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Cooperative Path Planning for a Class of Carrier-Vehicle Systems.

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***Abstract*— In this work we concentrate on the problem of path planning in a scenario in which two different vehicles with complementary capabilities are employed cooperatively to perform a desired task in an optimal way. In particular we consider the case in which a vehicle carrier, typically slow but with virtually infinite operativity range, and a carried vehicle, which on the contrary is typically fast but with a shorter operative range, can be controlled together to pursuit a certain mission while minimizing a pre-defined cost function. In particular we will concentrate on a particular scenario, which we denoted as "fast-rescue" problem, providing optimal and heuristic solutions to various cases.**

I. Introduction

The complexity of many applications envisioned for future autonomous vehicle networks, ranging from planetary exploration to security missions, requires a broad range of capabilities for individual units—ranging from air, ground or sea mobility, to sophisticated multi-modal sensor suites and actuation devices—which cannot be implemented on a single platform class. Rather, it may be necessary to coordinate diverse specialized units to attain complex objectives in a reliable, timely, and efficient fashion [14]. While considerable progress has been made on cooperative control of networks of homogeneous vehicles (see for example [1], [2], [6], [8], heterogeneous networks are still relatively poorly understood. In particular, it is of interest to understand how to optimally exploit the different capabilities of individual vehicles.

In this paper, we concentrate on a very simple system of heterogeneous vehicles, arising from the combination of (i) a slow autonomous surface carrier (typically a ship), with long range operational capabilities, and (ii) a faster vehicle (typically an helicopter, an UAV or an offshore vehicle) with a limited operative range. The carrier is able to transport the faster vehicle, as well as to deploy, recover, and service it. Even though this two-vehicle system is very simple, many interesting problems can be stated, involving optimization and coordination problems [7], [12], [13].

Here we will deal mostly with the so-called "fast rescue problems." By "fast rescue problems" we mean those scenarios in which one (or eventually many) "targets" with known and non-changing position have to be visited in the shortest possible time. This is the case of rescue or event monitoring missions in which it is important to arrive quickly at one or more desired locations.

As a preliminary work, in this paper we studied such problems by assuming holonomic dynamic models to represent the behavior of both the carrier and carried vehicles.

The paper is organized as follows. Section 2 introduces the carrier-vehicle systems, its dynamics and the constraints related. In Paragraph 3 the rescue problems are stated and in Paragraph 4 some solutions to the most common cases are proposed. Finally some conclusion and some indication on future research end the paper.

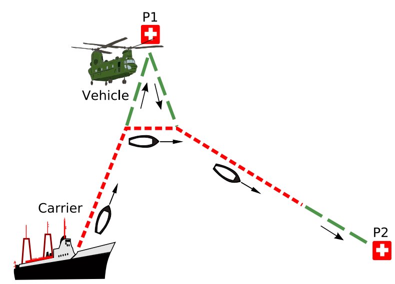

Fig. 1. The *carrier-vehicle* system on a rescue scenario.

II. The Carrier-Vehicle Hybrid System

The system we are going to deal with is composed by two different vehicles, a *vehicle carrier* (also denoted in the following as *carrier*), whose variables and functions will be denoted by subscript $\cdot_c$, and a *carried vehicle*, denoted by subscript $\cdot_v$. In the following we will refer to the combined system as the *carrier-vehicle system*.

To derive a mathematical model for the system, we will consider the vehicles as points belonging to the Euclidean space $\mathbb{R}^2$. The admissible path followed by each vehicle is a continuous curve $\Gamma : [0, t_f] \mapsto \mathbb{R}^2$. Let us fix an inertial frame $F_i = \{O_i, \overrightarrow{i}, \overrightarrow{j}\}$ and define with the notation v^i vectors $v \in \mathbb{R}^2$ expressed in F_i. The following state variables are introduced

$$p_c = \left[\begin{array}{cc} x_c^i & y_c^i \end{array} \right]^T \quad p_v = \left[\begin{array}{cc} x_v^i & y_v^i \end{array} \right]^T . \qquad (1)$$

with $p_c(t) \in \mathbb{R}^2$, $p_v(t) \in \mathbb{R}^2$ respectively the position of the vehicle carrier and the vehicle at time t in the inertial frame F_i. We will assume that the position of the carrier $p_c(t)$ evolves accordingly to the first order O.D.E

$$\dot{x}_c = V_c \cos(\phi_c) \quad \dot{y}_c = V_c \sin(\phi_c) \qquad (2)$$

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with $V_c \in \mathbb{R}^+$ the fixed velocity of the carrier and $\phi_c \in \mathbb{R}$ the control input.

To model the dynamics of the carried vehicle we distinguish between two different situations:

1) when it is not carried it evolves following its free behavior:

$$\dot{x}_v = V_v \cos(\phi_v) \quad \dot{y}_v = V_v \sin(\phi_v) \tag{3}$$

with $V_v \in \mathbb{R}^+$, $V_v > V_c$ and $\phi_v \in \mathbb{R}$ the control input for the vehicle.

2) when it is carried is a function of the carrier dynamics and in particular it will coincide with the carrier position, $p_v(t) = p_c(t)$.

