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A Lyapunov approach to second-order sliding mode controllers and observers

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***Abstract*— In this paper a strong Lyapunov function is obtained, for the first time, for the supertwisting algorithm, an important class of second order sliding modes (SOSM). This algorithm is widely used in the sliding modes literature to design controllers, observers and exact differentiators. The introduction of a Lyapunov function allows not only to study more deeply the known properties of finite time convergence and robustness to strong perturbations, but also to improve the performance by adding linear correction terms to the algorithm. These modification allows the system to deal with linearly growing perturbations, that are not endured by the basic supertwisting algorithm. Moreover, the introduction of Lyapunov functions opens many new analysis and design tools to the Higher Order Sliding Modes research area.**

I. INTRODUCTION

Sliding mode approach to control and observation is widely used due to its attractive characteristics of finite-time convergence and robustness to uncertainties. Sliding mode control has been thoroughly studied, from both practical and theoretical point of view [1], [3], [11], [12], [14], [16], [17], [21]. Subjects of increasing interest are the sliding modes based observers [4], [7], [8], [20], [22].

In most cases, sliding modes are obtained by means of injecting a non-linear discontinuous term, depending on the output error, into the controlling or observing system. The discontinuous injection must be designed such that the trajectories of the system are forced to remain on some sliding surface in the error space. The resulting motion is referred to as sliding mode [21]. This discontinuous term is the one which enables the system to reject disturbances and also some classes of mismatches between the actual system and the model used for design [20].

Levant [14], [16] defines the sliding mode order by means of a smooth dynamic system with a smooth output function σ, closed by some possibly dynamical discontinuous feedback, with an output σ to be forced to zero. Then, provided the successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$ is non-empty and consists of trajectories in the sense of Filippov [9], the motion on the set $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$ is said to be an *rth-order sliding mode*. The *r*th derivative $\sigma^{(r)}$ is supposed to be discontinuous or non existent.

The standard sliding mode is of the first order (i.e. $\dot{\sigma}$ is discontinuous) [14], [16]. It is known as robust and very accurate with respect to various classes of internal and external perturbations [17], but it is restricted to the case in which the output relative degree is 1, i.e. the discontinuous injection appears already in $\dot{\sigma}$. Besides, the high frequency switching that produces the sliding mode may cause chattering effect [3], [17]. Higher order sliding modes (HOSM) appears sometimes in systems with traditional sliding mode control or they are deliberately introduced because it has been found that finite time convergent HOSMs preserve the features of the first order sliding modes and can improve them, if properly designed, by eliminating the chattering [14], [16].

For the first order sliding modes, it is common to deal with the issues of stability, robustness and convergence rate of the equilibrium by means of a Lyapunov approach [21], [22], [20]. For higher order sliding modes, a similar treatment has not been developed until now. Instead, it is usual to use majorant curves or homogeneity based methods to study convergence [7], [8], [11], [15].

In [7], [8] a second order sliding mode (SOSM) observer, based on the super twisting algorithm [11] is studied. There, its finite time convergence is proved by means of a majorant curve, and its robustness to bounded perturbations is analyzed. For this algorithm we study its stability and finite time convergence characteristics by means of Lyapunov functions. This approach allowed us to extend the class of perturbations and uncertainties originally admitted by SOSM to include root square growing ones. Besides, by means of the addition of linear terms to SOSM (SOSML), linear growing perturbations are included too. Another advantage of the use of Lyapunov functions is that it is possible to obtain explicit relations for the design parameters. Note that the basic algorithm studied here is useful for control and observation, although we will emphasize the interpretation for the observer design.

In the following, we first present Lyapunov functions for basic and perturbed SOSM (section II). In section III, we propose an improvement for SOSM that enables it to deal with linear growing perturbations (SOSML). A simple example of the application of SOSML is presented in IV, and at last, some conclusions are presented.

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II. A LYAPUNOV FUNCTION FOR SOSM

An important algorithm for control and observation using SOSM is the so called Super Twisting Algorithm [17], that is described by the differential inclusion

$$\begin{aligned} \dot{x}_1 = & \quad -k_1 \left|x_1\right|^{1/2} \operatorname{sign}\left(x_1\right) + x_2 + \varrho_1\left(x,t\right) \\ \dot{x}_2 = & \quad -k_3 \operatorname{sign}\left(x_1\right) + \varrho_2\left(x,t\right) \end{aligned} \tag{1}$$

where x_i are the scalar state variables, k_i are gains to be designed, and ϱ_i are perturbation terms. The solutions are trajectories in the sense of Filippov [9]. We propose, for the first time, a Lyapunov function to ensure the convergence in finite time of all trajectories of this system to zero, when the gains are adequately selected, and for some kinds of perturbations. We construct first for the unperturbed dynamics (with $\varrho_1 = \varrho_2 = 0$) a *strong* Lyapunov function, that is, one possessing a negative definite time derivative. In a second step we take advantage of this property of a strong Lyapunov function to show that, for certain class of perturbations, the Lyapunov function still has a negative definite time derivative.

