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Mixed $\mathcal{H}_-/\mathcal{H}_\infty$ Fault Detection Observer Design for LPV Systems

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***Abstract*— This paper addresses the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer design issue for a class of linear parameter varying (LPV) system. Based on the quadratic $\mathcal{H}_\infty$ performance and affine quadratic $\mathcal{H}_\infty$ performance concepts, as well as the corresponding quadratic $\mathcal{H}_-$ index performance and affine quadratic H_- index performance for measuring the worst-case fault sensitivity of the underlying LPV system, the existence conditions of such an observer are given in terms of linear matrix inequalities (LMIs). Iterative algorithms are given to achieve the solutions.**

I. Introduction

Model based fault detection has received much attention and significant progress has been achieved, see [2][3][10] and the references therein. One of the particular interesting techniques among all the model based techniques is the observer based fault detection filter design. It has been shown that it is very effective in detecting sensor, actuator, and system component faults. Nevertheless, finding systematic design methods for system subjected to unknown disturbances and model uncertainties has been proven to be difficult. Since both disturbance and faults contribute to the residual generated by the observer, it is essential to isolate their effects to the residual. A fault detection observer should be robust to the disturbance but sensitive to the faults [3][7]. Some recent results aiming at this goal on this issue for LTI systems are reported in [3][10] [8][9] [6][5] and the references therein.

This paper concerns the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ observer design issue for a class of linear parameter (LPV) system. Analogous to the definition of the quadratic $\mathcal{H}_\infty$ performance for LPV systems [1] and the $\mathcal{H}_-$ index for linear time invariant systems, the $\mathcal{H}_-$ index for LPV systems is defined in terms of a linear matrix inequality (LMI). The first algorithm for designing the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ observer is proposed based on the quadratic stability condition and quadratic $\mathcal{H}_-$ index condition. For reducing the conservativeness of this algorithm, the affine quadratic stability (AQS) [4] and affine $\mathcal{H}_-$ index for LPV systems proposed in this paper are utilized. To this end, the robustness conditions and affine $\mathcal{H}_-$ index conditions for the underlying system are recast as parameter dependent LMIs. Furthermore, gridding technique and multiconvexity concept are applied to reduce the parameter dependent LMIs to a finite set of LMI constarints. Correspondingly, two algorithms are proposed and implemented by iterative LMI methods.

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II. Notations and Preliminaries

A. Notations

$\Xi_\infty(A,B,C,D,X,\gamma)$ and $\Xi^a_\infty(A,B,C,D,X,\gamma)$, respectively, denotes

$$\begin{pmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix}$$

and

$$\begin{pmatrix} A^TX+XA+\dot{X} & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix}$$

$\Xi_-(A,B,C,D,P,\beta)$ and $\Xi^a_-(A,B,C,D,P,\beta)$, respectively, indicates

$$\begin{pmatrix} A^TP+PA+C^TC & PB+C^TD \\ (PB+C^TD)^T & D^TD-\beta^2 I \end{pmatrix}$$

and

$$\begin{pmatrix} A^TP+PA+C^TC+\dot{P} & PB+C^TD \\ (PB+C^TD)^T & D^TD-\beta^2 I \end{pmatrix}$$

The symmetric entries below diagonal are denoted as $*$.

B. Quadratic $\mathcal{H}_\infty$ performance for LPV systems

Consider a general polytopic linear parameter-varying system of the following form

$$G(\theta):\begin{cases} \dot{x}=A(\theta)x+B(\theta)u \\ y=C(\theta)x+D(\theta)u \end{cases} \tag{1}$$

where $x\in R^{n_x}$ is the state vector, $u\in R^{n_u}$ is the input vector, $y\in R^{n_y}$ is the measurement vector and $\theta\in R^{n_\theta}$ is the scheduling variable measurable online.

Each parameter θ_i ranges between known extremal values $\underline{\theta_i}$ and $\overline{\theta_i}$, $\underline{\theta_i}\le\theta_i\le\bar{\theta}_i, i=1,2,\cdots,n_\theta$, which corresponds to a polytope $\mathcal{R}$ of vertices $\omega_1,\omega_2,....\omega_r$; that is $\theta\in\mathcal{R}:=Co\{\omega_1,\omega_2,\cdots,\omega_r, r=2^{n_\theta}\}$. Additionally, its time derivative $\nu=\dot{\theta}$ is bounded and satisfies $\underline{\nu_i}\le\nu_i\le\bar{\nu}_i, i=1,2,\cdots,n_\theta$. It defines an hyper-rectangle $\mathcal{V}:=Co\{\upsilon_1,\upsilon_2,\cdots,\upsilon_r, r=2^{n_\theta}\}$. The parameters of $A(\theta),B(\theta),C(\theta)$ and $D(\theta)$ are affine to the scheduling variable θ. For instance, $A(\theta)=A_0+A_1\theta_1+A_2\theta_2+\cdots+A_{n_\theta}\theta_{n_\theta}$ and it has the **polytopic property** $A(\theta)=\sum_{i=1}^{n_\theta}\alpha_iA(\omega_i)$ with $\sum_{i=1}^{n_\theta}\alpha_i=1$ and $\alpha_i\ge 0$.