From the above arguments it appears that the carried vehicle dynamics show an intrinsically hybrid behavior. For such a reason let us introduce a binary variable $c(t) \in \{0, 1\}$ with the following semantic:

1) $c(t) = 1$ if the vehicle is carried;
2) $c(t) = 0$ if it is not carried .

By means of $c(t)$ we define a 2 states automaton (see also [17]) ($c(t) = 0$ and $c(t) = 1$) that changes his status accordingly to the following guard conditions:

1) an input command, $u_w(t) \in \{0, 1\}$, that denote the "will" of change the actual state $c(t)$;
2) "compatibility constraints" that denote the conditions in term of system's state under which approach of the two vehicles are possible, nominally $p_c(t) = p_v(t)$.

Since one of the distinguish feature of the carried-vehicle is to have a finite operativeness (e.g. *fuel*), we introduce the internal dynamics

$$\dot{a}(t) = \begin{cases} -1 & \text{if } c(t) = 0 \\ 0 & \text{if } c(t) = 1 \end{cases} \tag{4}$$

where $a \in \mathbb{R}$ is the operativeness of the vehicle.

The operativeness of the vehicle is indeed decreasing when the vehicle is not carried ($c(t) = 0$), since it is assumed it has to use its own power, while it remains constant when the vehicle is on the deck of the carrier ($c(t) = 1$). For the sake of simplicity, it is supposed that anytime an approach occurs, $a(t)$ is restored to a certain constant default value $\bar{a}$. The faster vehicle is constrained to have non-negative operativeness; if this condition fails to be true an event occurs which bring the system to a fault state. This faulty situation is captured by adding a further state to the automaton describing the vehicle's dynamics, i.e. $c(t) = -1$ (faulty state).

III. The Rescue Problems Family

In this paper we are interested in studying and provide solutions for the so called "fast rescue problems. By fast rescue problems we mean those missions in which an ordered collection of n points, namely $q_1, \ldots, q_n$ where $q_i \in \mathbb{R}^2$, have to be visited ordinately in the "shortest possible time" by the fast vehicle, eventually satisfying a prescribed takeoff-visit-landing sequence.

More formally, if we define by $t_i \in \mathbb{R}^+$ the time in which q_i is visited, we can state the general rescue problem in the following form:

Given *a carrier-vehicle system* with *initial conditions* $p_c(0) = p_v(0) = p_{c,0} \in \mathbb{R}^2$ and $c(0) = 1$, *and a set of objective points* $\hat{q}_{list} = \{q_1, q_2, ..., q_n\}$ *with* $q_1 \in \mathbb{R}^2, i = 1, .., n$
find *the optimal inputs* $(\hat{u}(t))_{t:[0,\infty)}$
minimizing *a certain objective function* $obj(t_1, ..., t_n)$
such that *a certain given sequence of* $c(t)$ *state changes and visits events is satisfied*
and eventually *the vector* $\hat{t} = \begin{bmatrix} t_1 & ... & t_n \end{bmatrix}$ *belongs to a certain admissible set T*

where $\hat{u}(t) = [\phi_c(t), \phi_v(t), u_w(t)]^T$ is the vector representing all the inputs of the system.

In the next section we will introduce and solve some problems belonging to this family.

IV. Some solution

A. Fastest approach to a point

Given an initial point p_0 and a point to reach q_1, we want to find a trajectory for the carrier-vehicle system described above such that t_1 is minimized. Because the time spent in any trajectory between two points by each one of the two vehicles can be always equal or grater of the time to follow a straight line (we recall that the admissible path is a continuous curve), the problem can be reshaped into finding a take-off point p_{to} such that the carried vehicle has enough operativeness to take-off at p_{to}, and to reach q_1: $||p_{to} - q_1||_2 \leq V_v a$.

A formal definition of the problem is then

$$\begin{cases} \min\limits_{p_0} \; t_1 \\ \dfrac{||p_{to} - q_1||_2}{V_v} \leq \bar{a} \\ \dfrac{||p_o - p_{to}||_2}{V_c} + \dfrac{||p_{to} - q_1||_2}{V_v} \leq t_1 \end{cases} \tag{5}$$

A straightforward lower bound on the cost, that is valid in all the missions we will introduce here, is the optimal path of a vehicle that can always go at velocity V_c and it is allowed for certain amount of time ($\bar{a}$) to go at velocity V_v. Such a fastest path is obtained by means of a straight line between the starting point p_0 and the arrival point q_1 such that for the maximum allowed time $\bar{a}$ it goes to velocity V_v, and for the remaining time at velocity V_c, i.e.

$$t_{lb} = \\ = \min\left\{\bar{a}, \frac{||p_0 - q_1||_2}{V_v}\right\} + \max\left\{0, \frac{||p_0 - q_1||_2 - V_v\bar{a}}{V_c}\right\} \tag{6}$$

The optimal solution for problem (5) is then

$$p_{to} = \begin{cases} p_0 & if \; \dfrac{||p_0 - q_1||_2}{V_v} \leq \bar{a} \\ p_0 + V_v \, \bar{a} \; (q_1 - p_0) \end{cases} \tag{7}$$

since it's cost coincides with the lower bound (6).

