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Further Results on the Minimum Bi-Path Length for Rigid Objects with Dual Steering. Solution to a Problem posed by Ulam

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***Abstract*— The optimal motion of a rigid segment in the plane is determined for starting from some giving position and orientation and terminating at a prespecified position and orientation. Optimality is with respect to the sum of the distances traveled by the endpoints. As such, the solution is relevant for applications in optimal path planning for certain robotic vehicles. A new proof and geometric characterization is given. It adds to a growing repertoire of optimal path planning curves, and solves a problem posed by Ulam in 1960.**

I. Introduction

In [18, p.79] Ulam posed the following problem:

> Suppose two segments are given in the plane, each of length one. One is asked to move the first segment continuously, without changing its length to make it coincide at the end of the motion with a second given interval in such a way that the sum of the lengths of the two paths described by the end points should be a minimum. What is the general rule for this minimum motion?

While the Dubins [6], [15], [20], [16] and Reeds-Schepp problem come to mind with this simply posed problem, this problem is more intricate. The author has not found any discussion of this problem in the monumental works of LaValle [13] or Jurdjevic [11] and the collection [12]. This problem bears similarity to the Monge problem, but here rigidity is required throughout the entire motion. Envision the object as a bicycle-like robotic vehicle, but with two independent steering wheels. For simplicity we assume it has full actuation, unbounded acceleration, and is massless. It generalizes the problem solved by Balckom and Mason where the wheels were fixed transversally to the "bicycle" frame. We shall refer to the performance index for Ulam's problem as the *bi-path length*.

Under various assumptions on the class of extremals, a solution was given by Gurevich [9] and Goldberg [8]. Dubovitskii discussed the problem for the motion of a segment in $\mathbb{R}^n$, but for which the endpoints lie on prescribed surfaces. His solution, without any a priori assumptions, is based on the integral maximum principle by Dubovitskii and Milyutin [7]. A whole chapter is devoted to the construction of an "atlas" of all possible extremals. A new solution based entirely on Cauchy's surface area formula is given in [10] and provided interesting new insights. A variant, the *minimum time* problem with maximum velocity constraints, is solved in [4]. Optimality of postulated motions is shown algorithmically and by geometric means. However, velocity constrained minimum time paths are not the shortest bi-paths.

In this paper a new solution method is given by combining the classical maximum principle with geometric methods. It is shown that only two types of motions generate the extremals: Rotation about an endpoint and a Glide. The latter is a motion where both endpoints follow straight line paths (not necessarily parallel). In addition, we derive new geometric properties of the solution such as the *glide ellipse*, discussed in the text. These characterizations and properties add to a growing family of optimal path planning solutions and tools [12], [13], [14], [17], [19], and references therein.

II. Configuration space and Reachability

Considering the endpoints non-interchangeable (i.e., "colored"), we denote them as blue (B) and red (R) with coordinates (x_b, y_b) and (x_r, y_r). The rigidity constraint requires that at all times $(x_r - x_b)^2 + (y_r - y_b)^2 = 1$. Parameterizing the segment by the coordinates (x, y) of its blue end point (B) and its orientation with respect to the horizontal, θ. Thus (see Figure 1), B and R are given by

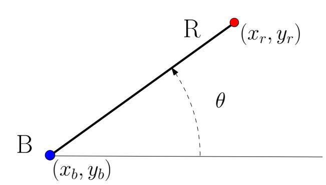

Fig. 1. Parameterization

$x_b = x, y_b = y, x_r = x + \cos\theta$, and $y_r = y + \sin\theta$. The natural state space is $\mathbf{X} = \mathbb{R}^2 \times S^1$. The kinematics of the segment are completely specified by the state equations $\dot{x} = u, \dot{y} = v$, and $\dot{\theta} = \omega$, where u, v and ω are the controls.