Definition 2.1: (Quadratic $\mathcal{H}_\infty$ performance)[1] The LPV system in (1) is said to have quadratic $\mathcal{H}_\infty$ performance γ

if and only if there exists a positive definite constant matrix $X > 0$ which satisfies the following LMI

$$\Xi_\infty(A(\theta), B(\theta), C(\theta), D(\theta), X, \gamma) < 0$$

for all values of the parameter vector $\theta \in \mathcal{R}$.

Quadratic $\mathcal{H}_\infty$ performance guarantees global asymptotic stability and $\mathcal{L}_2$-gain of the map from u to y less than γ for all possible parameter trajectories $\theta \in \mathcal{R}$, that is $\|y\|_2 < \gamma\|u\|_2$ where $\|y\|_2 := \left(\int_0^\infty y^T y dt\right)^{\frac{1}{2}}$.

It must be noticed that the quadratic $\mathcal{H}_\infty$ performance is more conservative than standard $\mathcal{H}_\infty$ performance for each fixed θ, since it requires the existence of a fixed Lyapunov function for the entire operation range.

Lemma 2.2: (Vertex property) [1] Considering the polytopic LPV plant described by (1), we can have the following equivalent statements:

i The LPV system is stable with quadratic $\mathcal{H}_\infty$ performance γ;

ii There exists a single matrix $X > 0$ such that

$$\Xi_\infty(A(\theta), B(\theta), C(\theta), D(\theta), X, \gamma) < 0$$

iii For all the vertex of the polytopic LPV system, there exists $X > 0$ such that

$$\Xi_\infty(A(\omega_i), B(\omega_i), C(\omega_i), D(\omega_i), X, \gamma) < 0,$$

where $i = 1, 2, ..., r$.

C. Quadratic $\mathcal{H}_-$ index for LPV systems

The $\mathcal{H}_-$ index as a sensitivity measure for LTI systems is stated as the following lemma in [7].

Lemma 2.3: Let $\beta > 0$ be a constant scalar, and denote $G(s) = C(sI - A)^{-1}B + D$ as a stable system. Then $\|G(jw)\|_-^{[0,\infty)} > \beta$, if and only if there exists a symmetric matrix P such that

$$\Xi_-(A, B, C, D, P, \beta) > 0$$

It is reasonable that we define the following $\mathcal{H}_-$ index for LPV systems as follows and it will be explained later.

Definition 2.4: (Quadratic $\mathcal{H}_-$ index performance for LPV systems) The LPV system (1) is said to have $\mathcal{H}_-$ index performance if there exists a symmetric constant matrix P such that

$$\Xi_-(A(\theta), B(\theta), C(\theta), D(\theta), P, \beta) > 0$$

for all $\theta \in \mathcal{R}$.

D. Affine quadratic $\mathcal{H}_\infty$ performance for LPV systems

Definition 2.5: [4](Affine Quadratic $\mathcal{H}_\infty$ performance) The LPV system in (1) has affine quadratic $\mathcal{H}_\infty$ performance γ if and only if there exist $n_\theta + 1$ symmetric matrix $X_0, X_1, \cdots, X_{n_\theta}$ such that

$$X(\theta) = X_0 + \theta_1 X_1 + \cdots + \theta_{n_\theta} X_{n_\theta} > 0$$

and

$$\Xi^a_\infty(A(\theta), B(\theta), C(\theta), D(\theta), X(\theta), \gamma) < 0$$

satisfied for all admissible parameter trajectory $\theta \in \mathcal{R}$.

Remark 2.6: As mentioned in [4], the affine quadratic $\mathcal{H}_\infty$ stability test is less conservative than the quadratic stability test.

Lemma 2.7: ([4]) (Vertex property) Consider the system (1) which depends on θ affinely. Assume that the parameter trajectories range in the hyper-rectangles $\mathcal{R}$ and $\mathcal{V}$. The system has affine quadratic $\mathcal{H}_\infty$ performance γ if there exist $n_\theta + 1$ symmetric matrices $X_0, X_1, \cdots, X_{n_\theta}$ such that

$$X(\theta) = X_0 + \theta_1 X_1 + \cdots + \theta_{n_\theta} X_{n_\theta} > 0$$

$$\Xi^a_\infty(A(\omega_i), B(\omega_i), C(\omega_i), D(\omega_i), X(\omega_i), \gamma) < 0$$

and

$$\begin{pmatrix} A_j^T X_j + X_j A_j & X_j B_j \\ B_j^T X_j & 0 \end{pmatrix} \geq 0$$

for all $(\omega, \nu) \in \mathcal{R} \times \mathcal{V}$ where $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, n_\theta$.