A. Unperturbed dynamics

For system (1) we will show that the function

$$V\left(x\right) = 2k_3\left|x_1\right| + \frac{1}{2}x_2^2 + \frac{1}{2}\left(k_1\left|x_1\right|^{1/2}\operatorname{sign}\left(x_1\right) - x_2\right)^2 \tag{2}$$

that is continuous everywhere but not differentiable at $x_1 = 0$, is a strong Lyapunov function for the system (1) without perturbations.

Remark 1: Since $V\left(x\right)$ (2) is continuous but not differentiable, a nonsmooth version of Lyapunov's theory is required (see [5], and the recent tutorial paper [6] for an overview). Note that the usual version for locally Lipschitz Lyapunov functions [9], [2], [19] is not appropriate here, since $V\left(x\right)$ is not locally Lipschitz. Therefore, the Proximal Subdifferential [5], and not the Generalized Gradient of $V\left(x\right)$, has to be employed. In the present case, however, a simpler method can be used, since the state trajectories $\varphi\left(t,x_0\right)$ of the differential inclusion (1) are absolutely continuous functions, and, therefore, $V\left(\varphi\left(t,x_0\right)\right)$ is a continuous function of time. Due to the lack of Lipschitzness of $V\left(x\right)$ it is not possible to assure the absolute continuity of $V\left(\varphi\left(t,x_0\right)\right)$, and its differentiability almost everywhere. However, $V\left(x\right)$ is continuously differentiable, except on the set $S = \left\{\left(x_1,x_2\right)\in\mathbb{R}^2 \mid x_1 = 0\right\}$. It is easy to see that the trajectories of the system (1) just cross the surface S and cannot stay on it, except when the origin $x = 0$ has been reached. This means that $V\left(\varphi\left(t,x_0\right)\right)$ is differentiable for almost every t, and on those points the derivative can be calculated in the usual way, applying the chain rule. This shows that in aplying Lyapunov's theorem one can just consider the points where $V\left(x\right)$ is differentiable. This argument is valid in all the proofs of the present paper, so that no further discussion of these issues will be required.

Theorem 2: Suppose that $k_1 > 0$ and $k_3 > 0$. Then all trajectories of the unperturbed system (1), with $\varrho_1 = \varrho_2 = 0$, converge in finite time to the origin $x = 0$, in a time $t\left(x_0\right)$ smaller than $T = 2V^{1/2}\left(x_0\right)/\gamma$, where x_0 is the initial state and γ is a constant depending on the gains k_1 and k_3. Moreover, $V\left(x\right)$, defined in (2) is a strong Lyapunov function assuring these properties.

Proof: Since (1) is a discontinuous differential equation its solutions are interpreted as the ones of the differential inclusion $\dot{x} \in f\left(x\right)$ obtained when $\operatorname{sign}\left(z\right)$ assigns the interval $[-1,1]$ to $z = 0$. Since $0 \in f\left(0\right)$ it follows that $x = 0$ is an equilibrium point. The proposed Lyapunov function can be written as a quadratic form $V\left(x\right) = \zeta^T P \zeta$ where $\zeta^T = \left[\left|x_1\right|^{1/2}\operatorname{sign}\left(x_1\right)\ ,\ x_2\right]$,

$$P = \frac{1}{2}\begin{bmatrix} 4k_3 + k_1^2 & -k_1 \\ -k_1 & 2 \end{bmatrix} .$$

Note that $V\left(x\right)$ is continuous but is not differentiable at $x_1 = 0$. It is positive definite and radially unbounded if $k_3 > 0$, i.e.

$$\lambda_{\min}\left\{P\right\}\left\|\zeta\right\|_2^2 \leq V\left(x\right) \leq \lambda_{\max}\left\{P\right\}\left\|\zeta\right\|_2^2\ , \tag{3}$$

where $\left\|\zeta\right\|_2^2 = \left|x_1\right| + x_2^2$ is the Euclidean norm of ζ. Its time derivative (see 1) along the solutions of the system is

$$\dot{V} = -\frac{1}{\left|x_1\right|^{1/2}}\zeta^T Q \zeta \leq -\frac{1}{\left|x_1\right|^{1/2}}\lambda_{\min}\left\{Q\right\}\left\|\zeta\right\|_2^2\ , \tag{4}$$

where

$$Q = \frac{k_1}{2}\begin{bmatrix} 2k_3 + k_1^2 & -k_1 \\ -k_1 & 1 \end{bmatrix} .$$