In view of what is to follow, we shall introduce two types of elementary motions: These are the pure *rotation* about one of the endpoints, and a *glide*. Denote the rotation about B over an angle θ by $\mathbf{Rot}_B(\theta)$. $\mathbf{Rot}_R(\theta)$ is similarly defined. A glide $\mathbf{Gli}_{(b,r)}$ is defined as the sliding of the endpoints B and R of the rod respectively along the fixed lines b and r (Figure 2). If b and r are parallel, at most 1 unit distant from each other, any glide $\mathbf{Gli}_{(b,r)}$ results in a new position B'R' of the rod, parallel to its initial position. However, a glide is defined also if b and r are not parallel. With such a glide, the rod has two limiting positions, because of the fixed length of the rod (rigidity). It is also important to point out that with a single glide, the motion of the endpoints is not necessarily unidirectional. A point P on b will be called a *turning point* if, after the endpoint B reaches P its motion reverses direction. Likewise, turning points on r may exist. For instance, the reader may easily be convinced that the glide from $(0, \pi/4)$ to $(1+\sqrt{2}, \pi)$ requires a turning point for endpoint B at $(2\sqrt{2}, 0)$ on b and a turning point at $(1+\sqrt{2}, -1)$ for endpoint R on r. Here, r is the line through $(\sqrt{2}/2, \sqrt{2}/2)$ and $(\sqrt{2}, 0)$.

Let us first consider the problem of *reachability*. Let the initial configuration be specified by endpoint B at P in $\mathbb{R}^2$, and $\theta \in S^1$. Can we find a motion such that endpoint B moves to B'=P' and the orientation is θ'? Clearly, translation and rotation invariance simplifies the problem. Thus assume that B lies initially at the origin (B=P=O), and that OB' is aligned with the x-axis. The reachable set under rotation about B is the submanifold $\{\mathrm{O}\} \times \mathrm{S}^1$.

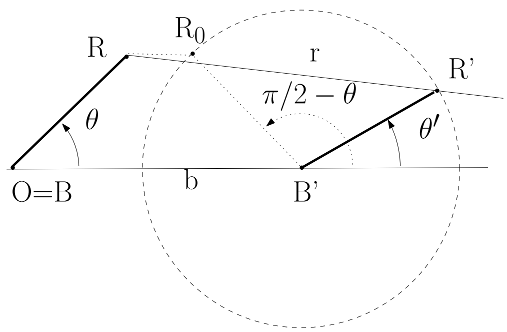

Fig. 2. Glide-reachable configuration (b fixed)

Which configurations in $\mathbf{X}$ are accessible by a single glide from $\{\mathrm{O}\} \times \{\theta\}$? Consider first an arbitrary but fixed endpoint B', different from B on the x-axis. (The line b, on which endpoint B lies, coincides with the x-axis.) If $\theta = \pm\pi/2$, the rod can only translate horizontally (r parallel to b). If $\theta \neq \pm\pi/2$, all but one configuration B'R' can be obtained by a glide $\mathbf{Gli}_{(b,r)}$, where r is the line RR'. The exception is the case with final angle $\theta' = \pi - \theta$. Indeed, if a single glide existed, it would have to be a glide with $r = \mathrm{RR}_0$ parallel to b. But as discussed, only a pure translation is possible then. Hence, for a fixed glide axis b, the reachable manifold contains the submanifold $\{(b \setminus \{\mathrm{O}\}) \times \theta' \,|\, \theta' \neq \pi - \theta\}$. If $\theta = \pm\pi/2$, all configurations on $b \times S^1$ are reachable by the glide. Furthermore, it is always possible to find a glide (with turning points, i.e., points where the notion of B and/or R becomes stationary) returning the rod with B'=B and giving it an arbitrary orientation. Since b may be chosen arbitrarily, the manifold, reachable by a single but arbitrary glide, is $\{O\} \times S^1 \cup \{(\mathrm{P}, \theta) \,|\, \mathrm{P} \in \mathbb{R}^2 \setminus \{O\}, \, \theta \in S^1 \setminus \{\arg \overline{OP}\}\}$. Projected on the hyperplane for a fixed nonzero distance, $\overline{OP}$, this is a torus with a path of winding number 1 taken out. This can be modeled by a Möbius strip. Combining one glide and a rotation, *all* points in $\mathbb{R}^2 \times S^1$ are reachable. Due to the occurence of turning points, the sum of the lengths of the segments $\overline{\mathrm{BB}'} + \overline{\mathrm{RR}'}$ may only be a lower bound for the bi-pth length from configuration BR to B'R'.