E. Affine quadratic $\mathcal{H}_-$ index performance for LPV systems

Here we define the affine quadratic $\mathcal{H}_-$ index for LPV system, which is a dual of the affine quadratic $\mathcal{H}_\infty$ performance.

Definition 2.8: (Affine quadratic $\mathcal{H}_-$ index performance for LPV systems) The LPV system (1)is said to have affine $\mathcal{H}_-$ index performance if there exist $n_\theta+1$ symmetric matrix $P_j, j = 1, 2, \cdots, n_\theta$ such that

$$\Xi^a_-(A(\theta), B(\theta), C(\theta), D(\theta), X(\theta), \beta) > 0$$

for $\theta \in \Theta$, where $P(\theta) = P_0 + \theta_1 P_1 + \cdots + \theta_{n_\theta} P_{n_\theta}$.

Remark 2.9: The two definitions of the $\mathcal{H}_-$ index performance for LPV systems actually can be derived from the definition of the $\mathcal{H}_-$ index of linear time varying systems stated in the following.

Definition 2.10: [5] For a stable linear time varying system $G : \omega \to y$,

$$G(t) := \begin{cases} \dot{x} = A(t)x + B(t)\omega \\ y = C(t)x + D(t)\omega \end{cases} \quad (2)$$

Its $\mathcal{H}_-$ index for the infinite horizon case is defined as

$$\|G\|_- = \inf_{\omega \in \mathfrak{L}_2} \frac{\|G\omega\|_2}{\|\omega\|_2} \quad (3)$$

Then we can have the following theorem, which is the modified version of theorem 3 in [7].

Theorem 2.11: The stable time varying system (2) can achieve the $\mathcal{H}_-$ index performance β, i.e. $\|G\|_- \geq \beta$ if there exists a symmetric matrix $P(t)$ such that

$$\Xi^a_-(A(t), B(t), C(t), D(t), P(t), \beta) > 0$$

The proof is omitted due to space limitation.

The following lemma is useful for the relaxation we need later on.

Lemma 2.12: [4] Consider a quadratic function of $\theta \in \mathcal{R}$

$$f(\theta_1, \theta_2, \cdots, \theta_K) = \alpha_0 + \sum_i \alpha_i \theta_i + \sum_{i<j} \beta_{ij}\theta_i\theta_j + \sum_i \lambda_i \theta_i^2 \quad (4)$$

The scalar function $f(\cdot)$ is positive in $\mathcal{R}$ if it takes positive values at the corners at the polytope $\mathcal{R}$ and $\lambda_i = \frac{\partial^2 f}{\partial \theta_i^2}(\theta) \leq 0$ for $i = 1, 2, \cdots, K$.

III. Problem Formulation

The considered linear parameter varying system described by state space equations of the form:

$$\Sigma(\theta) := \begin{cases} \dot{x} = A(\theta)x + B_d(\theta)d + B_u(\theta)u + B_f(\theta)f \\ y = C(\theta)x + D_d(\theta)d + D_u(\theta)u + D_f(\theta)f \end{cases} \quad (5)$$

where $x \in R^{n_x}$ is the state vector, $w \in R^{n_w}$ is the unknown input vector including modeling error, uncertain disturbance, process and measurement noises, $y \in R^{n_y}$ is the measurement vector, $f \in R^{n_f}$ is the fault vector, and $\theta \in R^{n_\theta}$ is the scheduling variable measurable online.

Assume that the designed LPV observer $F(\theta)$ has the following formulation:

$$F(\theta) := \begin{cases} \dot{\hat{x}} = A(\theta)\hat{x} + B_u(\theta)u + L(y - \hat{y}) \\ \hat{y} = C(\theta)\hat{x} + D_u(\theta)u \\ r = y - \hat{y} \end{cases} \quad (6)$$

Define $e = x - \hat{x}$, the residual error dynamic equations can be described by:

$$\begin{cases} \dot{e} = (A(\theta) - LC(\theta))e + (B_d(\theta) - LD_d(\theta))d \\ +(D_f - LD_f(\theta))f \\ r = C(\theta)e + D_d(\theta)d + D_f(\theta)f \end{cases} \quad (7)$$

The objective is to design the gain matrix L for the observer which maximizes the robustness against the disturbance d and also maximized the sensitivity to fault f.

IV. Mixed $\mathcal{H}_-/\mathcal{H}_\infty$ design methodology I

In this section, we concern the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ design based on the quadratic $\mathcal{H}_\infty$ performance and the quadratic $\mathcal{H}_-$ index performance for affine LPV systems.