$\dot{V}$ is negative definite if $Q > 0$, what is exactly the case if $k_1, k_3 > 0$. Using (3), (4) and the fact that

$$\left|x_1\right|^{1/2} \leq \left\|\zeta\right\|_2 \leq \frac{V^{1/2}\left(x\right)}{\lambda_{\min}^{1/2}\left\{P\right\}}$$

it follows that

$$\dot{V} \leq -\gamma V^{1/2}\left(x\right)\ ,$$

where

$$\gamma = \frac{\lambda_{\min}^{1/2}\left\{P\right\}\lambda_{\min}\left\{Q\right\}}{\lambda_{\max}\left\{P\right\}} .$$

Since the solution of the differential equation

$$\dot{v} = -\gamma v^{1/2}\ ,\ v\left(0\right) = v_0 \geq 0$$

is given by

$$v\left(t\right) = \left(v_0^{1/2} - \frac{\gamma}{2}t\right)^2 \tag{5}$$

it follows from the comparison principle [10] that $V\left(t\right) \leq v\left(t\right)$ when $V\left(x_0\right) \leq v_0$. From (5) one obtains that $V\left(x\left(t\right)\right)$, and therefore $x\left(t\right)$, converges to zero in finite time and reaches that value at most after $T = \frac{2V^{1/2}\left(x_0\right)}{\gamma}$ units of time. ∎

B. The perturbed dynamics

The following theorem assures the robustness of the globally asymptotic stability of the equilibrium of (1) in finite time when the gains are selected sufficiently high.

Theorem 3: Suppose that the perturbation terms of the system (1) are globally bounded by

$$|\varrho_1| \leq \delta_1 |x_1|^{1/2} \ , \ |\varrho_2| \leq \delta_2 \ , \tag{6}$$

for some constants $\delta_1, \delta_2 \geq 0$. Then the origin $x = 0$ is an equilibrium point that is strongly globally asymptotically stable if the gains satisfy

$$\begin{aligned} & k_1 > 2\delta_1 \\ & k_3 > k_1 \frac{5\delta_1 k_1 + 6\delta_2 + 4(\delta_1 + \delta_2/k_1)^2}{2(k_1 - 2\delta_1)} \end{aligned} \tag{7}$$

Moreover, all trajectories converge in finite time to the origin, upperbounded by $\tilde{T} = \frac{2V^{1/2}(x_0)}{\tilde{\gamma}}$, where x_0 is the initial state and $\tilde{\gamma}$ is a constant depending on the gains k_1, k_3 and the perturbation coefficients δ_1, δ_2.

Proof: Using (2) as a candidate Lyapunov function for the perturbed system (1) its (upper right hand) time derivative along the solutions of the system is

$$\dot{V} = -\frac{1}{|x_1|^{1/2}} \zeta^T Q \zeta + \frac{\varrho_1}{|x_1|^{1/2}} q_1^T \zeta + \varrho_2 q_2^T \zeta \ ,$$

where

$$q_1^T = \left[\begin{array}{cc} \left(2k_3 + \frac{k_1^2}{2}\right) & -\frac{k_1}{2} \end{array} \right] \ , \ q_2^T = \left[\begin{array}{cc} -k_1 & 2 \end{array} \right] \ .$$

Using the bounds on the perturbation (6) it can be shown that

$$\dot{V} \leq -\frac{1}{|x_1|^{1/2}} \zeta^T \tilde{Q} \zeta \ ,$$

where

$$\tilde{Q} = \frac{k_1}{2} \left[\begin{array}{cc} 2k_3 + k_1^2 - \left(\frac{4k_3}{k_1} + k_1\right)\delta_1 - 2\delta_2 & \bigstar \\ -\left(k_1 + 2\delta_1 + \frac{2\delta_2}{k_1}\right) & 1 \end{array} \right] .$$

$\dot{V}$ it is negative definite if $\tilde{Q} > 0$. It is easy to see that this is the case if the gains are as in (7). By the same arguments as before in the unperturbed case the state converges to zero in finite time, at most after $\tilde{T} = \frac{2V^{1/2}(x_0)}{\tilde{\gamma}}$ units of time, where $\tilde{\gamma} = \frac{\lambda_{\min}^{1/2}\{P\}\lambda_{\min}\{\tilde{Q}\}}{\lambda_{\max}\{P\}}$. ∎