III. Some Preliminary Results

The performance index is the sum of the path lengths traveled by B and R: $J = \int_0^1 (|\mathrm{d}S_B| + |\mathrm{d}S_R|)$. In terms of the chosen states and controls this is

$$J = \int_0^1 \left(\sqrt{u^2 + v^2} + \sqrt{(u - \omega \sin\theta)^2 + (v + \omega \cos\theta)^2} \right) \mathrm{d}t. \quad (1)$$

In order to illustrate the nontriviality of the problem, we consider first some specific cases, which can be solved in a purely geometric way.

Consider the initial configuration BR and desired final configuration B'R' in Figure 3. It is clear that the direct

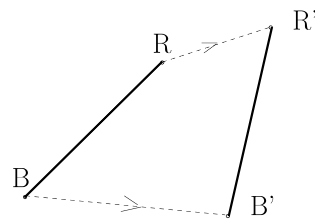

Fig. 3. Nontrivial glide P$\rightarrow$ P' and Q $\rightarrow$ Q'.

glide B $\rightarrow$ B' and R $\rightarrow$ R' is feasible. Moreover since both BB' and RR' are straight lines, it is not possible to find a shorter bi-path (in the absence of turning points). Now consider a final configuration with B'$\equiv$ B. Let BR and BR' making an angle β. A simple rotation about B, gives a path along the arc RR' of length β. Alternatively, consider the glide $\mathbf{Gli}_{RR',PQ}$. Point Q is the intersection of the bisector of (BR,BR') and the straight line RR'. Let P be the point on the bisector at distance 1 from Q (Figure 4). A glide path,

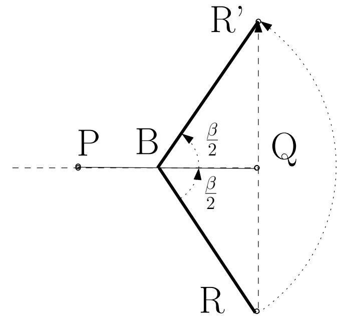

Fig. 4. Glide versus rotation P$\equiv$ P' and Q $\rightarrow$ Q'.

R→ Q→ R', B→ P→ B exists. The path taken by B has length $2|\overline{PB}| = 2(1-\cos\frac{\beta}{2})$, while the path by R takes $2\sin\frac{\beta}{2}$. The ratio of this total bi-path length with β is

$$\frac{D_{glide}}{D_{rot}} = \sqrt{2}\,\frac{\sin\frac{\beta}{4}\cos(\frac{\beta-\pi}{4})}{\frac{\beta}{4}}$$

For all β in the interval $(0,\pi)$, this ratio is larger than 1, thus proving that rotation is more efficient than the suggested glide (but this does not yet prove the rotation to be optimal.)

Lemma 3.1: If B and B' coincide, the optimal bi-path is symmetrical about the bisector PQ.
Proof: By contradiction.
The following theorem is then easily proven:

Theorem 3.1: If one of the endpoints at the end of the motion must coincide with its initial position, the optimal bi-path is a pure rotation about that endpoint.

A. Turning Points

A detailed discussion of turning points is simplified by first considering a simpler related problem: Find the optimal bi-path to align the rod with a given line ℓ.