The simple geometric intuition behind such a solution is the following.

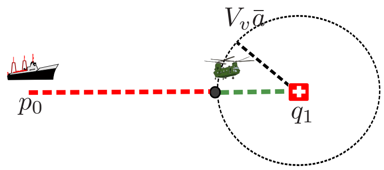

Fig. 2. Geometry of the fastest approach to a point problem.

The carried object can be launched only after that the vehicle is inside a circle of radius $V_v\bar{a}$. Then, because carried vehicle is faster of the carrier, as soon as the carrier is inside this circle, it has to launch the carried one that has to follow a straight line to the objective. It is trivial to see that no other choice is faster then this one. Moreover by simple geometric consideration, the shortest path for the carrier to approach the circle is a straight line directed to its center.

Remark - Notice that since $V_v > V_c$, the optimal solution always impose $c(t) = -1, \forall t > t_1$, The reason is that it has to use all the operativeness a to reach q_i as fast as possible. This implies that there is no *fuel* to come back to the carrier that being slower can't arrive in that point at the same time. □

B. Fastest approach to a point and re-entry mission

Given an initial point p_0 and a point to reach q_1 we want to find a trajectory for the carrier-vehicle system described above such that the arrival time at q_1 is minimized and, after the point is visited, the fast vehicle is able to come back to the carrier. The problem can be reshaped in finding a take-off point p_{to} and the landing point p_l and trajectories such that the following constraints are satisfied:

1) the carried vehicle trajectory is such that it has enough operativeness to take-off at p_{to}, to reach the objective and to arrive in p_l. This implies that the points have to be such that $||p_{to} - q_1||_2 + ||q_1 - p_l||_2 \leq V_v\bar{a}$;
2) the carrier vehicle trajectory is such that it is able to arrive from p_{to} to p_l in a time that is not greater than the maximum operativeness $\bar{a}$. This implies that the two points have to be at distance $||p_{to} - p_l||_2 \leq V_c\bar{a}$.

We want to minimize the "safe" arrival time t_l which is the sum of two contributions:

1) the time $t_{1,0}$ used by the carrier vehicle to reach p_{to} from p_0. Because for such a kind of vehicle the fastest path is always the straight line, $t_{1,0} = \frac{1}{V_c}||p_0 - p_{t0}||_2$;
2) the time $t_{2,0}$ used by the carried vehicle to arrive at q_1. Again the fastest path is the straight line then $t_{1,1} = \frac{1}{V_v}||p_{t0} - q_1||_2$.

As a consequence we can reduce the core of the problem to the following optimization problem

$$\begin{cases} \min \dfrac{1}{V_v}||p_{t0} - q_1||_2 + \dfrac{1}{V_c}||p_0 - p_{t0}||_2 \\ ||p_{to} - q_1||_2 + ||q_1 - p_l||_2 \leq V_v\bar{a} \\ ||p_{to} - p_l||_2 \leq V_c\bar{a} \end{cases} \quad (8)$$

Even in this case is possible to avoid numerical solutions and instead to find a closed form solution. Let us start from the geometry of the problem.

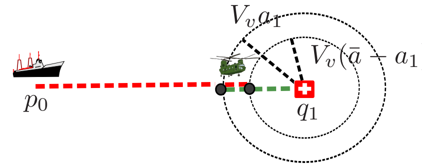

Fig. 3. Geometry of the fastest approach to a point an re-entry mission problem.

By exploiting the symmetry of the "operativeness circle", we know that the carried vehicle will be launched when it will enter in a certain circle of radius $V_v a_1 \leq V_v\bar{a}$. From that point the launched vehicle will follow the fastest trajectory to q_1, i.e. the straight line. Exactly like in the previous case, the optimal path for the carrier between the starting and the launching points is the straight line between p_0 and q_1. It remains now to discuss the choice of a_1 and how to build a feasible re-entry trajectory.

The main idea is that the optimum is obtained if the time to follow the straight line between p_o and q_1 is minimized, or, equivalently, if we choose the maximum a_1 such that the recovery is possible. Then, the carrier at time $t_{1,0} + \bar{a}$ has to be as near as possible to q_1. For such a reason it will continue on the straight line to q_1. By following this idea, we have then

$$a_1 = \min\left\{\frac{(V_c + V_v)}{2V_v}\bar{a}, \frac{||p_0 - q_1||_2}{V_v}\right\}$$

Remark - This is because the space the carrier will cover during the flight of the carried vehicle $V_c\bar{a}$ has to be equal to the difference of space between the "go" and the "come back" path of the launched vehicle i.e. $V_v a_1 - V_v(\bar{a} - a_1) = 2V_v a_1 - V_v\bar{a}$. □

The only further case to be analyzed is when q_1 is already inside the circle of radius a_1, in which trivially there is an instantaneous take-off. From the above discussion it follows that the minimum time to reach the objective q_1 is

$$\begin{aligned} t_1 = \max&\left\{0, \frac{||p_0 - q_1||_2}{V_c} - \frac{V_c + V_v}{2V_c}\bar{a}\right\} + \\ &+ \min\left\{\frac{V_c + V_v}{2V_v}\bar{a}, \frac{||p_0 - q_1||_2}{V_v}\right\}. \end{aligned} \quad (9)$$