III. An improved SOSM algorithm

The supertwisting algorithm (1) studied before is a nonlinear version of the following basic linear algorithm

$$\begin{aligned} \dot{x}_1 &= -k_2 x_1 + x_2 + \varrho_1(x,t) \\ \dot{x}_2 &= -k_4 x_1 + \varrho_2(x,t) \ , \end{aligned} \tag{8}$$

although their properties are very different. By means of the smooth Lyapunov function candidate

$$V(x) = k_4 x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}(k_2 x_1 - x_2)^2$$

we will study some of these properties. It is easy to see that $V(x)$ is positive definite and radially unbounded if $k_4 > 0$. Its derivative is

$$\dot{V} = -x^T P_0 x + \varrho_1 q_0^T x + \varrho_2 q_1^T x$$

where

$$\begin{aligned} P_0 &= k_2 \left[\begin{array}{cc} (k_2^2 + k_4) & -k_2 \\ -k_2 & 1 \end{array} \right] \\ q_0^T &= \left[\begin{array}{cc} (k_2^2 + 2k_4) & -k_2 \end{array} \right] \\ q_1^T &= \left[\begin{array}{cc} -k_2 & 2 \end{array} \right] . \end{aligned}$$

For the nominal case, when the perturbation terms vanish, i.e. $\varrho_1 = \varrho_2 = 0$, the state x converges exponentially fast to zero if $P_0 > 0$, that is, if $k_2 > 0$ and $k_4 > 0$. If we suppose that the perturbation terms are globally bounded by

$$\begin{aligned} |\varrho_1| &\leq \delta_3 |x_1| \ , \\ |\varrho_2| &\leq \delta_4 |x_1| \ , \end{aligned}$$

for some constants $\delta_3, \delta_4 \geq 0$, then

$$\dot{V} \leq -x^T \left(P_0 - \tilde{Q}\right) x$$

where

$$\tilde{Q} = \left[\begin{array}{cc} (k_2^2 + 2k_4)\delta_3 + k_2\delta_4 & \bigstar \\ \frac{1}{2}(k_2\delta_3 + 2\delta_4) & 0 \end{array} \right] .$$

There is exponential convergence if $\left(P_0 - \tilde{Q}\right) > 0$, i.e. if

$$\begin{aligned} & k_2 > 2\delta_3 \\ & k_4 > \frac{2\delta_3 k_2^3 + \left(\frac{1}{4}\delta_3^2 + 3\delta_4\right)k_2^2 + \delta_3\delta_4 k_2 + \delta_4^2}{k_2(k_2 - 2\delta_3)} \ . \end{aligned}$$

From the previous analysis of both algorithms, the linear (8) and the nonlinear one (1), two notable differences can be observed. First, the two algorithms have quite different converging properties: the linear system converges *exponentially*, whereas the trajectories of the SOSM algorithm converge in *finite time*. This is due to the lack of local Lipschitzness of the SOSM algorithm at the origin, that is, its behavior around the zero state is very strong compared to the linear case. On the other side, the linear correction terms are stronger than the ones of the SOSM algorithm far from the origin. These differences causes another striking difference between both algorithms: the kind of perturbations that each one is able to tolerate. The main difference is that the linear system can deal with perturbations that are stronger very far away from the origin and weaker near the orign than the ones that are endured by the SOSM algorithm. So, for example, the SOSM algorithm is not able to endure (globally) a linearly growing perturbation, but the linear algorithm can deal with it easily. However, the linear algorithm is not able to support a strong perturbation near the origin, what is one of the main advantadges of the SOSM.

A. A Modified SOSM: SOSML, unperturbed case

In what follows we will propose a new algorithm, that combines the linear and the nonlinear correction terms, so that it inherits the best properties of both. Consider a modified SOSM (SOSML), described by the following differential inclusion

$$\begin{aligned} \dot{x}_1 = & \quad -k_1 \left|x_1\right|^{1/2} \operatorname{sign}\left(x_1\right) - k_2 x_1 + x_2 \\ \dot{x}_2 = & \quad -k_3 \operatorname{sign}\left(x_1\right) - k_4 x_1 \end{aligned} \tag{9}$$

where x_i are scalar state variables and k_i are the constant gains to be designed. The next theorem shows that SOSML has finite time convergence as does the SOSM. We are able to provide a strong Lyapunov function to ensure these properties.