We shall here only consider the special case when one of the endpoints of the rod lies already on ℓ, keeping B' free. Let BR make an angle β with ℓ, assuming first that $0 < \beta < \pi/2$ (Figure 5). We solve the problem in two steps. First, require that the endpoint R' coincides with a fixed location M on ℓ. This allows us to solve a family of optimization problems parameterized by M. In doing this we characterize the turning points. Then proceed to find the minimizing bi-path length in case the position along ℓ is free, i.e., M may be chosen arbitrarily. Let the path RM have angle α with the perpendicular PR to the line ℓ. Put the origin of the coordinate system at the initial point B. The endpoint R has coordinates $(\sin\beta, \cos\beta)$. Let M have coordinates $1+y$, so that when R has moved to M, the other endpoint is at B' with coordinates $(0, y)$. Let the point S on RM be parameterized by $s \in [0,1]$, i.e., $S(s) = ((1-s)\sin\beta, s(1+y) + (1-s)\cos\beta)$, thus with S(0)≡ R and S(1) ≡ M. While S moves on MR, B glides along ℓ to U(s). Let $|\overline{\mathrm{PU}(s)}| = p(s)$. From the quadratic equation $|\overline{\mathrm{US}}| = 1$, we get, picking the smallest root, $p(s) = s(1+y) + (1-s)\cos\beta - \sqrt{1-(1-s)^2\sin^2\beta}$. A turning point is a stationary point for $p(s)$. Straightforward manipulations yield the coordinate of U:

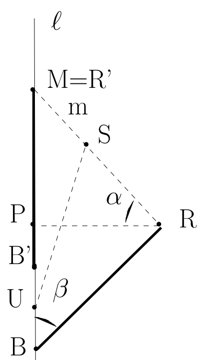

Fig. 5. Optimal bi-path to a wall.

$$p^* = p(s^*) = 1+y-\frac{1}{\sin\beta}\sqrt{(1+y)^2-2(1+y)\cos\beta+1}.$$

Theorem 3.2: The turning point conditions corresponds to a parameter s such that $\overline{\mathrm{SU}}$ is perpendicular to $\overline{\mathrm{MR}}$.

The following cases (noting that B=B' corresponds to $\alpha = \beta/2$) can be discerned:

1) $\alpha > \beta$ There is no turning point on RM. Here $y > 0$ and the optimal bi-path length is $J = |\overline{\mathrm{RM}}| + y$.
2) $\beta/2 < \alpha < \beta$. The turning point lies below B, and $y > 0$.
3) $0 < \alpha < \beta/2$. Similar to the previous case, except that now B' lies between B and U (with $y < 0$).
4) $\beta - \pi/2 < \alpha < 0$ Now $y < 0$ and $J = |\overline{\mathrm{RM}}| + |y|$.

The optimal bi-path distance with gliding along RM and ℓ respectively requires a total bi-path length

$$J^*(y) = y+2+(1-\frac{2}{\sin\beta})\sqrt{(1+y)^2-2(1+y)\cos\beta+1}$$

It is clear that $y < 0$ cannot be minimizing. The angle α relates to y by $\frac{1+y-\cos\beta}{\sin\beta} = \tan\alpha$. The optimal α occurs in the interval $(0,\beta)$: $\alpha^*(\beta) = \arctan\frac{\sin(\beta)}{2\sqrt{1+\sin(\beta)}}$. The case $\pi/2 < \beta < \pi$ is similar and omitted. We conclude:

Theorem 3.3: The bi-path length for the alignment of the rod with a wall, when one endpoint is already positioned along the wall is a pure rotation about this endpoint.

IV. Main Result

Following standard optimal control methods (PMP) [3], the Hamiltonian for Ulam's problem with performance index (1) is obtained by adjoining the dynamical equations with Lagrange multiplier functions (the co-states).

$$H = \sqrt{u^2+v^2} + \sqrt{(u-\omega\sin\theta)^2+(v+\omega\cos\theta)^2} + +\lambda_x u + \lambda_y v + \lambda_\theta\omega. \quad (2)$$

A. Minimality of the Hamiltonian

The Hamiltonian (2) as function of u and v is convex, and differentiable except at the tips (u=v=0) and at ($u = \omega\sin\theta$ and $v = -\omega\cos\theta$). It follows that, as function of u and v, the minimum occurs either at one of these tips, or at a point satisfying $\frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = 0$. Thus motivated, we consider first the nondifferentiable cases:

Case 1: $u = v = 0$. The Hamiltonian evaluates to $H = |\omega| + \lambda_\theta\omega$. There are 5 subcases for ω^*, the optimal ω:

$$\begin{array}{rr|l} \text{i)} & \lambda_\theta < -1 & \omega^* \to -\infty \\ \text{ii)} & \lambda_\theta = -1 & \omega^* \in [0,\infty) \\ \text{iii)} & |\lambda_\theta| < 1 & \omega^* = 0 \\ \text{iv)} & \lambda_\theta = 1 & \omega^* \in (-\infty, 0] \\ \text{v)} & \lambda_\theta > 1 & \omega^* \to \infty. \end{array} \quad (3)$$

Subcases i) and v) are obviously meaningless, as the resulting path is discontinuous. The remaining cases lead to the optimal $H^* = 0$. However, subcase iii) implies that the rod remains motionless, clearly a non-solution. This leaves only the two border line cases, and we note that the corresponding rate of rotation is only constrained by its sign.

Case 2: Next we consider the tip $u = \omega \sin\theta, v = -\omega \cos\theta$. The Hamiltonian is: $H = |\omega| + [\lambda_\theta + (\lambda_x \sin\theta - \lambda_y \cos\theta)]\omega$. which is of Case 1 form, but with λ_θ augmented by $(\lambda_x \sin\theta - \lambda_y \cos\theta)$. The conclusion is then similar, and only the cases $\lambda_\theta + (\lambda_x \sin\theta - \lambda_y \cos\theta) = \pm 1$ make the optimization problem meaningful, resulting in $H^* = 0$.

If $u = v = 0$, but $\omega \neq 0$, it is a rotation of the rod about the endpoint B, the other case a rotation about R. We conclude that $\mathbf{Rot}_B$ and $\mathbf{Rot}_A$ are potentially parts of an optimal solution.

Consider now the *entire line segment* between the two tips, and set $u = k\omega \sin\theta$ and $v = -k\omega \cos\theta$, for $0 \leq k \leq 1$. The full Hamiltonian evaluates on this line segment to

$$\begin{aligned} H &= k|\omega| + (1-k)|\omega| + [\lambda_\theta + k(\lambda_x \sin\theta - \lambda_y \cos\theta)]\omega \\ &= |\omega| + [\lambda_\theta + k(\lambda_x \sin\theta - \lambda_y \cos\theta)]\omega \end{aligned} \quad (4)$$

Since this is *affine* in k, an indifferent case (k anywhere in [0,1]) occurs if $\lambda_x \sin\theta - \lambda_y \cos\theta = 0$. In this case *all* points of the segment connecting $(0,0)$ to $(\omega \sin\theta, -\omega\cos\theta)$ are potential minima for fixed ω and θ. To optimize the Hamiltonian (4), constrained to this line segment, note first that it is of the form $H = |\omega| + \lambda\omega$, where we set $\lambda = \lambda_\theta + k(\lambda_x \sin\theta - \lambda_y \cos\theta)$. If $|\lambda| < 1$, then $\omega^* = 0$. If $|\lambda| > 1$, no minimum exists (the infimum is $-\infty$). If $\lambda = 1$, then $\omega^* \leq 0$, and for $\lambda = -1$, $\omega^* \geq 0$ are minimizing. What is the character of these potential solutions? Note that $\omega^* = 0$ implies in this case that $u^* = v^* = 0$, so that the object is not moving at all. Obviously, this cannot be part of a solution to Ulam's problem. Also, the case when only the infimum, but not a minimum exists, cannot correspond to a solution, as this would allow an impulsive control which would render the path discontinuous.

This leaves only the cases $|\lambda| = 1$ as potential solutions. Indeed, choosing an $\omega' \neq 0$ satisfying the corresponding constraints, a nonzero value for u and v results. The velocity components for B are then $u_B = k\omega' \sin\theta$, and $v_B = -k\omega' \cos\theta$, and the velocity components for R are

$$\begin{aligned} u_R &= u - \omega' \sin\theta = -(1-k)\omega' \sin\theta \\ v_R &= v + \omega' \cos\theta = (1-k)\omega' \cos\theta. \end{aligned}$$

Geometrically, this means that the line BR *rotates* with angular velocity ω' about a point between B and R, exactly k units from B and $1-k$ from R.

This indicates that an instantaneous *turn about B or R or any point between B and R* is a potential optimal segment of the solution.