C. Fastest approach to 2 far points with re-entry mission

Let us consider the case in which the carried vehicle has to reach two different points in an ordered way, minimizing the time t_2 and providing that, after each visit, it comes back to the carrier. Hereafter, in order to avoid the analysis of cases of low practical interest, it will be supposed that the starting point and the two points to reach are "far enough" with respect to the carried vehicle operativeness, nominally we have $d_{0,1} = ||q_1 - p_0||_2 > V_v\bar{a}$. and $d_{1,2} = ||q_2 - q_1||_2 > 2V_v\bar{a}$.

Because the shortest path between two different points in the Euclidean space $\mathbb{R}^2$ is represented by the straight line and because both the vehicles have the ability to "wait in a point", it is possible to recast the original problem of finding the optimal trajectory as the problem of fixing 4 (distinguished) points: the two take-off points p_{to1}, p_{to2} and the two landing points p_{l1}, p_{l2}. We can then state the following optimization problem:

$$\begin{array}{l} \min\ t_{p_0,p_{to1}} + t_{p_{to1},p_{l1}} + t_{p_{l1},p_{to2}} + t_{p_{to2},q_2} \\ \quad \frac{\|p_0 - p_{to1}\|_2}{V_c} = t_{p_0,p_{to1}} \quad \frac{\|p_{to1} - p_{l1}\|_2}{V_c} \leq t_{p_{to1},p_{l,1}} \\ \quad \frac{\|p_{to,1} - q_1\|_2}{V_v} + \frac{\|p_{to1} - q_1\|_2}{V_v} \leq t_{p_{to1},p_{l1}} \\ \quad \frac{\|p_{l1} - p_{to2}\|_2}{V_c} = t_{p_{l1},p_{t-o,2}} \quad \frac{\|p_{to2} - q_2\|_2}{V_v} \leq t_{p_{to2},q_2} \\ \quad \frac{\|q_2 - p_{l2}\|_2}{V_v} \leq t_{q_2,p_{l2}} \quad \frac{\|p_{l,2} - p_{to,2}\|_2}{V_c} \leq t_{p_{to,2},p_{l,2}} \\ \quad 0 \leq t_{p_{to1},p_{l1}} \leq \bar{a} \quad 0 \leq t_{p_{to2},p_{l2}} \leq \bar{a} \\ \quad 0 \leq t_{p_{to2},q_2} + t_{q_2,p_{l2}} \leq \bar{a} \end{array} \tag{10}$$

With an eye at (10), we observe that the last two terms of the cost function represent the objective of the 1-point fastest approach problem (8) with an initial condition that corresponds to the given landing point p_{l1}. For this reason the last term of the objective function can be optimized independently, allowing to further reduce the problem to the determination of two points p_{to1} and p_{l1}, i.e.

$$\begin{array}{l} \min\ t_{p_0,p_{to1}} + t_{p_{to1},p_{l1}} + t_{q2} \\ \quad \frac{\|p_0 - p_{to1}\|_2}{V_c} = t_{p_0,p_{to1}} \quad \frac{\|p_{to1} - p_{l1}\|_2}{V_c} \leq t_{p_{to1},p_{l,1}} \\ \quad \frac{\|p_{to1} - q_1\|_2}{V_v} + \frac{\|p_{to1} - q_1\|_2}{V_v} \leq t_{p_{to1},p_{l1}} \\ \quad \frac{\|p_{l1} - q_2\|_2}{V_c} - \frac{V_c + V_v}{2V_c}\bar{a} + \frac{V_c + V_v}{2V_v}\bar{a} = t_{q2} \\ \quad 0 \leq t_{p_{to1},p_{l1}} \leq \bar{a} \end{array} \tag{11}$$

The solution of such an optimization problem can be computationally hard. For such a reason, geometric solutions (eventually sub-optimal) have been investigated. In order to proceed we first derive an upper bound and two distinguished lower bounds to be used in the analysis.

An upper bound to the optimal cost can be found considering the case in which the carried vehicle never takes off (or equivalently has an instantaneous take-off and landing) and goes straight to the first point. Then, to reach the last point, it follows the optimal 1-point strategy with cost given by (9). The overall cost of this solution is given by

$$t_{up} = \frac{d_{0,1}}{V_c} + \frac{d_{1,2}}{V_c} - \bar{a}\left(\frac{V_c + V_v}{2V_c}\right) + \bar{a}\left(\frac{V_c + V_v}{2V_v}\right). \tag{12}$$

Remark - The upper bound (12) coincides with the optimal solution when the starting point p_0 and the objective points q_1 and q_2 are placed on the same line. □