Theorem 4: The origin $x = 0$ is an equilibrium point of the system (9) that is strongly globally asymptotically stable if $k_i > 0$, $i = 1, \cdots, 4$, and $4k_3k_4 > \left(8k_3 + 9k_1^2\right) k_2^2$. Under the same conditions

$$V(x) = 2k_3 |x_1| + k_4 x_1^2 + \frac{1}{2} x_2^2 + \\ + \frac{1}{2}\left(k_1 |x_1|^{1/2} \operatorname{sign}(x_1) + k_2 x_1 - x_2\right)^2$$

is a continuous Lyapunov function ensuring this property. Moreover, all trajectories converge in finite time to the origin, where the convergence time is upperbounded by $T^* = \frac{2V^{1/2}(x_0)}{\gamma_1}$, where x_0 is the initial state and γ_1 is a constant depending on the gains k_i.

Proof: The proposed Lyapunov function can be written as a quadratic form $V(x) = \xi^T \Pi \xi$ where

$$\xi = \begin{bmatrix} |x_1|^{1/2} \operatorname{sign}(x_1) \\ x_1 \\ x_2 \end{bmatrix}, \\ \Pi = \frac{1}{2} \begin{bmatrix} \left(4k_3 + k_1^2\right) & k_1 k_2 & -k_1 \\ k_1 k_2 & \left(2k_4 + k_2^2\right) & -k_2 \\ -k_1 & -k_2 & 2 \end{bmatrix}.$$

Note that $V(x)$ is continuous but not differentiable at $x_1 = 0$. Moreover, it satisfies

$$\lambda_{\min}\{\Pi\} \|\xi\|_2^2 \leq V(x) \leq \lambda_{\max}\{\Pi\} \|\xi\|_2^2 \ , \tag{10}$$

where $\|\xi\|_2^2 = |x_1| + x_1^2 + x_2^2$ is the Euclidean norm of ξ.

The time derivative along the trajectories of the system is

$$\dot{V} = -\frac{1}{|x_1|^{1/2}} \xi^T \Omega_1 \xi - \xi^T \Omega_2 \xi$$

where

$$\Omega_1 = \tfrac{k_1}{2} \begin{bmatrix} \left(2k_3 + k_1^2\right) & 0 & -k_1 \\ 0 & \left(2k_4 + 5k_2^2\right) & -3k_2 \\ -k_1 & -3k_2 & 1 \end{bmatrix}, \\ \Omega_2 = k_2 \begin{bmatrix} \left(k_3 + 2k_1^2\right) & 0 & 0 \\ 0 & \left(k_4 + k_2^2\right) & -k_2 \\ 0 & -k_2 & 1 \end{bmatrix}.$$

It is negative definite if (note that this is only a sufficient condition) $\Omega_1 > 0$, and $\Omega_2 > 0$. It is not difficult to show that this will be the case if $k_i > 0$, $i = 1, \cdots, 4$, and

$$4k_3k_4 > \left(8k_3 + 9k_1^2\right) k_2^2 \ .$$

Since

$$\dot{V} \leq -\frac{1}{|x_1|^{1/2}} \lambda_{\min}\{\Omega_1\} \|\xi\|_2^2 - \lambda_{\min}\{\Omega_2\} \|\xi\|_2^2 \tag{11}$$

and using (10) and the fact that

$$|x_1|^{1/2} \leq \|\xi\|_2 \leq \frac{V^{1/2}(x)}{\lambda_{\min}^{1/2}\{\Pi\}}$$

it follows that

$$\dot{V} \leq -\gamma_1 V^{1/2}(x) - \gamma_2 V(x) \ ,$$

where

$$\gamma_1 = \frac{\lambda_{\min}^{1/2}\{\Pi\} \lambda_{\min}\{\Omega_1\}}{\lambda_{\max}\{\Pi\}} \ , \ \gamma_2 = \frac{\lambda_{\min}\{\Omega_2\}}{\lambda_{\max}\{\Pi\}} \ .$$

By the comparison lemma it follows easily that $V(x(t))$, and therefore $x(t)$, converges to zero in finite time and reaches that value at most after $T^* = \frac{2V^{1/2}(x_0)}{\gamma_1}$ units of time. ■

B. SOSML: behavior under perturbations

In this paragraph it will be shown that when perturbations terms are present, i.e.

$$\begin{aligned} \dot{x}_1 = & \quad -k_1 |x_1|^{1/2} \operatorname{sign}(x_1) - k_2 x_1 + x_2 + \varrho_1(x,t) \\ \dot{x}_2 = & \quad -k_3 \operatorname{sign}(x_1) - k_4 x_1 + \varrho_2(x,t) \ , \end{aligned} \tag{12}$$

the modified SOSM algorithm inherits the robustness properties of both linear and nonlinear algorithms, i.e. it is able to endure strong perturbations near the origin and linearly growing perturbations far from the equilibrium.