Another viewpoint is obtained from: $v_R/u_R = u_B/v_B = -\tan\theta$. The heading of both endpoints is along the line perpendicular to BR, but in opposite directions. Their speeds are respectively $k\omega'$ and $(1-k)\omega'$. This means indeed a rotation about the point k units from B.

Differentiable Case: If a minimizer exists outside the segment (0,0) to $(\omega \sin\theta, -\omega\cos\theta)$, where H is convex and differentiable, it must be unique. It is found by setting the partial derivatives of H w.r.t. u, v and ω zero.

B. Adjoint Equations

The adjoint equations are

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \quad (5)$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 \quad (6)$$

$$\dot{\lambda}_\theta = -\frac{\partial H}{\partial \theta} = \frac{(u - \omega\sin\theta)\omega\cos\theta + (v + \omega\cos\theta)\omega\sin\theta}{\sqrt{(u-\omega\sin\theta)^2 + (v+\omega\cos\theta)^2}} \quad (7)$$

First, (5) and (6) tell us that λ_x and λ_y are constant along the optimal solution. Substituted in the *smooth* optimality conditions, they imply the constancy of both $\frac{u}{\sqrt{u^2+v^2}} + \frac{u-\omega\sin\theta}{\sqrt{(u-\omega\sin\theta)^2+(v+\omega\cos\theta)^2}}$ and $\frac{v}{\sqrt{u^2+v^2}} + \frac{v+\omega\cos\theta}{\sqrt{(u-\omega\sin\theta)^2+(v+\omega\cos\theta)^2}}$.

But, from the geometry, $\frac{u}{\sqrt{u^2+v^2}} = \cos\alpha_b$, and $\frac{v}{\sqrt{u^2+v^2}} = \sin\alpha_b$, where α_b is the heading of the blue endpoint B.

Likewise, expressing $\cos\alpha_r$ and $\sin\alpha_r$ the heading of R is also expressible in terms of u, v, ω and θ, giving

$$\cos\alpha_b + \cos\alpha_r = \xi \quad (8)$$

$$\sin\alpha_b + \sin\alpha_r = \eta, \quad (9)$$

where ξ and η are constants. This sets the stage for the geometric interpretation. There are either zero or two solutions for these equations, as shown in Figure 6. The circles centered at O, and (ξ, η), of radius one either intersect or are disjoint. If $\xi^2 + \eta^2 > 2$ no solution is possible. If $0 < \xi^2 + \eta^2 < 2$, then if (α_b^*, α_r^*) is one solution, then its permutation, (α_r^*, α_b^*) is the second solution. The boundary case gives a unique solution with equal angles. If $\xi = \eta = 0$,

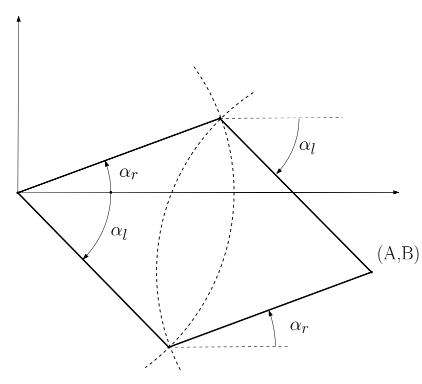

Fig. 6. Path Angles

then $\alpha_b = \pi + \alpha_r$ is the only constraint, and the value of α_b

is entirely free. Apart from an infinitesimal rotation about a point on the rod, such a glide cannot satisfy the rigidity constraint.

It follows from this that the corresponding path is such that the *endpoints travel in straight line segments*. The double solution indicates that abrupt switching *may* be a possibility. The degenerate case (equal angles) corresponds to a parallel translation of the rod.