A first lower bound to the optimal cost can be calculated assuming that the trajectory of the system is characterized by a speed V_v for $\bar{a}$ seconds, in the shortest-path between p_0 and q_1, and a speed V_c for the remaining time. Again from q_1 to the second objective q_2 we can make use of the 1-point optimal solution (9). The cost of such a lower bound is

$$t_{low} = \frac{d_{0,1}}{V_c} + \bar{a}\left(1 - \frac{V_v}{V_c}\right) + \frac{d_{1,2}}{V_c} - \bar{a}\left(\frac{V_c + V_v}{2V_c}\right) + \bar{a}\left(\frac{V_c + V_v}{2V_v}\right). \tag{13}$$

A second straightforward lower bound is the cost of the 1-point optimal solution between p_0 and q_2, discarding the fact we need to visit q_1. Such a lower bound has cost:

$$t_{low} = \frac{||q_2 - p_0||_2}{V_c} - \bar{a}\left(\frac{V_c + V_v}{2V_c}\right) + \bar{a}\left(\frac{V_c + V_v}{2V_v}\right) \tag{14}$$

In many significant cases it is possible to find a geometric solution whose cost is equal to those two lower bounds, and then optimal.

The main idea, underling the geometric constructions that we will present here, is to take advantage of the possibility of the carrier to launch the carried vehicle (that will visit q_1) and to take a shortcut in order to have a *rendezvous* somewhere on the straight line to q_2 before the carried vehicle has exhausted the *fuel*. This idea is depicted in Fig. 4.

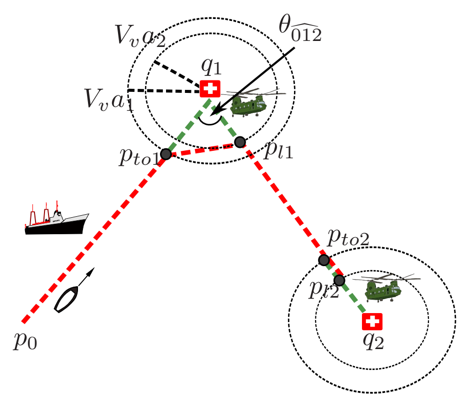

Fig. 4. The geometric construction behind Proposition 1.

The following propositions can be stated:

Proposition 1: Let $\theta_{\widehat{012}}$ be the smallest angle formed by the segments $\overline{p_0 q_1}$ and $\overline{q_1 q_2}$. If $\theta_{\widehat{012}} \leq \text{acos}\left(1 - 2(V_c^2/V_v^2)\right)$ then it exists at least one optimal solution to the problem such that the total cost is equal to lower bound (13)

Proof. Let us proceed with a constructive proof. As shown in Fig. 4, the idea is that the carried vehicle will be released at distance $V_v a_1$ from q_1 on the line segment $\overline{p_0 q_1}$ and will be recovered at distance $V_v a_2$ from q_1 on the straight path $\overline{q_1 q_2}$. In order to satisfy the hypothesis of the lower bound (13) and the operativeness constraints,

$$a_1 + a_2 = \bar{a} \tag{15}$$

has to be imposed.

As it is clear from the Fig. 4, between the take-off and the landing events, the carrier will "cut the edge" by following the straight segment $\overline{q_{to1}, q_{l1}}$ whose length will be denoted by $V_c t_{carrier}$. Because of operativeness constraints, this proposed solution will be a feasible one to the given problem if

$$t_{carrier} \leq \bar{a}. \tag{16}$$

By simple triangle consideration it is possible to see that $V_c t_{carrier} \leq V_v a_1 + V_v a_2 = V_v \bar{a}$. In particular by using Carnot theorem it is possible to write that

$$(V_c t_{carrier})^2 = (V_v a_1)^2 + (V_v a_2)^2 - 2V_v^2 a_1 a_2 \cos\left(\theta_{\widehat{012}}\right) . \quad (17)$$

Because of the condition (15), it is of interest, for a given angle $\theta_{\widehat{012}}$, to choose a_1 and a_2 in order to minimize $t_{carrier}$. Let us rewrite (17) by using (15)

$$\begin{aligned}(t_{carrier})^2 = &\frac{1}{V_c}\left[(V_v a_1)^2 + (V_v(\bar{a}-a_1))^2 + \right.\\ &\left.-2V_v^2 a_1(\bar{a}-a_1)\cos\left(\theta_{\widehat{012}}\right)\right] = \\ = &\frac{1}{V_c}\left[2V_v^2 a_1^2 + V_v^2\bar{a} - 2V_v^2\bar{a}a_1 - 2V_v^2\bar{a}a_1\cos\left(\theta_{\widehat{012}}\right) + \right.\\ &\left.-2V_v^2 a_1^2\cos\left(\theta_{\widehat{012}}\right)\right]\end{aligned} \quad (18)$$

Being $t_{carrier} \geq 0$, the minimum is reached when the minimum of $t_{carrier}^2$ is reached. Then by nullifying the derivative of (18) w.r.t. a_1, we have

$$\left[4a_1 - 2\bar{a} - 2\bar{a}\cos\left(\theta_{\widehat{012}}\right) - 4a_1\cos\left(\theta_{\widehat{012}}\right)\right] = 0 .$$