A. LPV Observer Synthesis Based on the Robustness Conditions

The observer error dynamic system without faults described by

$$\begin{cases} \dot{e} = (A(\theta) - LC(\theta))e + (B_d(\theta) - LD_d(\theta))d \\ r = C(\theta)e + D_d(\theta)d \end{cases} \quad (8)$$

Lemma 4.1: Consider the error dynamical system (8), the following robustness conditions are equivalent:

1) The quadratic $\mathcal{H}_\infty$ performance γ from d to r is achieved.
2) There exists a matrix L and a symmetric matrix $X > 0$ such that

$$\Xi_\infty(A(\theta) - LC(\theta), B_d(\theta) - LD_d(\theta), C(\theta), D_d(\theta), X, \gamma) < 0 \quad (9)$$

for $\theta \in \mathcal{R}$

3) There exists a matrix L and a symmetric matrix $X > 0$ such that

$$\Xi_\infty(A(\omega_i) - LC(\omega_i), B_d(\omega_i) - LD_d(\omega_i), C(\omega_i), D_d(\omega_i), X, \gamma) < 0 \quad (10)$$

where $\omega_i, i = 1, 2, ..., r$ are the vertices.

4) There exists a matrix F_d and symmetric matrix $X > 0$ such that

$$\begin{pmatrix} Q & W^T & C^T(\omega_i) \\ W & -\gamma I & D_d^T(\omega_i) \\ C(\omega_i) & D_d(\omega_i) & -\gamma I \end{pmatrix} < 0 \quad (11)$$

where $Q = A(\omega_i)^T X - C(\omega_i)^T F_d^T + XA(\omega_i) - F_d C(\omega_i)$, $W = B_d(\omega_i)^T - D_d(\omega_i)F_d^T$, $i = 1, 2, ..., r$ and the observer filter

$$L = X^{-1}F_d \quad (12)$$

5) There exist matrices L, L_0 and a symmetric matrix $X > 0$ such that

$$\begin{pmatrix} M_1 & M_3 & M_4 & X \\ * & M_2 & 0 & -D_d^T(\omega_i)L^T \\ * & * & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \quad (13)$$

where

$$\begin{aligned} M_1 &= XA(\omega_i) + A(\omega_i)^T X + C^T(\omega_i)C(\omega_i) + \\ &\quad (L_0C(\omega_i))^T(LC(\omega_i)) - (LC(\omega_i))^T \\ &\quad (L_0C(\omega_i)) + (L_0C(\omega_i))^T(L_0C(\omega_i) \\ &\quad -2X_0X - 2XX_0 + 2X_0X_0 \\ M_2 &= D_d(\omega_i)^T D_d(\omega_i) - \gamma^2 I - \\ &\quad (LD_d(\omega_i))^T(L_0D_d(\omega_i)) - (LD_d(\omega_i))^T \\ &\quad (L_0D_d(\omega_i)) + (L_0D_d)^T((L_0D_d(\omega_i))) \\ M_3 &= XB_d(\omega_i) + C(\omega_i)^T D_d(\omega_i) \\ M_4 &= X - (LC(\omega_i))^T \end{aligned}$$

where $i = 1, 2, ..., r$

(The proof is omitted.)

B. $\mathcal{H}_-$ Index Sensitivity Conditions

Consider the observer error dynamics without disturbance

$$\begin{cases} \dot{e} = (A(\theta) - LC(\theta))e + (B_f(\theta) - LD_f(\theta))f \\ r = C(\theta)e + D_f(\theta)f \end{cases} \quad (14)$$

then we can have the following lemma.

Lemma 4.2: Consider the error dynamical system (14), for a given β, its $\mathcal{H}_-$ index performance is greater than β if one of the following conditions is satisfied:

1) There exists a symmetric matrix P and a gain matrix L such that

$$O := \begin{pmatrix} H_1 & H_2 \\ H_2^T & D_f^T(\omega_i)D_f(\omega_j) - \beta^2 I \end{pmatrix} > 0 \quad (15)$$

where $H_1 = P(A(\omega_i) - LC(\omega_i)) + (A^T(\omega_i) - LC(\omega_i))P + C^T(\omega_i)C(\omega_j)$, $H_2 = PB_f(\omega_i) + C^T(\omega_i)D_f(\omega_j)$, $i = 1, 2, ..., r, j = 1, 2, \cdots, r$.

2) There exist symmetric matrix P and F_f such that

$$\begin{pmatrix} H_3 & H_4 \\ H_4^T & D_f^T(\omega_i)D_f(\omega_j) - \beta^2 I \end{pmatrix} > 0 \quad (16)$$

where $H_3 = PA(\omega_i) - F_f C(\omega_i) + A^T(\omega_i)P - C^T(\omega_i)F_f + C^T(\omega_i)C(\omega_j)$, $H_4 = PB_f(\omega_i) + C^T(\omega_i)D_f(\omega_j)$, $i = 1, 2, ..., r, j = 1, 2, \cdots, r$, $F_f = PL$.

