mathbb{R}^n$, then achieving rendezvous for the agents is equivalent to them achieving a consensus on their locations in $\mathbb{R}^n$. This relationship between consensus and rendezvous has not been unnoticed to researchers [13]–[15]. Nonetheless, an explicit description of the nature of such relationship was not presented until [16]. In that work, we show that a broad family of rendezvous algorithms can be seen as an application of the consensus protocol, thus inheriting the well-studied convergence properties of this family of protocols.

Most existing studies of the rendezvous assume that the evolution of the system is deterministic: there are no random influences on the measurements and the evolution of the state. This assumption is difficult to justify in real life applications, where both measurements and the evolution of the system have some degree of uncertainty. Some studies [17], [18] have analyzed the effect of noise for a particular version of the consensus protocol, and [19] considers the effect of uniformly distributed measurement noise for a particular class of rendezvous algorithms. We generalize these results and study a general class of the consensus (and rendezvous) algorithms, where the only assumption on the noise is that it is zero-mean and bounded.

In this work, we show how noisy rendezvous can be reduced to noisy consensus (for which the general solution still is, to the best of our knowledge, an open problem). We show that in the presence of bounded noise, the consensus is *almost attained*, meaning that the agents will converge to some finite ball. We then discuss how the radius of the ball can be reduced given that the robots can only traverse finite distances in finite time. We conclude the paper with simulation results showing that our bounds are tight, but that they are typically conservative.

II. PRELIMINARIES AND NOTATION

A. Consensus algorithms

Consensus protocols were introduced by Tsitsiklis in 1984 [5], and then re-discovered independently by the control community with the work of Jadbabaie *et al.* [8]. Subsequent research inspired by this work led to the continuous time version of the protocol [9], and was generalized in [10]. We refer the interested reader to the survey [20] and the references therein. For purposes of this paper, we will focus on the discrete time consensus algorithm.

Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a vector, and let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix with the following properties:

1) $\mathbf{A}$ is primitive, i.e. there is a positive integer k such that $\mathbf{A}^k$ has all its entries positive, and
2) $\mathbf{A}$ is stochastic, i.e. all its entries are non-negative, and the sum of the entries in each column is equal to 1.

It follows from Geršgorin's circle theorem [21] and the Perron-Frobenius Theorem for primitive matrices [22] that $\mathbf{A}$ has all but one of its eigenvalues in the interior of the complex unit circle and that the remaining eigenvalue is equal to 1 and has $\mathbf{1}$, the vector in $\mathbb{R}^n$ which has all its entries equal to 1, as its associated eigenvector. From here it follows that the discrete time linear system given by $\mathbf{x}_m = \mathbf{A}^m\mathbf{x}_0$ is stable, and converges to an equilibrium point which is a scalar multiple of $\mathbf{1}$, the eigenvector associated with the eigenvalue 1 [23]. We call such matrix $\mathbf{A}$ a *consensus matrix*.

It is shown, among others in [5], [10], that if $\{\mathbf{A}_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ are all consensus matrices, and the matrix $\mathbf{A}_i$ is such that its positive entries are uniformly bounded below by a

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positive real number α (i.e., independently of i), then

$$\lim_{m\to\infty}\prod_{i=1}^{m}\mathbf{A}_i=\frac{1}{n}\mathbf{1}\mathbf{1}^T. \tag{1}$$

The following lemma is taken from [22].

Lemma 1: Let $\mathbf{A},\,\mathbf{B}\in\mathbb{R}^{n\times n}$ be two non-negative matrices. If both $\mathbf{A}$ and $\mathbf{B}$ have its zero and positive entries in the same positions, then either both matrices are primitive, or none of them is. I.e., for non-negative matrices the condition of being primitive depends only on the profile of the matrix.

B. Proximity graphs

The concept of *proximity graph* can be used to introduce the notion of *neighbor* which will be fundamental for our algorithm. We assume the reader is familiar with the basic concepts of graph theory, as presented for instance in [24], [25]. During this section we will be following the presentation in [2].

Let $\mathbb{F}(\mathbb{R}^n)$ be the set of finite point sets in $\mathbb{R}^n$. We denote by $\mathcal{P}=\{p_1,\ldots,p_m\}\subset\mathbb{R}^n$ a typical element of $\mathbb{F}(\mathbb{R}^n)$, where $p_1,\ldots,p_m$ are distinct points. Let $\mathbb{G}(\mathbb{R}^n)$ be the set of undirected graphs whose vertex set belongs to $\mathbb{F}(\mathbb{R}^n)$.

A *proximity graph function* $\mathcal{G}:\mathbb{F}(\mathbb{R}^n)\to\mathbb{G}(\mathbb{R}^n)$ is a map that assigns to each element $\mathcal{P}\in\mathbb{F}(\mathbb{R}^n)$ an undirected graph with vertices given by the elements of $\mathcal{P}$, and with the set of edges $\mathcal{E}$ being defined by the function $\mathcal{E}_{\mathcal{G}}:\mathbb{F}(\mathbb{R}^n)\to\mathbb{F}(\mathbb{R}^n\times\mathbb{R}^n)$ contained in $\mathcal{P}\times\mathcal{P}\setminus\text{diag}(\mathcal{P})$, where $\text{diag}(\mathcal{P})=\{(p,p)|p\in\mathcal{P}\}$.