Theorem 5: Suppose that the perturbation terms of the system (12) are globally bounded by

$$\begin{aligned} |\varrho_1| &\leq \delta_1 |x_1|^{1/2} + \delta_3 |x_1| \ , \\ |\varrho_2| &\leq \delta_2 + \delta_4 |x_1| \ . \end{aligned} \tag{13}$$

for some constants $\delta_1, \delta_2, \delta_3, \delta_4 \geq 0$. Then the gains k_i can be selected high enough so that the origin $x = 0$ is an equilibrium point that is strongly globally asymptotically stable, and all trajectories converge in finite time to the origin.

Proof: Using the same Lyapunov function its derivative can be written as

$$\dot{V} = -\frac{1}{|x_1|^{1/2}} \xi^T \Omega_1 \xi - \xi^T \Omega_2 \xi + \omega_1^T \xi + \frac{1}{|x_1|^{1/2}} \omega_2^T \xi$$

where

$$\omega_1^T = \left[k_1 \left(\tfrac{3k_2}{2} \varrho_1 - \varrho_2\right), \left[\left(k_2^2 + 2k_4\right) \varrho_1 - k_2 \varrho_2\right], -k_2 \varrho_1\right]$$

$$\omega_2^T = \varrho_1 \left[\ \left(2k_3 + \tfrac{k_1^2}{2}\right) \quad 0 \quad -\tfrac{k_1}{2} \ \right].$$

Using the bounds (13) of the perturbation terms then

$$\frac{1}{|x_1|^{1/2}} \omega_2^T \xi \leq \frac{\delta_1}{|x_1|^{1/2}} \xi^T \Delta_1 \xi + \delta_3 \xi^T \Delta_1 \xi$$

where

$$\Delta_1 = \begin{bmatrix} \left(2k_3 + \frac{k_1^2}{2}\right) & 0 & \frac{k_1}{4} \\ 0 & 0 & 0 \\ \frac{k_1}{4} & 0 & 0 \end{bmatrix},$$

and also

$$\omega_1^T \xi \leq \frac{1}{|x_1|^{1/2}} \xi^T \Delta_2 \xi + \xi^T \Delta_3 \xi$$

where

$$\Delta_2 = \begin{bmatrix} \delta_2 k_1 & 0 & 0 \\ 0 & \left[k_1\left(\frac{3k_2}{2}\delta_3 + \delta_4\right) + \left(k_2^2 + 2k_4\right)\delta_1\right] & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} k_2\left(\delta_2 + \frac{3}{2}k_1\delta_1\right) & 0 & \frac{1}{2}k_2\delta_1 \\ 0 & \left(k_2^2 + 2k_4\right)\delta_3 + k_2\delta_4 & \frac{1}{2}k_2\delta_3 \\ \frac{1}{2}k_2\delta_1 & \frac{1}{2}k_2\delta_3 & 0 \end{bmatrix}.$$

The derivative of the Lyapunov function can be then written as

$$\dot{V} = -\frac{1}{|x_1|^{1/2}} \xi^T \left(\Omega_1 - \Delta_2 - \delta_1\Delta_1\right)\xi - \xi^T\left(\Omega_2 - \Delta_3 - \delta_3\Delta_1\right)\xi$$

$\left(\Omega_1 - \Delta_2 - \gamma_1\Delta_1\right) > 0$ if

$$\begin{array}{l} k_1 > 2\max\left(\delta_1, \sqrt{\delta_2}\right) \\ k_2 > \frac{3}{8}\delta_3 + \frac{1}{4}\sqrt{\frac{9}{4}\delta_3^2 + 8\delta_4} \\ k_3 > k_1 \frac{\delta_1 k_1 + \frac{1}{8}\delta_1^2 + \delta_2}{2\left(\frac{1}{2}k_1 - \delta_1\right)} \\ k_4 > \frac{k_1\left[\frac{1}{2}k_1\left(k_1 + \frac{1}{2}\delta_1\right)^2\left(2k_2^2 - \frac{3}{2}\delta_3 k_2 - \delta_4\right) + \left(\frac{5}{2}k_2^2 + \frac{3}{2}\delta_3 k_2 + \delta_4\right)p_1\right]}{2\left(p_1 - \frac{1}{2}k_1\left(k_1 + \frac{1}{2}\delta_1\right)^2\right)\left(\frac{1}{2}k_1 - \delta_1\right)} - \frac{1}{2}k_2^2 \end{array} \tag{14}$$

where

$$p_1 \triangleq k_1\left(\frac{1}{4}k_1^2 - \delta_2\right) + \left(\frac{1}{2}k_1 - \delta_1\right)\left(2k_3 + \frac{1}{2}k_1^2\right)$$