Consequently the choice of a_1 and a_2 that minimize $t_{carrier}$ is

$$a_1 = a_2 = \left(\frac{1+\cos(\theta)}{2+2\cos(\theta)}\right)\bar{a} = \frac{1}{2}\bar{a}$$

and the minimum $t_{carrier}$ is:

$$t_{carrier} = \bar{a}\frac{V_v}{V_c}\sqrt{\frac{1}{2} - \frac{\cos\left(\theta_{\widehat{012}}\right)}{2}}$$

By imposing condition (16) it is possible to find that the maximum angle for which this solution is feasible: $\sqrt{0.5\left(1-\cos\left(\theta_{\widehat{012}}\right)\right)} < V_c/V_v$. This inequality holds true if $\cos\left(\theta_{\widehat{012}}\right) \leq 1 - 2(V_c^2/V_v^2)$ that for $\theta_{\widehat{012}} \in [0, \pi)$ is $\theta_{\widehat{012}} \leq \text{acos}\left(1 - 2V_c^2/V_v^2\right)$ □

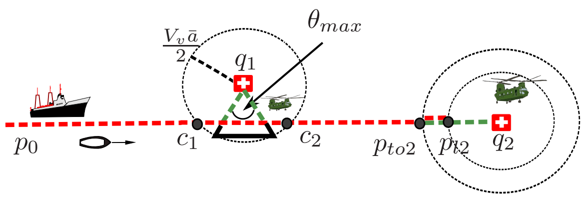

Fig. 5. The geometric interpretation behind Proposition 2.

Proposition 2: If the straight line between p_0 and q_2 touches in 2 points, respectively c_1 and c_2, the circle of center q_1 and radius $V_v\bar{a}/2$ and if the smallest angle defined by the segments $\overline{c_1 q_1}$ and $\overline{q_1 c_2}$ is bigger or equal than $\text{acos}\left(1 - 2\frac{V_c^2}{V_v^2}\right)$ then it exists at least one solution such that the total cost is equal to lower bound (14) .

Proof. The carrier will go directly to the last point through a straight line. By hypothesis, such a straight line will intersect a triangle, built like the one in Fig. 5, with angle $\theta_{max} = \text{acos}\left(1 - 2\frac{V_c^2}{V_v^2}\right)$. Following the proof of Proposition 1, it is known that the time used by the carrier to follow the basis of that triangle is equal to the time the fast vehicle uses to follow the other two sides. Then using those intersection point as take-off and rendezvous point, a feasible solution of cost (14) is reached. □

For the case not covered by the two proposition above some heuristics have been developed.

A first heuristic can be simply obtained by using the 1-point solution. The idea is two perform two time a "single point" iteration. Such an idea it is very simple and has guaranteed results both for the 2-point cases and for the generalized one and will be described in the next subsection.

A second heuristic here proposed is built in the following way. Let us build a triangle with one vertex on q_1 and the other two on the circle of center q_1 and radius $\frac{1}{2}\bar{a}V_v$ and whose angle on q_1 is equal to θ_{max}, such that it is included in the angle $\theta_{\widehat{012}}$ and such that it does not intersect the segment $\overline{p_0 q_2}$ as depicted in Fig. 6. If we fix the two points determined by the intersection of such a triangle and the circle as the take-off point and the rendezvous one

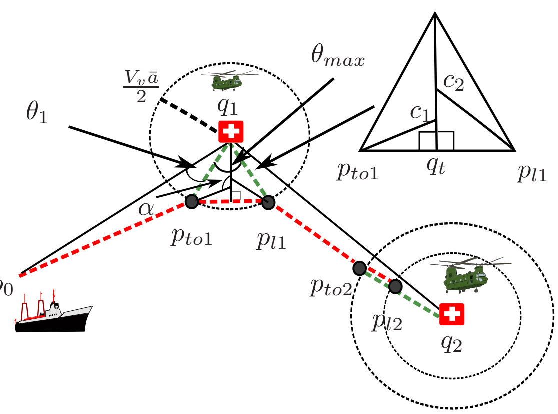

Fig. 6. The geometric interpretation behind the algorithm used for the 2 point case.

the total cost of the strategy will be

$$\begin{aligned}t_{eu} = &\frac{\sqrt{\left(\frac{1}{2}\bar{a}V_v\right)^2 + d_1^2 - (\bar{a}V_v)\,d_1\cos(\theta_1)}}{V_c} + \bar{a} + \\ &+\frac{\left(\frac{1}{2}\bar{a}V_v\right)^2 + d_2^2 - (\bar{a}V_v)\,d_2\cos\left(\theta_{\widehat{012}} - \theta_{\max} - \theta_1\right)}{V_c} + \\ &-\bar{a}\left(\frac{V_c+V_v}{2V_c}\right) + \bar{a}\left(\frac{V_c+V_v}{2V_v}\right)\end{aligned}$$

where $\theta_1 \in \left[0, \theta_{\widehat{012}} - \theta_{\max}\right]$ is the one defined in Fig. 6. In order to determine an optimal θ_1, a simple numerical optimization can be performed, as in Fig. 7.