3) There exists matrices L, L_0, and symmetric matrices P and P_0 such that

$$\begin{pmatrix} \check{N}_{11j} & \check{N}_{12j} & \check{N}_{13j} & P \\ * & \check{N}_{22j} & 0 & D_f(\omega_i)^T L^T \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix} > 0 \quad (17)$$

where

$$\begin{aligned} \check{N}_{11j} &= 2P_0P + 2PP_0 - 2P_0P_0 + C(\omega_i)^T \\ & \quad L_0^T LC(\omega_j) + C(\omega_i)^T L^T L_0 C(\omega_j) \\ & \quad + PA(\omega_i) + A(\omega_i)^T P + C(\omega_i)^T C(\omega_j) \\ \check{N}_{12j} &= PB_f(\omega_i) + C(\omega_i)^T D_f(\omega_j) \\ \check{N}_{13j} &= P + C(\omega_i)^T L^T \\ \check{N}_{22j} &= D_f(\omega_i)^T D_f(\omega_j) - \beta^2 I + D_f(\omega_i)^T \\ & \quad L_0^T L D_f(\omega_j) + D_f(\omega_i)^T L^T L_0 D_f(\omega_j) \\ & \quad - D_f(\omega_i)^T L_0^T L_0 D_f(\omega_j) \end{aligned}$$

where $i = 1, 2, \cdots, r, j = 1, 2, \cdots, r$.

(The proof is omitted.)

C. $\mathcal{H}_-/\mathcal{H}_\infty$ Fault Detection Observer Design for LPV systems

Analogous to the theorem in [10], we can have the following theorem on the $\mathcal{H}_-/\mathcal{H}_\infty$ observer design for the underlying LPV system.

Theorem 4.3: Given scalars $\gamma > 0$, $\beta > 0$ and consider the error dynamical system (7), its $\mathcal{H}_\infty$ performance is smaller than γ and $\mathcal{H}_-$ index greater than β if there exists matrices L, L_0, symmetric matrix P_0 and P, $X > 0$ and $X_0 > 0$ such that inequalities (13) and (17) hold.

Algorithm 4.4: **Mixed index observer design algorithm I:** Given the system model $\Sigma(\theta)$ as in (5).

1) Maximize β subject to $P < 0$, (11) and (16) with $F_d = -F_f$ and $P = -X$. Then calculate the optimal filter gain matrix L_{opt} using (12). Let $L_0 = L_{opt}$.
2) With $L = L_0$, maximize β subject to (15) to get P_{opt}, and minimize $Tr(X)$ subject to $X > 0$ and (10) to get $X_{opt} > 0$. Let $X_0 = X_{opt}$ and $P_0 = P_{opt}$.
3) With L_0, X_0 and P_0, maximize $\gamma^2 - \beta^2$ subject to $X > 0$, (13) and (17) to get L_{opt}, $X_{opt} > 0$, P_{opt} and β_{opt}. Let $L_0 = L_{opt}$, $X_0 = X_{opt}$, $P_0 = P_{opt}$.
4) Repeat step 3 till a certain number of iterations are reached or $\gamma^2 - \beta^2$ reaches almost a constant.

V. Mixed $\mathcal{H}_-/\mathcal{H}_\infty$ design methodology II

In this section, we use the affine quadratic $\mathcal{H}_\infty$ performance which bases on parameter dependent Lyapunov function and affine quadratic $\mathcal{H}_-index$ performance techniques to design the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ observer.

A. Robustness Conditions Based on Parameter Dependent Lyapunov Functions

Lemma 5.1: Consider the error dynamical system (8), the following robustness conditions are equivalent:

1) The error dynamical system has the affine quadratic $\mathcal{H}_\infty$ performance γ.
2) (Bounded Real Lemma) if there exist $n_\theta + 1$ matrix $X_0, X_1, \cdots, X_{n_\theta}$ such that

$$\begin{aligned} X(\theta) = X_0 + \theta_1 X_1 + \cdots + \theta_{n_\theta} X_{n_\theta} > 0 \\ \Xi^a_\infty(A(\theta) - LC(\theta))^T, B_d(\theta) - LD_d(\theta) \\ C(\theta), D_d(\theta)(\theta), X(\theta), \gamma) < 0 \end{aligned}$$

for all admissible parameter trajectory $\theta \in \mathcal{R}$.