We say that $\mathbf{A}_{\mathcal{G}}\in\mathbb{R}^{n\times n}$ is the matrix induced by the proximity graph $\mathcal{G}(\mathcal{P})$ if its entries are non-negative, has non-zero diagonal terms, and the entry $a_{ij}\neq 0$ if and only if $(p_i,p_j)\in\mathcal{E}$. In the case that the proximity graph is undirected, then $(p_i,p_j)\in\mathcal{E}\Rightarrow(p_j,p_i)\in\mathcal{E}$. In an abuse of notation, we will denote the edges of the graph $\mathcal{G}$ by either $\mathcal{E}$ or the function $\mathcal{E}_{\mathcal{G}}$ that defines the set.

Lemma 2: If $\mathcal{G}(\mathcal{P})$ is connected, then $\mathbf{A}_{\mathcal{G}}$ is primitive.

Proof: By Lemma 1 it is enough to show the result when the positive entries of the matrix $\mathbf{A}_{\mathcal{G}}$ are equal to 1. Observe that the (i,j) entry in the product $\mathbf{A}_{\mathcal{G}}^k$ is given by $\sum_{(l_1,l_2,\ldots l_{k-1})\subseteq\{1,\ldots,n\}^k}a_{il_1}a_{l_1l_2}\ldots a_{l_{k-1}j}$, which will be positive if and only if it is possible to arrive from position p_i to position p_j, passing by at most k different vertices (the entry in the position (i,j) will be the number of ways of doing so). Since $\mathcal{G}(\mathcal{P})$ is connected, the result follows. ■

C. Contraction rate for a consensus matrix

Let $\mathbf{C}\in\mathbb{R}^{n\times n}$ be a consensus matrix. If the matrix represents a connected graph, and its positive entries are bounded below by $\epsilon>0$, we characterize the set $\mathcal{C}_\epsilon^n$ that describes the elements of such matrices as

$$\mathcal{C}_\epsilon^n=\mathcal{B}_\epsilon^n\cap\mathcal{S}_\epsilon^n\cap\mathcal{P}_\epsilon^n\cap\mathcal{D}_\epsilon^n, \tag{2}$$

where for $1\leq i,\,j\leq n$

$$\mathcal{B}_\epsilon^n=\{a_{i,j}:\,a_{i,j}\in\{0\}\cup[\epsilon,\,1]\},$$
$$\mathcal{S}_\epsilon^n=\{a_{i,j}:\,\sum_{k=1}^{n}a_{i,k}=1\},$$
$$\mathcal{P}_\epsilon^n=\{a_{i,j}:\,\sum_{l_1,\ldots,l_{n-1}=1}^{n}a_{il_1}a_{l_1l_2}\ldots a_{l_{n-1}j}\geq\epsilon^n\},$$
$$\mathcal{D}_\epsilon^n=\{a_{i,j}:a_{i,i}\in[\epsilon,\,1]\},$$

The set $\mathcal{C}_\epsilon^n$ characterizes all the consensus matrices we are interested in. Observe that since $\mathcal{C}_\epsilon^n$ is the finite intersection of closed sets, it is closed. Since $a_{i,j}\in\{0\}\cup[\epsilon,1]$, which is a bounded set, so is $\mathcal{C}_\epsilon^n$. Hence this set is compact.

For a particular $\mathbf{C}\in\mathcal{C}_\epsilon^n$, we denote its n eigenvalues (not necessarily distinct) by $\lambda_1,\ldots,\lambda_n$, where $|\lambda_1|\leq\cdots\leq|\lambda_{n-1}|<\lambda_n=1$. As a consequence of Rouché's theorem and the inverse mapping theorem for analytic functions [26], the roots of a polynomial are continuous functions of its coefficients. Since the coefficients of the characteristic polynomial of a matrix are a continuous function of its entries, the set of eigenvalues λ_i for the family of consensus matrices we are interested in is a continuous function on $\mathcal{C}_\epsilon^n$ which is compact. Henceforth, there exists $\rho<1$ such that $|\lambda_{n-1}|\leq\rho$, for every $\mathbf{C}\in\mathcal{C}_\epsilon^n$.

Let $\mathbf{v}_i$ be the eigenvector associated to the eigenvalue λ_i. If λ_i has multiplicity greater than 1, we let $\mathbf{v}_i$ be its extended eigenvector of the multiplicity of its eigenvalue. The set $\mathcal{V}=\{\mathbf{v}_i\}_{i=1}^n$ is linearly independent, hence span $(V)=\mathbb{R}^n$.