It is possible to prove that it always exist at least a choice of $\theta_1 \in \left[\theta_{\widehat{012}} - \theta_{\max}\right]$ such that this solution, in the case Proposition 1 & 2 doesn't apply, it is always lower then the upper bound (12). For this purpose let us consider the particular solution $\theta_1 = \left[\theta_{\widehat{012}} - \theta_{\max}\right]/2$ as depicted in Figure 6, where, by construction, the angle denoted as α is always grater than $\pi/2$

Then let us draw the lines $\overline{p_0 p_{to1}}$ and $\overline{q_2 p_{l1}}$ until they touch the height segment of the triangle. Let us denote by c_1 and c_2 such an intersection point and by q_t the center of the basis of the triangle.

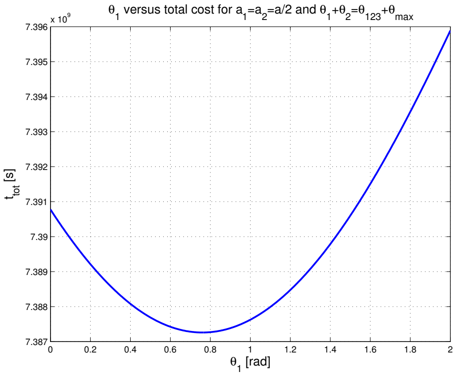

Fig. 7. The cost resulting from local search techniques applied to the free paramenter θ_1.

Because $\alpha > \pi/2$ then, from the properties of the triangles, $\|c_1 - p_0\|_2 < \|q_1 - p_0\|_2$ and $\|c_c - p_1\|_2 < \|q_2 - q_1\|_2$. Let us concentrate on $\|c_1 - p_0\|_2 < \|q_1 - p_0\|_2$ (the same will be on the other inequalities, by symmetry). We observe that $\|p_{to1} - p_0\|_2 + \|c_1 - p_{to1}\|_2 < \|q_1 - p_0\|_2$.

Since $\|c_1 - p_{to1}\|_2$ is the hypotenuse of a square triangle of basis $1/2\,\bar{a}V_c$, we have also that $\frac{1}{2}\bar{a}V_c < \|c_1 - p_{to1}\|_2$ and finally $\frac{1}{2}\bar{a}V_c + ||c_1 - p_0|| < ||q_1 - p_0||_2$. By using this result it is straightforward to prove that this heuristic gives always results that are lower then (12).

D. Fastest approach to n far points with re-entry mission

As well as the 2 points case, to visit a certain number n of ordered point q_1, q_2, ...q_n starting from an initial point p_0 is not an easy problem.

Let us denote with $d_{i,i+1}$ the Euclidean distance between the point i and the next desired point $i+1$, where for sake of simplicity $d_{0,1}$ denotes the distance between the initial position of the carrier p_0 and q_1. An upper bound to the optimal solution can be given for the case when the carried vehicle never takes-off, except to reach the last point, and the carrier itself visits all the points. Such an upper bound has the following total cost

$$t_{n,up} = \left[\sum_{i=1}^{n} \frac{d_{i-1,i}}{V_c}\right] - \frac{(V_c+V_v)}{2V_c}\bar{a} + \frac{(V_c+V_v)}{2V_v}\bar{a} \quad (19)$$

The lower bound can be reached in the same way of the 2 points case by supposing that the carrier-carried couple can be seen as a vehicle able to go at velocity V_v for a time $\bar{a}$ for each point to reach. This would mean that the carrier is always able to "cut the edge": just like in the 2 points case this optimal solution is practically reachable only for small angles between the lines connecting 2 consecutive points). Such a lower bound has the following cost

$$\begin{aligned} t_{n,low} &= \sum_{i=1}^{n-1}\left[\frac{d_{i-1,i} - \bar{a}V_v}{V_c} + \bar{a}\right] + \\ &+ \left[\frac{d_{n-1,n}}{V_c} + \left(\frac{(V_c+V_v)}{2}\bar{a}\right)\left(\frac{1}{V_v} - \frac{1}{V_c}\right)\right] \\ &= \sum_{i=1}^{n-1}\left[\frac{d_{i-1,i}}{V_c}\right] + (n-1)\,\bar{a}\left(1 - \frac{V_v}{V_c}\right) + \\ &+ \left(\frac{(V_c+V_v)}{2}\bar{a}\right)\left(\frac{1}{V_v} - \frac{1}{V_c}\right) \end{aligned} \quad (20)$$

While the problem itself can be hard to solve, it is possible to build heuristic solutions able to guarantee some performances. Here we will introduce and discuss one of them based on the 1-point fastest approach with re-entry mission solution. Further possible better heuristic will be only mentioned in this paper and can be build based on the results proposed for the 2 points case.