Consider now (7), which states that along the optimal path

$$\lambda_\theta = \sin(\theta - \alpha_r). \tag{10}$$

An equivalent geometric form of (7) is

$$\dot{\lambda}_\theta = \omega(\cos\alpha_r \cos\theta + \sin\alpha_r \sin\theta) = \omega\cos(\theta - \alpha_r),$$

and we note that the quantity $\alpha_r - \theta$ is the *heading* of R *with respect to the body* BR of the rod. Hence, upon combining, we obtain a first order nonlinear differential equation

$$\lambda_\theta^2 + \frac{1}{\omega^2}\dot{\lambda}_\theta^2 = 1, \tag{11}$$

with solution

$$\arcsin\lambda_\theta(t) - \arcsin\lambda_\theta(0) = \pm\Omega(t), \tag{12}$$

where $\dot{\Omega} = \omega$, thus implying that $\Omega - \theta$ is constant. Thus again,

$$\lambda_\theta(t) = \sin(\arcsin\lambda_\theta(0) \pm \Omega(t)), \tag{13}$$

and from (10)

$$\sin\big(\theta(t) - \alpha_r(t)\big) = \sin\big(\theta(0) - \alpha_r(0) \pm \Omega(t)\big).$$

This gives a *piecewise linear* law, leaving the possibility of switches

$$\theta(t) - \alpha_r(t) = \theta(\tau) - \alpha_r(\tau) \pm \Omega(t) - \Omega(\tau).$$

But assume there is a switch from α_r^* to α_b^* at time τ. By the continuity of the above solution, this requires that

$$\theta(\tau+) - \alpha_r(\tau+) = \theta(\tau+) - \alpha_b(\tau-) = \theta(\tau-) - \alpha_b(\tau-).$$

Unless $\alpha_r = \alpha_b$ at the time of the switch, τ, this creates a discontinuity in θ at τ, which cannot occur with a smooth motion.

At once this precludes a zig zag motion (i.e. one for which $\alpha_b = -\alpha_r$) from being optimal. In view of the constancy of the sum of the cosines and the sum of the sines of the angles, $\alpha_b = -\alpha_r$ implies indeed that $\xi = \cos\alpha_b + \cos\alpha_b = 2\cos\alpha_b$ and $\eta = \sin\alpha_b + \sin\alpha_r = 0$, which indicates a zig-zag motion of the segment in the direction of its initial axis. But, it is clear that a simple parallel translation outperforms the zig zag.

In the nondifferentiable case, if, for $k \in [0,1]$, $u_B = k\omega' \sin\theta$ and $v_B = -k\omega' \cos\theta$, then the costate equation corresponding to θ gives

$$\dot{\lambda}_\theta = 0. \tag{14}$$

In such potentially optimal solutions, which all correspond to rotations about a point B, R or somewhere in between (for the degenerate case) the costate λ_θ must remain constant. But we already found that this constant is either +1 or -1 for a rotation about B. In view of the constancy of the Hamiltonian (and $H^* = 0$), while θ changes, under the rotation, this implies also that λ_θ has magnitude 1, and that $\lambda_x = \lambda_y = 0$. We have thus shown

Theorem 4.1: Partial paths obtained by pure rotations $\mathbf{Rot}_B$ or $\mathbf{Rot}_R$ about an endpoint, or such hat the headings of the endpoints are constant, i.e., glides $\mathbf{Gli}_{(b,r)}$, are candidate optimal paths.

Finally we note for arbitrary μ, constant or not, $H(\mu u, \mu v, \mu\omega) = \mu H(u, v, \omega)$. Without loss of generality, we may restrict the problem to $\omega| = 1$ and $\omega = 0$. In this case (13) leads to a piecewise linear law.

V. Geometry

There are two *modes* of optimal motion: rotations and glides. The behavior of λ_θ is the key. In the setup of Ulam's problem, initial and final configurations are specified. Consequently, the initial and final value of the co-state are not specified. Now it follows from the piecewise linear law, that $\lambda_\theta(t)$ varies as the sine of a linearly (in time) increasing or decreasing angle. Then either the motion is finished before $\lambda_\theta(t)$ can reach the values +1 or -1, in which case the solution is just a glide. Alternatively, after some intermediate time, $\lambda_\theta(t)$ may reach one of the boundary values +1 or -1. In this case $\dot{\lambda}_\theta$ is zero, and thus λ_θ keeps the value for the rest off the motion. But, we have seen that this corresponds to a rotation.