Let $\Delta=\text{span}(\mathbf{1})$ be the diagonal on $\mathbb{R}^n$. Given $\mathbf{C}\in\mathcal{C}_\epsilon^n$, we denote by $\Delta_{\mathbf{C}}^{C}$ the complement of Δ which is invariant under the action of $\mathbf{C}$; i.e. $\Delta_{\mathbf{C}}^{C}=\text{span}(\mathcal{V}\setminus\{\mathbf{1}\})$. Note that $\mathbf{C}$ acts as the identity on Δ.

Recall that $\mathbb{R}^n=\Delta\oplus\Delta_{\mathbf{C}}^{C}$. Let $\mathbf{v}\in\mathbb{R}^n$. Write $\mathbf{v}=\mathbf{v}_\Delta+\mathbf{v}_{\Delta_{\mathbf{C}}^{C}}$. Observe that for the elements of $\mathbf{v}$ to be in consensus, we need $\mathbf{v}_{\Delta_{\mathbf{C}}^{C}}=0$. If $v_i,\,v_j$ denote the i and j entries of $\mathbf{v}$, then

$$\begin{aligned}|v_i-v_j| &= \left|\left((v_\Delta)_i+\left(v_{\Delta_{\mathbf{C}}^{C}}\right)_i\right)-\left((v_\Delta)_j+\left(v_{\Delta_{\mathbf{C}}^{C}}\right)_j\right)\right|\\ &= \left|\left(v_{\Delta_{\mathbf{C}}^{C}}\right)_i-\left(v_{\Delta_{\mathbf{C}}^{C}}\right)_j\right|\leq\left|\left(v_{\Delta_{\mathbf{C}}^{C}}\right)_i\right|+\left|\left(v_{\Delta_{\mathbf{C}}^{C}}\right)_j\right|\\ &\leq \sqrt{2}\left\|\mathbf{v}_{\Delta_{\mathbf{C}}^{C}}\right\|.\end{aligned} \tag{3}$$

Given $\mathbf{v}$, its norm in $\Delta_{\mathbf{C}}^{C}$ is a continuous function of $\mathbf{C}\in\mathcal{C}_\epsilon^n$. Since $\mathcal{C}_\epsilon^n$ is compact, there is a matrix $\mathbf{C}^\star\in\mathcal{C}_\epsilon^n$ for which the norm of $\mathbf{v}$ in $\Delta_{\mathbf{C}^\star}^{C}$ is maximum. Let $\mathbf{r}\in\Delta_{\mathbf{C}}^{C}$. Since $\Delta_{\mathbf{C}}^{C}$ is invariant under $\mathbf{C}$, then we can restrict the norm $\|\cdot\|$ in $\mathbb{R}^n$ to a norm in this invariant subspace. We will denote such norm as $\|\cdot\|_{\Delta_{\mathbf{C}}^{C}}$. Observe that

$$\|\mathbf{C}\mathbf{r}\|_{\Delta_{\mathbf{C}}^{C}}\leq\|\mathbf{C}\|\cdot\|\mathbf{r}\|_{\Delta_{\mathbf{C}}^{C}}\leq\rho\|\mathbf{r}\|_{\Delta_{\mathbf{C}}^{C}}. \tag{4}$$

Therefore, if $\mathbf{v}'=\mathbf{C}\mathbf{v}$, from (3) and (4) we obtain

$$\left|(v')_i-(v')_j\right|\leq\sqrt{2}\,\|(\mathbf{C}\mathbf{v})\|_{\Delta_{\mathbf{C}}^{C}}\leq\sqrt{2}\rho\left\|\mathbf{v}_{\Delta_{\mathbf{C}^\star}^{C}}\right\|. \tag{5}$$

This gives us a uniform bound for the decay rate between any two elements in $\mathbf{v}$. In particular, we have this bound for the elements that attain the *diameter* of $\mathbf{v}$ (the ones that maximize the left-hand side in 3). This allow us to establish the following result:

Theorem 3: The rate of convergence to consensus under matrices in $\mathcal{C}_\epsilon^n$ is at least exponential of rate ρ.

This result is well known in the literature on consensus algorithms, and for some particular matrices tighter convergence bounds have been obtained in the past. Some partial results on the convergence of the algorithm when the topology of the graph changes were first proposed in [27], [28]. More recently, [29] establishes better bounds for convergence. For the case when noise is present, in [18] some convergence rates are derived, but the authors exploit a very particular configuration of the evolution matrix they are considering in their work.

III. MODEL

Let $\mathcal{R}$ be a robotic network as defined in [30]. Let $\{a_i\}_{i=1}^N$ denote the set of agents and let $\{q_i\}_{i=1}^N \in \mathbb{F}(\mathbb{R}^n)$ be their positions with respect to a fixed coordinate frame $\mathcal{Q}$. In an abuse of notation, we will denote the set of points, with respect to the frame $\mathcal{Q}$, simply as $\mathcal{R}$. We will assume that the agents have no knowledge about $\mathcal{Q}$.