The basic idea of the "1-step heuristic" is the one of using, at each time step, the 1-step solution: the carrier will proceed straight to the next objective point and the vehicle will take-off only when at a distance $d_{to} = (V_c + V_v)\bar{a}/2$ from the objective. The carrier will keep going straight while the carried vehicle will visit the point and come back. The rendezvous point will be on the same line at distance $d_l = (V_v - V_c)\,\bar{a}/2$ from the objective. From this rendezvous point the ship will go towards to the next point to visit. By exploiting Carnot theorem, the total cost for such a strategy can be expressed in closed form

$$t'_n = \frac{d_1 - d_l + \sum_{i=2}^{n-1}\left[\sqrt{d_l^2 + d_i^2 - 2d_i d_l \cos(\theta_i)} - d_l\right]}{V_c} + \\ + \frac{\sqrt{d_l^2 + d_n^2 - 2d_n d_l \cos(\theta_i)} - d_{to}}{V_c} + \frac{d_{to}}{V_v} \quad (21)$$

Where θ_i is the angle resulting from the line between $\overline{p_{i-1}p_i}$ and $\overline{p_i p_{i+1}}$. Simple computations show that the cost (21) has the interesting property of satisfying the following inequality

$$t_{n,low} \le t'_n \le t_{n,up}\,.$$

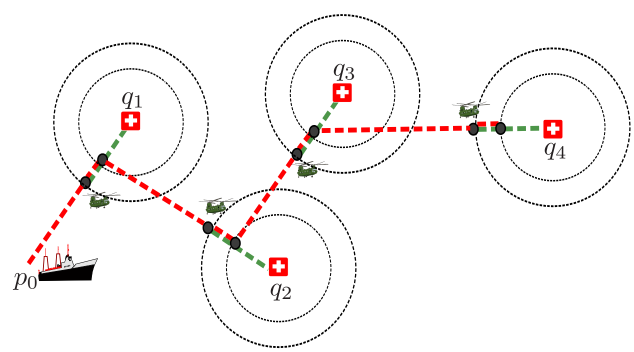

Fig. 8. The geometric interpretation behind the algorithm proposed to visit a set of n (4 in the figure) ordered points.

Inspired by the the results for the 2 points case, we propose also a second euristic solution which is based on the idea of satisfying the condition of Proposition 1 by suitably "slow down" the velocity of the vehicle such that the rendezvous is possible. Let us denote with $\theta_{list} = \{\theta_1, \theta_2, ...\theta_{n-1}\}$ the set of the $n-1$ angles $\in [0, \pi]$ between the straight lines that connect consecutive points. For each element i of θ_{list} we compute the *maximum rendezvous velocity* $\bar{v}_i$ according to the following algorithm

- if $\theta_i \equiv 0$, $\bar{v}_i = V_v$
- else $\bar{v}_i = \min \left\{ V_v, \sqrt{\dfrac{2V_c^2}{1-\cos\theta_i}} \right\}$

The *maximum rendezvous velocity* $\bar{v}_i$ is then used in order to bound the velocity of the vehicle to reach the desired point i following the shortest path depicted in Fig. 9. The cost of this solution can be computed as

$$\begin{aligned} t"_n &= \sum_{i=1}^{n-1} \left[\frac{d_{i-1,i} - \bar{a}\bar{v}_i}{V_c} + \bar{a} \right] + \\ &+ \left[\frac{d_{n-1,n}}{V_c} + \left(\frac{(V_c+V_v)}{2} \bar{a} \right) \left(\frac{1}{V_v} - \frac{1}{V_c} \right) \right] \end{aligned} \quad (22)$$

Simple computations shows that the cost (22) verify

$$t_{n,low} \leq t"_n \leq t_{n,up} \, .$$

and moreover if $\bar{v}_i = V_v$ for all $i = 1, ..n-1$, which in turns reduces to conditions on the angles and the ratio between the velocity of the carrier and the vehicle, then it coincides with the lower bound (20).

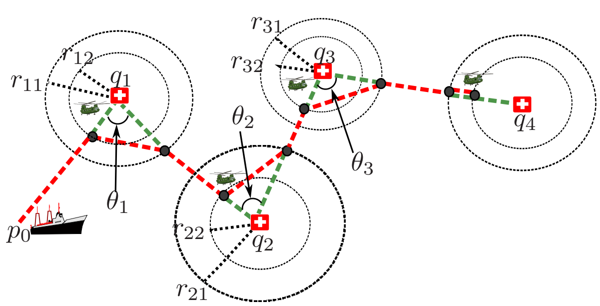

Fig. 9. The geometric interpretation behind the second algorithm proposed to visit a set of n (4 in the figure) ordered points. Notation: $r_{11} = \bar{v}_1 a_{11}$, $r_{12} = \bar{v}_1 a_{12}$, $r_{21} = \bar{v}_2 a_{21}$, $r_{22} = \bar{v}_2 a_{22}$, $r_{31} = \bar{v}_3 a_{31}$, $r_{32} = \bar{v}_3 a_{31}$, with $a_{i,1} + a_{i,2} = \bar{a}$.

V. Conclusions and Future Work

In this paper we dealt with the problem of path planning for two complementary vehicles: a slow long-range carrier and a fast vehicle with a limited operativeness. In particular we focused on the "fast rescue problems" i.e. the family of problems in which one has to visit an ordered set of point in minimum time. Several results about optimal solutions and approximation heuristics have been presented. Work is being currently done to extend the results to vehicles with nonholomic constraints.

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