The second term in the inequality can be written as $\xi^T\left(\Omega_2 - \Delta_3 - \gamma_3\Delta_1\right)\xi = \zeta^T\Gamma_1\zeta + x^T\Gamma_2 x$, where

$$\Gamma_1 = \begin{bmatrix} k_2\left(\left(k_3 + 2k_1^2\right) - \delta_2 - \frac{3}{2}k_1\delta_1\right) - \left(2k_3 + \frac{k_1^2}{2}\right)\delta_3 & \cdots \\ -\frac{1}{2}\left(k_2\delta_1 + \frac{1}{2}k_1\delta_3\right) & \end{bmatrix}$$

$$\begin{bmatrix} \cdots -\frac{1}{2}\left(k_2\delta_1 + \frac{1}{2}k_1\delta_3\right) \\ \frac{1}{2}k_2 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} k_2\left(k_4 + k_2^2\right) - \left(k_2^2 + 2k_4\right)\delta_3 - k_2\delta_4 & \star \\ -k_2\left(k_2 + \frac{1}{2}\delta_3\right) & \frac{1}{2}k_2 \end{bmatrix}$$

This term is positive definite if

$$\begin{array}{l} k_2 > 2\delta_3 \\ k_3 > \frac{\left(k_2\delta_1 + \frac{1}{2}k_1\delta_3\right)^2}{2k_2\left(k_2 - 2\delta_3\right)} + \frac{\left(\delta_2 + \frac{3}{2}\delta_1 k_1\right)k_2 - 2\left(k_2 - \frac{1}{4}\delta_3\right)k_1^2}{\left(k_2 - 2\delta_3\right)} \\ k_4 > k_2 \frac{\left[k_2\left(k_2 + 3\delta_3\right) + \frac{1}{2}\delta_3^2 + \delta_4\right]}{k_2 - 2\delta_3} \end{array} \tag{15}$$

One can see that it is always possible to select $k_i > 0\ ,\ i = 1, \cdots, 4$, so that both sets of inequalities (14,15) are satisfied for every $\delta_i > 0\ ,\ i = 1, \cdots, 4$.

Under the previous conditions

$$\dot{V} \leq -\frac{1}{|x_1|^{1/2}} \xi^T\left(\Omega_1 - \Delta_2 - \delta_1\Delta_1\right)\xi$$

and the global asymptotic stability and the finite time convergence follows using the same arguments as in the previous proofs. ∎

IV. EXAMPLE: A SOSML BASED OBSERVER

As an illustration, here we design a SOSML based observer for a simple nonlinear system, and the observation results are compared with SOSM, first order sliding modes, and linear observers. Consider a pendulum which state model is given by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= \tfrac{1}{J}u - \tfrac{MgL}{2J}\sin\left(x_1\right) - \tfrac{V_s}{J}x_2 + \rho \\ y &= x_1. \end{aligned}$$

where $x_1 = \theta$ is the angle of oscilation, $x_2 = \dot{\theta}$ is the angular velocity, M is the pendulum mass, g is the gravitational force, L is the pendulum lenght, $J = ML^2$ is the arm inertia, V_s is the pendulum viscous friction coefficient, and ρ is a bounded perturbation that for simulation purposes is modeled as

$$\rho\left(t\right) = 0.5\sin\left(2t\right) + 0.5\cos\left(5t\right).$$

This system have been used in [7], [8] for the design of a super twisting observer. Here, a SOSML observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1\left|e_1\right|^{1/2}\operatorname{Sign}\left(e_1\right) + k_2 e_1 \\ \dot{\hat{x}}_2 &= \tfrac{1}{J}u - \tfrac{MgL}{2J}\sin\left(\hat{x}_1\right) - \tfrac{V_s}{J}\hat{x}_2 + k_3\operatorname{Sign}\left(e_1\right) + k_4 e_1, \end{aligned} \tag{16}$$

is proposed, where $e_1 = x_1 - \hat{x_1}$. The observation error dynamic is given by