We assume that each one of the robots is capable of identifying those agents that satisfy a certain criteria $\mathfrak{C}$, which induces the relationship $\sim$ that defines the edges in the proximity graph $\mathcal{G}(\mathcal{R})$: $(p_i, p_j) \in \mathcal{E}_{\mathcal{G}(\mathcal{R})}$ if $p_i \sim p_j$; i.e. if p_j satisfies the criteria $\mathfrak{C}$ with respect to p_i[1]. Observe that $\sim$ is not necessarily a symmetric relation. For each agent a_i, we denote the set of all the agents $a_j,\, j \neq i$ that satisfy the criteria $\mathfrak{C}$ by $\mathcal{N}_i :== \{a_j \in \mathcal{R} | a_i \sim a_j\}$.

We will refer to this set as the set of *neighbors* of a_i, or simply as *the neighbors* when the agent a_i is clear from the context. Some examples of $\mathfrak{C}$ are *being closer than certain distance* d or *being neighbors in the sense of Voronoi*. In [16] we assumed that each agent a_i was able to correctly estimate the positions of its neighbors with respect to an arbitrary coordinate frame $\mathcal{Q}_i$ defined by itself at each time instant. We showed that convergence to rendezvous was independent of such reference frame. In this paper, we focus on the case when if $a_i \sim a_j$, then the position p_j of agent a_j is estimated by agent a_i as $p_{i,j} = p_j + n_{i,j}$ where $n_{i,j}$ is some measurement noise. We will show that when $n_{i,j}$ is *small*, then convergence to *almost* rendezvous is guaranteed.

Based on the information the agents gather from the observations of their neighbors they will update their position, with respect to the coordinate frame $\mathcal{Q}$, as

$$q_i[m+1] = q_i[m] + u_i[m], \tag{6}$$

where the control law u_i for the motion of agent a_i is based on distributed consensus and is described next.

IV. CONSENSUS-BASED RENDEZVOUS

[1]We will either say $p_i \sim p_j$ or $a_i \sim a_j$ to denote this relation.

Algorithm 1 Consensus-Based Rendezvous

Require: Agent a_i at time m

1: Identify the set of neighbors $\mathcal{N}_i = \left\{a_{i_1}, a_{i_2}, \ldots, a_{i_{r_i(m)}}\right\}$.
2: Evaluate the position p_{i,i_j}, $1 \leq j \leq r_i(m)$ of each neighbor, and its own position p_{i,i_0} with respect to an arbitrary coordinate frame $\mathcal{Q}_i$.
3: Compute $p_i = \sum_{j=0}^{r_i(m)} \lambda_{i,j} p_{i,i_j}$, where $\lambda_{i,j} > \epsilon > 0$ and $\sum_{j=0}^{r_i(m)} \lambda_{i,j} = 1$.
4: Set $u_i[m] = \varrho\,(p_i - p_{i,i_0})$, where $0 < \varrho < 1$.

Algorithm 1 is the *Consensus-Based Rendezvous* (CBR) as presented in [16] when no noise is present in the measurements. The main change in the algorithm would be that if noise is present in the estimation, then line 3 would become

$$p_i = \sum_{j=0}^{r_i(m)} \lambda_{i,j} p_{i,i_j} + \sum_{j=0}^{r_{i(m)}} \lambda_{i,j} n_{i,j}, \tag{7}$$

where $n_{i,i} = 0$. The only assumption we will make about the noise is that it is zero mean and it has bounded support.

As in [16], we can show that in the noisy scenario the updates can be made invariant with respect to the coordinate frame that each agent chooses. The proof is analogous to Lemma 3 in that paper, hence omitted. The orthonormality in the matrix that defines the change of coordinates implies that the noise levels remain invariant.

Lemma 4 (Lemma 3 in [16]): The evolution of each agent is independent of the local frame $\mathcal{Q}_i$ it chooses to implement the CBR algorithm.

Due to this lemma, we can stack together the equations for each agent, and write the evolution of the system in a matrix form as $\mathbf{q}[m+1] = \mathbf{I}\,\mathbf{q}[m] + \mathbf{U}[m] + \mathfrak{N}[m]$, where $\mathbf{I} \in \mathbb{R}^{N\times N}$ is the identity matrix, $\mathbf{q} \in \mathbb{R}^{N\times n}$, $\mathbf{U} \in \mathbb{R}^{N\times n}$ and the noise matrix $\mathfrak{N}[m] \in \mathbb{R}^{N\times n}$. Under the assumption that the proximity graph $\mathcal{G}(\mathcal{R})$ is connected, then the induced matrix $\mathbf{A}_\mathcal{G}$, where the entry $a_{i,j} = \lambda_{i,j}$, is a consensus matrix. This makes $\mathbf{U}[m] = \varrho\,(\mathbf{A}_\mathcal{G} - \mathbf{I})\,\mathbf{q}[m]$, $\varrho \in (0,1)$ and thus we can rewrite the discrete time system as

$$\mathbf{q}[m+1] = [(1-\varrho)\mathbf{I} + \varrho\mathbf{A}_\mathcal{G}]\,\mathbf{q}[m] + \mathfrak{N}[m], \tag{8}$$

Since the matrix $[(1-\varrho)\mathbf{I} + \varrho\mathbf{A}_\mathcal{G}] = \mathbf{C}_\mathcal{G}[m]$ is also a consensus matrix, we rewrite (8) as

$$\mathbf{q}[m+1] = \mathbf{C}_\mathcal{G}[m]\mathbf{q}[m] + \mathfrak{N}[m], \tag{9}$$

Remark: Observe that although the derivation of (9) is made under the assumption of a uniform ϱ for each agent, it is possible to derive an equivalent formulation if each robot has its own ϱ_i.