3) There exist matrices L, L_0 and symmetric matrix $X_0, X_1, \cdots, X_{n_\theta}$ and $X_0^0, X_1^0, \cdots, X_{n_\theta}^0$ such that

$$X(\theta) = X_0 + \theta_1 X_1 + \cdots + \theta_{n_\theta} X_{n_\theta} > 0 \quad (18)$$

$$\begin{pmatrix} J_1(\theta) & J_3(\theta) & J_4(\theta) & P(\theta) \\ * & J_2(\theta) & 0 & -D_d^T(\theta) L^T \\ * & * & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \quad (19)$$

where

$$\begin{aligned} J_1(\theta) &= X(\theta)A(\theta) + A^T(\theta)X(\theta) + C^T(\theta)C(\theta) - \\ & \quad (L_0C(\theta))^T(LC(\theta)) - (LC(\theta))^T(L_0C(\theta)) \\ & \quad + (L_0C(\theta))^T(L_0C(\theta)) + 2X^0(\theta)X(\theta) - \\ & \quad 2X(\theta)X^0(\theta) + 2X^0(\theta)X^0(\theta) \\ & \quad + \Sigma_{j=1}^{n_\theta} X_j \dot{\theta}_j. \\ J_2(\theta) &= D_d^T(\theta)D_d(\theta) - (LD_d(\theta))^T(L_0D_d(\theta)) \\ & \quad - (LD_d(\theta))^T(L_0D_d(\theta)) - \gamma^2 I + \\ & \quad (L_0D_d(\theta))^T((L_0D_d(\theta))) \\ J_3(\theta) &= X(\theta)B_d + C^T(\theta)D_d(\theta) \\ J_4(\theta) &= X(\theta) + (LC(\theta))^T \end{aligned}$$

where $X^0(\theta) = X_0^0 + \theta_1 X_1^0 + \cdots + \theta_{n_\theta} X_{n_\theta}^0 > 0$

Proof: The proof is omitted due to space limitation. ■

The robustness conditions derived in lemma 5.1 are non-convex problem in general and there are infinite set of LMI constraints. To reduce the robustness conditions to a finite set of LMI, we propose two methods. As suggested in [11], we can use the ***gridding method***. It has the risk of achieving an unreliable result if the grid density is not high enough. It also has the computation burden issue if the density is too high. A trade off must be done between the reliability and the computation complexity. The other strategy is to use the so-called multiconvexity condition to reduce the infinite LMI set to a finite set. This is stated in the following lemma.

Lemma 5.2: Consider the error dynamical system (8), it has the affine quadratic $\mathcal{H}_\infty$ performance γ if one of the following conditions is satisfied:

1) There exists a matrix L and $n_\theta+1$ symmetric matrices $X_j>0, j=0,1,\cdots n_\theta$ such that

$$X(\omega_i)>0 \quad (20)$$

$$\begin{pmatrix} T_1 & T_2 & C^T(\omega_i) \\ T_2^T & -\gamma I & D_d^T(\omega_i) \\ C(\omega_i) & D(\omega_i) & -\gamma I \end{pmatrix} < 0 \quad (21)$$

where $T_1 = (A(\omega_i) - LC(\omega_i))^T X(\omega_i) + X(\omega_i)(A(\omega_i) - LC(\omega_i)) + \sum_{j=1}^{n_\theta} X_j\{\underline{\nu}_j, \bar{\nu}_j\}$. Here the $\Sigma_{j=1}^{n_\theta} X_j\dot{\theta}_j$ is replaced by $\sum_{j=1}^{n_\theta} X_j\{\underline{\nu}_j, \bar{\nu}_j\}$ since $\dot{\theta}$ is affine to the inequality and we only need to check their extremal points. This notation is also used in [11]. In the following the same replacement applies if the θ in $\Sigma_{j=1}^{n_\theta} X_j\dot{\theta}_j$ needs to be replaced by the vertex ω_i. $T_2 = X(\omega_i)(B_w(\omega_i) - LD_d(\omega_i))$, $X(\omega_i) = X_0 + \theta_1^i X_1 + \theta_2^i X_2 + \cdots + \theta_{n_\theta}^i X_{n_\theta}$, $i=1,2,...,r$. and

$$\begin{pmatrix} T_j & X_j(B_{d,j} - LD_{d,j}) \\ * & 0 \end{pmatrix} \geq 0 \quad (22)$$

where $T_j = (A_j - LC_j)^T X_j + X_j(A_j - LC_j)$,$j = 1,2,...,n_\theta$.

2) There exist matrices L, L_0 and $n_\theta+1$ symmetric matrix $X_j, j=1,2...,n_\theta$ such that

$$X(\omega_i)>0 \quad (23)$$

$$\begin{pmatrix} J_1(\omega_i) & J_3(\omega_i) & J_4(\omega_i) & X(\omega_i) \\ * & J_2(\omega_i) & 0 & J_5 \\ * & * & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \quad (24)$$

and

$$\begin{pmatrix} \check{S}_{11} & X_j B_{d,j} + C_j^T D_{d,j} \\ * & \check{S}_{22} \end{pmatrix} \geq 0 \quad (25)$$

where

$$\begin{aligned} J_5 &= -D_d^T(\omega_i) L^T \\ \check{S}_{11} &= A_j^T X_j + X_j A_j - (L_0 C_j)^T (LC_j) + C_j^T C_j \\ &\quad -(LC_j)^T (L_0 C_j) - (L_0 C_j)^T (L_0 C_j) \\ \check{S}_{22} &= D_{d,j}^T D_{d,j} - (LD_{d,j})^T (L_0 D_{d,j}) \\ &\quad -(L_0 D_{d,j})^T (LD_{d,j}) + (L_0 D_{d,j})^T (L_0 D_{d,j}) \end{aligned}$$

where $i=1,2,...,r$, $j=1,2,...,n_\theta$

Proof: The proof of this lemma is straightforward by using the multiconvex conditions in [4]. ■