$$\begin{aligned} \dot{e}_1 &= e_2 - k_1\left|e_1\right|^{1/2}\operatorname{Sign}\left(e_1\right) - k_2 e_1 \\ \dot{e}_2 &= \frac{MgL}{2J}\left(\sin\left(\hat{x}_1\right) - \sin\left(x_1\right)\right) - \frac{V_s}{J}e_2 - k_3\operatorname{Sign}\left(e_1\right) + \\ &\quad - k_4 e_1 + \rho \\ &= -k_3\operatorname{Sign}\left(e_1\right) - k_4 e_1 + \varrho_2\left(e, t\right) \end{aligned}$$

where $\varrho_2\left(e,t\right) = \rho - \frac{MgL}{2J}\cos\left(z\left(t\right)\right)e_1 - \frac{V_s}{J}e_2$, is the perturbation term. Note that the perturbation contains a linear term in e_1 (obtained using the mean value theorem), a bounded perturbation ρ and a linear term in e_2. Observe that this error dynamics is similar to (12), and therefore, we use here the relations shown in III-B to obtain k_i. These values were used in the observer structure (16), and a simulation of the behavior of the observer was obtained. For this simulation, the initial conditions were zero for the observer and $x_1 = -1$, $x_2 = 3$ for the pendulum. For simulation numeric values $M = 1.1$(kg), $g = 9.815$(m/s^2), $L = 1$(m), and $V_s = 0.18$(kg.m/s^2) were used. If one make zero k_2 and k_4, the SOSML observer reduces to SOSM observer. A common linear observer and a first order sliding modes observer [22] were designed too for comparison purposes, and the results are shown in Figures 1 and 2.

The linear observer shows exponential convergence, and it cannot cope with the bounded disturbance. The first order SM observer endures the perturbation, but it has a characteristic chattering, that is avoided by the two second order SM observers. Besides, SOSM and SOSML has finite

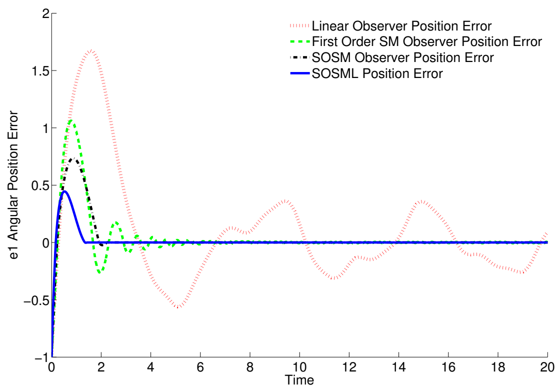

Fig. 1. Error in the estimated position with SOSML, SOSM, First Order SM and Linear Observers

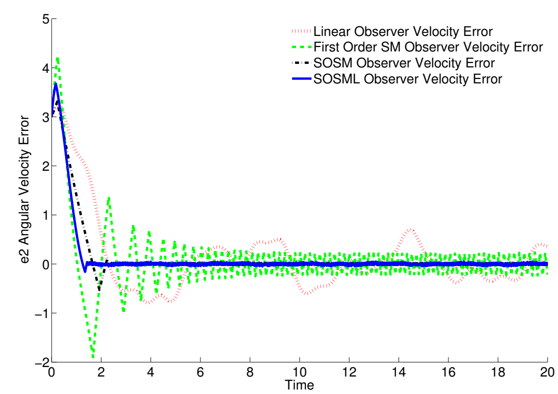

Fig. 2. Error in the estimated velocity with SOSML, SOSM, First Order SM and Linear Observers

time convergence that is improved by the linear term in the case of SOSML.

V. Conclusions

In this paper we have provided, for the first time, strong Lyapunov functions for a class of algorithms of Second Order Sliding Modes, the supertwisting algorithm. This is a very important law used for controller, observer and exact differentiator design with outstanding properties: convergence in finite time and robustness to strong perturbation terms. Compared to the first order sliding modes algorithms, well known in the literature for possessing such properties, the SOSM trajectories are smoother, avoiding the strong chattering effect of the classical sliding modes. The use of strong Lyapunov functions allows to study more deeply the convergence and robustness properties of these algorithms, it permits the combination of SOSM with other algorithms, and the incorporation of a very fruitful and powerful method in the higher order sliding modes research area. As a first step in this direction, we have proposed a modified supertwisting algorithm, that combines the benefits of the SOSM and those of a linear algorithm. This is an easy task, when a strong Lyapunov function can be provided, as is the case in this paper.

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