For simplicity in the notation, we are going to drop the dependency on the proximity graph $\mathcal{G}$. Under the action of $\mathbf{C}$, the evolution of the formation can be viewed as the joint evolution of n individual consensus systems in $\mathbb{R}^N$, all of them sharing the same consensus matrix. From now on, we will consider a single vector in $\mathbb{R}^N$, knowing that the results extend trivially to $\mathbb{R}^{N\times n}$.

Suppose that at time $m+1$, the system will evolve according to the matrix $\mathbf{C} \in \mathcal{C}_{\epsilon}^{N}$. Let Δ be the diagonal in $\mathbb{R}^N$ and let $\Delta_{\mathbf{C}}^{C}$ be the complement of Δ invariant under the action of $\mathbf{C}$. Since $\mathbb{R}^N = \Delta \oplus \Delta_{\mathbf{C}}^{C}$

$$\begin{aligned} \mathbf{q}[m+1]_{\Delta} &= (\mathbf{C}[m]\mathbf{q}[m])_{\Delta} + \mathfrak{N}[m]_{\Delta}, \\ \mathbf{q}[m+1]_{\Delta_{\mathbf{C}}^{C}} &= (\mathbf{C}[m]\mathbf{q}[m])_{\Delta_{\mathbf{C}}^{C}} + \mathfrak{N}[m]_{\Delta_{\mathbf{C}}^{C}}. \end{aligned}$$

Observe that for the purpose of reaching the consensus, the norm $\|\cdot\|_{\Delta_{\mathbf{C}}^{C}}$ indicates how *far* the formation is from consensus, so is enough to focus on this subspace. We say that the formation reaches *δ-consensus* if $\|\mathbf{q}\|_{\Delta_{\mathbf{C}}^{C}} < \delta$. For purposes of reaching consensus, the effects of $\mathfrak{N}[m]_{\Delta}$ are negligible. Since $\Delta_{\mathbf{C}}^{C}$ is invariant under $\mathbf{C}$, we can thus concentrate only on

$$\mathbf{q}[m+1]_{\Delta_{\mathbf{C}}^{C}} = (\mathbf{C}[m]\mathbf{q}[m])_{\Delta_{\mathbf{C}}^{C}} + \mathfrak{N}[m]_{\Delta_{\mathbf{C}}^{C}}. \quad (10)$$

For simplicity in the notation, we will drop the index $\Delta_{\mathbf{C}}^{C}$, although from now on we restrict ourselves to this subspace. Consider $\mathbf{C}\mathbf{q}[m] + \mathfrak{N}[m]$. From (4) we have that

$$\|\mathbf{q}[m+1]\| \leq \rho\|\mathbf{q}[m]\| + \|\mathfrak{N}[m]\|. \quad (11)$$

Thus we have the following lemma:

Lemma 5: If at time m it is true that

$$\|\mathfrak{N}[m]\| < (1-\rho)\|\mathbf{q}[m]\|, \quad (12)$$

then $\|\mathbf{q}[m+1]\| < \|\mathbf{q}[m]\|$.

We now show a stronger result: as long as the noise is bounded (uniformly in time), the formation will converge to a finite ball. This assumption is quite natural since the noise is the result of the measurement errors due to imperfect sensors, which have a finite range.

Theorem 6: Suppose that the noise is uniformly bounded by σ, i.e. $\|\mathfrak{N}[m]\| \leq \sigma < \infty$ for every time m. Then, as long as $\|\mathbf{q}\| > \sigma/(1-\rho)$, $\|\mathbf{q}\|$ will be decreasing, and the formation will converge to $\sigma/(1-\rho)$-rendezvous.

Proof: This follows from Lemma 5. ■

Note that Theorem 6 guarantees that the formation will converge to a ball if it is not already inside it; once inside, it can escape, but it will be then driven back into the ball. We will elaborate on this in Section VI.

If we assume that the locations of the neighbors are estimated correctly when they are closer than some threshold r, from (3), by making $r = \sqrt{2}\sigma/(1-\rho)+\epsilon$ for any $\epsilon > 0$, we can guarantee perfect observations among all the agents in the formation after some finite time T. We will show that this bound for r can be actually improved if physical constraints on the motion of the agents are taken into account.

V. Robustness of the algorithm with respect to the proximity graph

So far we assumed that the proximity graph $\mathcal{G}(\mathcal{R})$, induced by the formation at time m, $m \geq 0$, is always connected and that each agent is always capable to estimate the locations of all the neighbors up to a certain error. We will now focus on what happens when the agent occasionally fails to detect one or more of its neighbors, possibly making the induced graph by the matrix $\mathbf{C}$ disconnected.