B. $\mathcal{H}_-$ Index Sensitivity Conditions

Lemma 5.3: Consider the dynamical system (14),it is said to have the $\mathcal{H}_-$ index performance of the LPV system great than β if one of the following conditions is satisfied:

1) There exists $n_\theta+1$ symmetric matrixes $P_j, j = 1,2\cdots,n_\theta$ and L such that

$$\begin{aligned} \Xi_-^a(A(\theta) - LC(\theta), B_f(\theta) - LD_f(\theta), \quad (26) \\ C(\theta), D_f(\theta), P(\theta), \beta) > 0 \end{aligned}$$

where $P(\theta) = P_0 + \theta_1 P_1 + ... + \theta_{n_\theta} P_{n_\theta}$.

2) There exist matrices L, L_0, and $n_\theta+1$ symmetric matrices $P_j, j=1,2\cdots,n_\theta$ and $P^0(\theta)$ such that

$$\begin{pmatrix} \check{N}_1(\theta) & \check{N}_3(\theta) & \check{N}_4(\theta) & P(\theta) \\ * & \check{N}_2(\theta) & 0 & D_f(\theta)^T L^T \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix} > 0 \quad (27)$$

where

$$\begin{aligned} \check{N}_1(\theta) &= 2P^0(\theta)P(\theta)(\theta) + 2P(\theta)P^0(\theta) \\ &\quad -2P^0(\theta)P^0(\theta) + C(\theta)^T L_0^T LC(\theta) + \\ &\quad C(\theta)^T L^T L_0 C(\theta) + P(\theta)A(\theta) + \\ &\quad A(\theta)^T P(\theta) + C(\theta)^T C(\theta) + \Sigma_{j=1}^{n_\theta} P_j \dot{\theta}_j \\ \check{N}_2(\theta) &= D_f(\theta)^T D_f(\theta + D_f(\theta)^T L_0^T LD_f(\theta) \\ &\quad +D_f(\theta)^T L^T L_0 D_f(\theta) - \\ &\quad D_f(\theta)^T L_0^T L_0 D_f(\theta)) - \beta^2 I \\ \check{N}_3(\theta) &= P(\theta)B_f(\theta) + C(\theta)^T D_f(\theta) \\ \check{N}_4(\theta) &= P(\theta) + C(\theta)^T L^T \\ P(\theta) &= P_0 + \theta_1 P_1 + ... + \theta_{n_\theta} P_{n_\theta} \\ P^0(\theta) &= P_0^0 + \theta_1 P_1^0 + ... + \theta_{n_\theta} P_{n_\theta}^0 \end{aligned}$$

Proof: The proof is omitted. ■

The affine $\mathcal{H}_-$ index sensitivity conditions above are parameter dependent and they can be reduced to finite LMI set constraints by the multiconvexity technique.

Lemma 5.4: The dynamical system (14) can achieve a $\mathcal{H}_-$ index performance β if one of the following conditions is satisfied:

1) There exists $n_\theta+1$ symmetric matrix $P_j, j = 1,2\cdots,n_\theta$ and L such that

$$\begin{aligned} \Xi_-^a(A(\omega_i) - LC(\omega_i), B_f(\omega_i) - LD_f(\omega_i), \quad (28) \\ C(\omega_i), D_f(\omega_i), P(\omega_i), \beta) > 0 \end{aligned}$$

and

$$\begin{pmatrix} H_j & P_j(B_{f,j} - LD_{f,j}) \\ * & -D^T D_j \end{pmatrix} \leq 0 \quad (29)$$

where $H_j = (A_j - LC_j)^T P_j + P_j(A_j - LC_j) + C_j^T C_j$,$i=1,2,...,n_\theta$.

2) There exist matrices L, L_0, and $n_\theta+1$ symmetric matrices $P_j, j=1,2\cdots,n_\theta$ and $P^0(\omega_i)$ such that

$$\begin{pmatrix} \check{N}_1(\omega_i) & \check{N}_3(\omega_i) & \check{N}_4(\omega_i) & P(\omega_i) \\ * & \check{N}_2(\omega_i) & 0 & \check{N}_5 \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix} > 0 \quad (30)$$

and

$$\begin{pmatrix} \check{B}_{11} & P_j B_{f,j} \\ * & \check{B}_{22} \end{pmatrix} \leq 0 \quad (31)$$

$$\begin{aligned}\check{N}_5 &= D_f(\omega_i)^T L^T \\ \check{B}_{11} &= A_j^T P_j + P_j A_j - (L_0 C_j)^T (L C_j) \\ &\quad -(L C_j)^T (L_0 C_j) - (L_0 C_j)^T (L_0 C_j) \\ \check{B}_{22} &= D_{f,j}^T D_{f,j} - (L D_{f,j})^T (L_0 D_{f,j}) \\ &\quad -(L_0 D_{f,j})^T (L D_{f,j}) + (L_0 D_{f,j})^T (L_0 D_{f,j})\end{aligned}$$

where $i = 1, 2, ..., r$ and $j = 1, 2, ..., n_\theta$.