Hence, it is now clear that the entire solution takes either the form of a glide, a rotation, a glide sandwiched between two rotations, a glide combined with a rotation, or at most a glide followed by a rotation, followed by the 'permuted' glide. *There can never be more than two glides*! These are indeed all cases described in the atlas of Dubovitskii [7], and makes the solution much more intricate than for the Dubins problem which involves at most a rotation followed by a translation, followed by a rotation.

A. Glide-ellipse

We derive here another geometric characterization of the glide $\mathbf{Gli}_{(b,r)}$, by analyzing the path of the *midpoint* of the segment BR. We consider two cases:

1) Crossing Paths: Consider Figure 7.

We assume that the paths of the endpoints are crossing at O. For convenience, we introduce a coordinate system with the origin at O, and the x-axis along the bisector of the the paths. Hence the two paths have the equation $y = \pm ax$ for some a. It is easily shown, that the midpoint lies on the ellipse specified by

$$(2ax_M)^2 + \left(\frac{2y_M}{a}\right)^2 = 1. \tag{15}$$

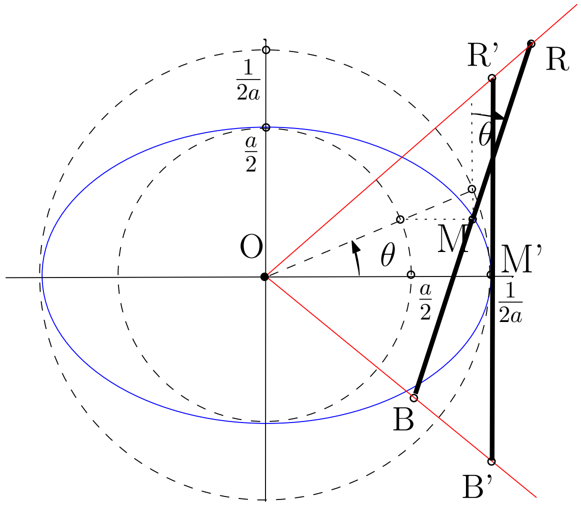

Fig. 7. Path Geometry

Let's refer to this ellipse as the *glide ellipse*. This ellipse has semi-axes of length $\frac{1}{2a}$ and $\frac{a}{2}$ respectively along the x and y direction. This implies also the remarkable property that the *area* of this glide ellipse is $\frac{\pi}{4}$, and therefore is independent of the angle between r and ℓ.

Note that in general each point on this ellipse corresponds in a one-to-one way with a position and orientation of the segment along the potential optimal path.

2) Parallel Paths: If the paths are parallel, obviously, their distance must be less than 1. There are two possibilities, the begin and end configurations, BR and B'R', are parallel, or they are symmetrical. In the first case it follows that the segment can only slide parallel to itself. The midpoint M travels along the line parallel to and halfway between the endpoint paths. This is a degenerate ellipse.
Points along this degenerate ellipse correspond with two possible orientations of the segment along the potential optimal path. However, these two cannot be mapped into each other along an optimal path.

Omitting intricate details, it follows from the previous section that the optimal bi-path consists of one line of sight connection between endpoints, while the other endpoint follows a glide-rotation-glide. The rotation connects the turning points on the glide path straight path. During this rotation, the corresponding midpoint follows a circular arc tangent to the glide ellipses.

VI. Conclusions and Extensions

We have solved a problem posed by Ulam. It was shown that any optimal bi-path consists of either a glide, a rotation, a combination of the two, or - at most - , a glide sandwiched between two rotations or a rotation sandwiched between two glides related by permutation of their glide angles.
We combined geometry with the maximum principle in establishing this solution, as the maximum principle itself does not directly lead to an analytic solution. We gave also a new characterization of the glide solution by its glide ellipse and the invariance of its area. We also discussed a special case: the optimal bi-path to a given line. For a robot with dual steering, this is a parallel parking problem.
Finally, interesting similar work in [5] and [1] was brought to our attention. In the first, the endpoints are constrained to move perpendicular to the rod. The second studies a related problem of minimal orbit length for an arbitrary but fixed point on the rod.
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