We will denote the state of the formation $\mathcal{R}$ at time m by $\mathcal{R}[m]$. We will denote the proximity graph induced at time m by $\mathcal{G}(\mathcal{R})[m]$, and its set of edges by $\mathcal{E}_{\mathcal{G}}[m]$.

We define the *real proximity graph* of the formation $\mathcal{R}$ at time m as the graph which vertices are the agents in the formation, and which edge set $\mathcal{E}'_{\mathcal{G}}[m] \subseteq \mathcal{E}_{\mathcal{G}}[m]$ contains the pair (a_i, a_j) if and only if the agent a_i does identify the agent a_j as one of its neighbors. Since communication is not allowed among the agents, the fact that agent a_i misses a_j does not imply that a_j also misses a_i even when the relationship $\sim$ happens to be symmetric. Observe that in case all neighbors are correctly detected the *real proximity graph* will coincide with $\mathcal{G}(\mathcal{R})[m]$.

We say that the formation is *strongly connected* if there exists a positive integer K such that, for every $m > 0$, the graph with vertices induced by $\mathcal{R}$, and edge set defined by the union $\mathcal{E} = \bigcup_{i=0}^{K-1} \mathcal{E}'_{\mathcal{G}}[m+i]$, is connected.

Note that the matrix $\mathbf{C}[m]$ having its entry $c_{i,j} = \lambda_{i,j}$, the weight assigned by agent a_i to the position of agent a_j by the CBR algorithm (Algorithm 1), is always stochastic. Under the assumption that the underlying graph for $\mathcal{R}$ (i.e. the real proximity graph) is strongly connected, we can state the following lemma:

Lemma 7: Assume the formation is strongly connected. Let $m > K$. Then

$$\prod_{j=1}^{K} \mathbf{C}[m-j] = \mathbf{C}_{\star}[m] \ \in \mathcal{C}_{\epsilon^K}^{N}. \quad (13)$$

Proof: The result follows from the assumption on the formation to be strongly connected. ■

Our intuition is that if we focus on the evolution of our system each K time instants, grouping the partial noises into a new noise component $\mathfrak{N}'$, we should obtain a new system which is equivalent to the one we studied before. We now proceed to formalize this point.

Let $l \geq 0$. Consider the evolution of the system between times $lK+1$ and $(l+1)K$. We will keep denoting the matrix that represents the evolution of the system at time m by $\mathbf{C}[m]$ although the induced graph might not necessarily be connected. From (9) we have

$$\begin{aligned} \mathbf{q}[lK+1] &= \mathbf{C}[lK]\mathbf{q}[lK] + \mathfrak{N}[lK] \\ \mathbf{q}[lK+2] &= \mathbf{C}[lK+1]\mathbf{q}[lK+1] + \mathfrak{N}[lK+1] \\ \vdots \quad & \vdots \quad \vdots \\ \mathbf{q}[(l+1)K] &= \mathbf{C}[(l+1)K-1]\mathbf{q}[(l+1)K-1] + \\ & \quad + \mathfrak{N}[(l+1)K-1], \end{aligned}$$

so

$$\mathbf{q}[(l+1)K] = \mathbf{C}_{\star}[(l+1)K]\mathbf{q}[lK] + $$
$$+ \sum_{j=1}^{K-1} \prod_{r=1}^{j} \mathbf{C}[(l+1)K-r]\mathfrak{N}[(l+1)K-j-1] + \mathfrak{N}[(l+1)K-1]$$

or, equivalently,

$$\mathbf{q}[(l+1)K] = \mathbf{C}_\star[(l+1)K]\mathbf{q}[lK] + \mathfrak{N}'[(l+1)K-1]. \quad (14)$$

Observe that $\|\mathfrak{N}'[(l+1)K-1]\| \leq K\sigma$ and has bounded support. If we make $\mathbf{v}[m+1] = \mathbf{q}[(m+1)K]$, $\mathbf{B}[m] = \mathbf{C}_\star[(m+1)K]$ and $\eta[m] = \mathfrak{N}'[(m+1)K-1]$, we can rewrite (14) as

$$\mathbf{v}[m+1] = \mathbf{B}[m]\mathbf{v}[m] + \eta[m], \quad (15)$$

which is equivalent to (9). The convergence analysis above thus carries over to this scenario.

Clearly, if the proximity graph is not induced by a strongly connected formation, we cannot guarantee that all the agents would converge to rendezvous but we do have analogous claims for each of the strongly connected components.

VI. Reduction of σ

Although we obtain δ-consensus as a function of $\|\mathfrak{N}\|$, if ρ is close to 1, then $\sigma/(1-\rho)$, although finite, might be large. We can actually reduce this bound by considering the physical constrains on the system.