Proof: The proof is straightforward by using multiconvex lemma 2.12. ∎

C. $\mathcal{H}_-/\mathcal{H}_\infty$ Fault Detection Observer Design for LPV systems

Theorem 5.5: Consider the system (5), the fault detection observer (6) and the associated residual error dynamics (7), then the observer has the robustness performance γ from d to r, and the $\mathcal{H}_-$ index performance β from f to r if there exist matrix L , L_0, symmetric matrices $X(\theta) > 0$, $X^0(\theta) > 0$, $P^0(\theta)$ and $P(\theta)$ such that inequalities (19) and (27) are satisfied.

The inequalities (19) and (27) have infinite LMI set to be solved. By using the Gridding Method in [11], a numerical solution can be achieved. In order to solve the problem iteratively, we need the start values of P^0, X^0. For this reason, we need to know all the initial values of $P_j^0, X_j^0, j = 0, 1, 2, ..., n_\theta$. These can not be trivially solved by the same trick in algorithm 4.4. However, we can use the method in algorithm 4.4 for a constant P and X to initialized P_0^0 and X_0^0 and set $P_j^0, X_j^0, j = 1, 2, ..., n_\theta$ as $\mathbf{0}$. At each griding point, a LMI set can be constructed. An algorithm can be derived similar to algorithm 4.4. We refer to this algorithm as **algorithm II** or gridding algorithm. The detailed steps are omitted due to space limitation.

The gridding algorithm provides a numerical solution for handling the infinity LMI set. It can achieve almost the best performance if the grid density is high enough. Clearly, the computation burden will be also high, especially when the system has a high order. An alternative strategy is to reduce the infinite LMI sets to be finite by using the multiconvex technique which are stated in lemma 5.2 and 5.4. Now we can have the following theorem and algorithm.

Theorem 5.6: Given scalars $\gamma > 0$, $\beta > 0$ and consider the error dynamical system (7), its $\mathcal{H}_\infty$ performance is less than γ and its $\mathcal{H}_-$ index performance greater than β if there exist matrices L, L_0, symmetric matrix $P^0(\omega_i)$ and $P(\omega_i)$, $X(\omega_i) > 0$ and $X^0(\omega_i) > 0$ such that inequalities (24), (25) and (30), (31) hold.

Algorithm 5.7: **Mixed index Observer design algorithm III:** Given the system model $\Sigma(\theta)$ as in (5)

1) Maximize β subject to $P < 0$, (11) and (16) with $F_d = -F_f$ and $P = -X$. Then calculate the optimal filter gain matrix L_{opt} using (12). Let $L_0 = L_{opt}$.
2) With $L = L_0$, maximize β subject to (15) to get P_{opt}, and minimize $Tr(X)$ subject to $X > 0$ and (10) to get $X_{opt} > 0$. Let $X_0^0 = X_{opt}$ and $P_0^0 = P_{opt}$ and $X_j^0 = \mathbf{0}, P_j^0 = \mathbf{0}, j = 1, 2, ..., n_\theta$.
3) With L_0, $X^0(\omega_i)$ and $P^0(\omega_i)$, maximize $\gamma^2 - \beta^2$ subject to $X(\omega_i) > 0$, (24), (25) and (30), (31) to get L_{opt}, $X_{j,opt}$, $P_{j,opt}$ and β_{opt}. Let $L_0 = L_{opt}$, $X^0(\omega_i) = X_{opt}(\omega_i)$, $P^0(\omega_i) = P_{opt}(\omega_i)$.
4) Repeat step 3 till a certain number of iterations are reached or $\gamma^2 - \beta^2$ reaches almost a constant.

VI. Conclusions

This paper has investigated the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ observer design for a class of linear parameter varying system. Three algorithms are proposed in terms of linear matrix inequalities. The first algorithm mainly bases on the vertex property of polytopic LPV systems. Since constant matrix X and P are used for the robustness and sensitivity conditions, it is a conservative algorithm. To reduce the conservatism, parameter dependent matrix $X(\theta)$ and $P(\theta)$ are introduced for the robustness condition and sensitivity condition, respectively. Gridding method is applied to achieve a numerical solution, which is our second algorithm. To avoid using the gridding method and reduce the computation burden, multiconvex technique is utilized to achieve a finite LMI set constraints (algorithm III).

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