For agent a_i, consider its next location p_i as defined in (7). Since we are dealing with physical agents with an upper bound on how fast they can move, there is a d so that $\|q_i[m+1] - q_i[m]\| \leq d$ for every m. This means that, although the point p_i obtained in (7) might satisfy $|p_i - q_i[m]| > d$, the maximum velocity constraint of the agent will take it as far as d units away from where it started in a single time interval. Let $|p_i - q_i[m]| = D > d$. The point $q_i[m+1]$ that the agent reaches is then at most $q_i[m+1] = q_i[m] + d\,(q_i[m] - p_i)/D$, or $(1-d/D)q_i[m] + (d/D)p_i$. In particular, this implies that the noise $\mathfrak{N}[m]_i = \sum_{j=0}^{r_{i(m)}} \lambda_i n_{i,\,j}$ will be, in $\Delta_{\mathbf{C}}^{C}$, reduced at least by a factor of $d/D < 1$.

Intuition here indicates that the smaller the time interval and the slower the robots are, the smaller the d and the more robust the system will be with respect to noise. In particular, considering again the case that the neighbors are located exactly when they are closer than some threshold r, we observe that for the noise in the neighbor location estimate to affect p_i, the neighbor needs to be further away than r. If we allow only pairwise updating between the agents (similar to what happens in the aggregation algorithm [12]), the results in the previous section and the discussion above suggest that noise will only affect the system when $D > \sqrt{2}\sigma/(1-\rho)$, and therefore its effect in the update will be bounded by $\frac{d}{D}\sigma < \frac{d}{\sqrt{2}\sigma/(1-\rho)}\sigma = d(1-\rho)/\sqrt{2} < d/\sqrt{2} < d$, which, surprisingly, is independent from σ.

We can pursue this even further: if the perfect localization threshold is $r = k\sigma$, for the agent a_i to obtain a noisy estimate of the position of agent a_j, their distance D needs to be larger than $k\sigma$, and thus the norm of the noise observed is bounded by $\frac{d}{D}\sigma < \frac{d}{k\sigma}\sigma = \frac{d}{k}$.

VII. Simulations

The previous equation, and Theorem 6 show that our results are only affected by a scaling factor when choosing the unit of length. For simplicity, we thus omit such unit in the following discussion.

For the simulations we present here, we first implemented our algorithm by deploying 30 agents uniformly distributed in a square region of side 12. We assumed a uniform noise distribution between $[-9,\,9]$ for the relative measurements between the agents (i.e. the $n_{i,\,j}$ in (7) rather than the $\mathfrak{N}[k]$), which is actually quite large compared to the size of the region. We chose the uniform distribution for the noise because among all the distributions with a given bounded support, this is the one that provides the least information about the process. We ran the system by setting $d = .1$, 1 and 100 and assumed a proximity graph induced by an r-disk graph with $r = 6$.

Figure 1 shows the evolution of the diameter for the same noise level and different values of d. As can be seen from the figure, for typical realizations our bound is not tight. The reason is that we derived the bound for the worst case scenarios in both the magnitude ρ of the largest eigenvalue in $(0,\,1)$, and the bound of the particular realization of the noise at each time instant. A conservative estimate for ρ would be $\rho > 2/3$, in which case we obtain, as in Theorem 6, a value of $r = \sigma/(1-\rho) > 27$. Nonetheless, the simulation shows that the formation converges to a δ-rendezvous with $\delta < r$, as observed in Figure 1.a. As expected, the δ-rendezvous level depends on d but, unlike our bound in (**??**), which gives a linear dependency, the relationship appears to be superlinear.

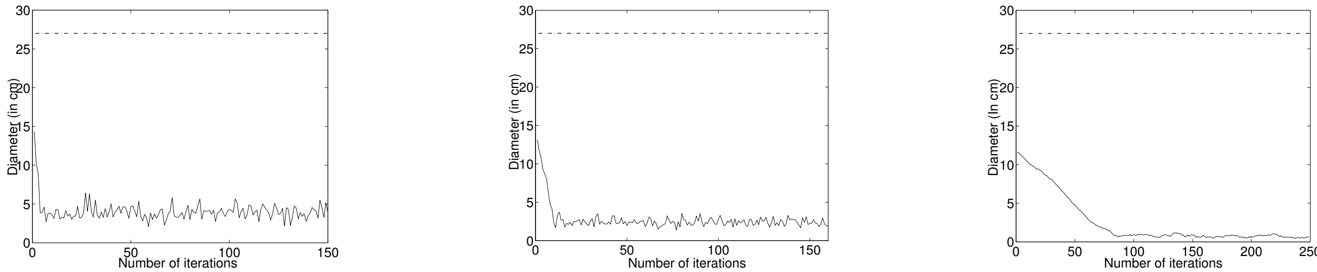
In Figure 2 we present a particular realization for which the bound in Theorem 6 is *almost* achieved. For the case in Figure 2.a, our bound is approximately 47.08 and for Figure 2.b it is approximately 5.23. The consensus matrix was constant (the proximity graph does not change over time), and the realization of the noise was the same for every time m. This shows that there are indeed situations in which, even though the measurement noise is small, the convergence radius of the formation is rather large.

VIII. Conclusion

We show that for the Consensus-Based Rendezvous (CBR) algorithm proposed in [16], under the effect of noise uniformly bounded in time, there is a finite δ such that the formation achieves δ-rendezvous. Furthermore, the rendezvous is achieved exponentially under the assumption of connectedness of the proximity graph induced by the formation. We showed that when the noise is bounded the formation always converges to δ-rendezvous and we derived the bound on δ. Simulations suggest that the bound is conservative for typical realizations, however we provide an example that shows that the bound can be achieved. Further work includes deriving probabilistic performance guarantees that can better describe typical simulation runs.

References

[1] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobilerobots with limited visibility," *IEEE Transactions on Robotics and Automation*, vol. 15, no. 5, pp. 818–828, 1999.



(a) Diameter evolution when $d = 100$. (b) Diameter evolution when $d = 1$. (c) Diameter evolution when $d = 0.1$.

Fig. 1. Evolution of the diameter in the formation when the noise is uniformly distributed in $[-9, 9]$, and the agents have different maximum speeds. Dashed line marks the theoretical bound. The larger the maximum traveled distance d, the less robust the system is to noise, but the faster it converges.

(a) Diameter evolution when $\|\mathfrak{N}\| = 9$. (b) Diameter evolution when $\|\mathfrak{N}\| = 1$.

Fig. 2. Evolution of the diameter on the formation for a *worst case scenario* noise for the formation. Dashed line marks the theoretical bound. Observe that although the diameter of the formation remains finite, it approaches the theoretically predicted bounds.

[2] J. Cortes, S. Martinez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Transactions in Automatic Control*, vol. 51, no. 8, pp. 1289–1298, 2006.

[3] J. Lin, A. Morse, and B. Anderson, "The multi-agent rendezvous problem - the asynchronous case," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, 2004, pp. 1926–1931.

[4] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of asynchronous robots with limited visibility," *Theoretical Computer Science*, vol. 337, no. 1-3, pp. 147–168, 2005.

[5] J. N. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 1984.

[6] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Transactions on Automatic Control*, vol. 31, no. 9, 1986.

[7] D. Bertsekas and J. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Prentice Hall, 1989.

[8] A. Jadbabaie, J. Lien, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transaction on Automatic Control*, vol. 48, no. 6, 2003.

[9] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions On Automatic Control*, pp. 1520–1533, Sep. 2004, special issue on Networked Control Systems.

[10] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, 2005.

[11] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Transactions on Information Theory*, vol. 52, no. 6, 2006.

[12] M. Jelasity and A. Montresor, "Epidemic-style proactive aggregation in large overlay networks," in *Proceedings of The 24th International Conference on Distributed Computing Systems (ICDCS 2004)*, 2004, pp. 102–109. [Online]. Available: citeseer.ist.psu.edu/jelasity04epidemicstyle.html

[13] L. Schenato and S. Zampieri, "On the performance of randomized communication topologies for rendezvous control of multiple vehicles," in *Proceedings of the* 17^{th} *International Symposium on Mathematical Theory of Networks and Systems*, 2007.

[14] L. Schenato and G. Gamba, "A distributed consensus protocol for clock synchronization in wireless sensor network," in *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, 2007.

[15] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, 2007.

[16] C. Caicedo-N. and M. Žefran, "Consensus-based rendezvous," in *IEEE Multiconference on Systems and Control*, 2008.

[17] M. Huang and J. H. Manton, "Stochastic approximation for consensus seeking: Mean square and almost sure convergence," in *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, 2007.

[18] F. Cucker and E. Mordecki, "Flocking in noisy environments," *Journal de Mathmatiques Pures et Appliques*, 2007, to appear.

[19] S. Martnez, "Practical rendezvous through modified circumcenter algorithms," in *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, 2007.

[20] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[21] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1990.

[22] E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd ed., ser. Springer Series in Statistics. Springer Verlag, 1981.

[23] C.-T. Chen, *Linear System, Theory and Design*, ser. The Oxford Series in Electrical and Computer Engineering. Oxford University Press, 1999.

[24] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts on Mathematics. Springer, 2001, no. 207.

[25] D. B. West, *Introduction to Graph Theory*. Prentice Hall, 2001.

[26] L. Ahlfors, *Complex Analysis*. McGraw-Hill, 1979.

[27] A. Olshevsky and J. N. Tsitsiklis, "Convergence rates in distributed consensus and averaging," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006.

[28] A. Olshevsky, "Convergence speed in distributed consensus and averaging," Master's thesis, Massachusetts Institute of Technology, 2006.

[29] D. Angeli and P.-A. Bliman, "Convergence speed of distributed consensus and topology of the associated information spread," in *Proceedings of the* 46^{th} *IEEE Conference on Decision and Control*, 2007.

[30] S. Martinez, F. Bullo, J. Cortes, and E. Frazzoli, "On synchronous robotic networks - Part I: Models, tasks and complexity notions," in *Proceedings of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05*, 2005.