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Port-Hamiltonian Formulation and Analysis of the LuGre Friction Model.

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***Abstract*— A port-Hamiltonian formulation of the LuGre friction model is presented that can be used as a building block in the physical modeling of systems with friction. Based on the dissipation structure matrix of this port-Hamiltonian LuGre model, an alternative proof can be given for the passivity conditions that are known in the literature. As a specific example, the interconnection of a mass with the port-Hamiltonian LuGre model is presented. It is shown that the lossless interconnection structure and dissipation structure of the port-Hamiltonian LuGre model are consistent with those of the interconnection. Additionally, to render the friction model continuously differentiable, a smooth re-parametrization of the friction curve is proposed that extends and simplifies the existing results.**

I. INTRODUCTION

Although friction is essential to almost every aspect of mechanical behaviour, dealing with the phenomenon remains a challenge in many engineering areas. In control systems engineering this is not different. Incorporating the complex and nonlinear behaviour of friction in a control system design has been a topic of research for decades (see [1] and the references therein). As many of the control solutions tend to be model-based, there has been a need for a faithful but relatively simple friction model.

Starting from static models, where the friction force is described as a function of the relative velocity between the two surfaces in contact, several extensions have led to dynamical friction models that capture both the nonlinear force-velocity relation with Coulomb friction, viscous friction, and the Stribeck effect, as well as transient behaviour and stiction without a logical rule.

The first dynamic friction model able to capture all these effects is the LuGre model, presented in [4] and [11]. Although the model has some inaccuracy in the pre-sliding (i.e., stiction) regime and was subsequently modified in [13], it has been embraced by many control engineers. Further developments and implementations of the LuGre model included longitudinal and combined-slip tyre models (see [5] and [6]), and observer-based friction compensation schemes [8].

In this paper, we present a port-Hamiltonian description of the LuGre model. Port-Hamiltonian systems are a class of system descriptions (linear or nonlinear) that arise naturally from network modeling of physical systems in a variety of domains (e.g. mechanical, electrical and thermodynamical). Exposing the relation between the energy storage, dissipation, and interconnection structure of the system, this framework underscores the physics of the system. The connection with network (bond-graph) modeling is further formalized with the notion of a so-called Dirac structure on the space of flows and efforts of the system. Although it might be argued that from a thermodynamical perspective the inclusion of irreversible processes in a Hamiltonian framework should be done via the use of port-contact structures (see [7]), in this paper we use the port-controlled Hamiltonian system with dissipation (PCHD) form as it is presented in [14].

An attractive aspect of the port-Hamiltonian formalism is that a power-preserving interconnection between port-Hamiltonian systems results in a new port-Hamiltonian (PH) system with composite energy, interconnection and dissipation structure. Based on this principle, complex, multi-domain systems can be modeled by interconnecting PH descriptions of its sub-systems. Moreover, control design methodologies are available that can be directly applied to such PH descriptions of complex nonlinear systems [12]. It is precisely in this context that a PH description of the LuGre model can be of great value. The PH description of the LuGre friction model that is presented in this paper can be used as a resistive element in PH descriptions of (complex) systems containing friction, which can then be used for controller synthesis.

Since friction is, in its very nature, a dissipation phenomenon (although the pre-sliding phase should ideally be conservative), it is clear that any model describing it faithfully has to be passive. In [4], passivity of the LuGre model was proven for the mapping from relative velocity to the virtual bristle state. However, it is correctly argued in [2] that the model should be passive in the mapping from relative velocity to friction force, which is the natural power-conjugate input-output pair. In the same paper, necessary and sufficient conditions are derived for this physically relevant passivity property. The proof is constructed using the direct analysis of passivity in terms of time integrals of power. The PH description of the LuGre model that we present here enables us to give a short alternative proof for this passivity condition.

The remainder of this paper is organized as follows. In Section II we give a description of the LuGre friction model with some of its important characteristics. We also reformulate the model to render it physically more attractive. In Section III we present the PH description of the model, after which we derive the passivity conditions in Section IV.

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A simple PH interconnection of a mass with Lugre friction is presented in Section V. Finally, in Section VI we propose an alternative re-parametrization of the static friction curve that can be incorporated in the LuGre model to render it continuously differentiable.

II. LuGre Friction Model

The LuGre friction model that is presented in [4] and [11] is a so-called bristle model, i.e., the dynamical part of the model describes a virtual bristle deflection. The model is described by the following set of equations

$$\begin{aligned} \dot{z} &= -\frac{\sigma_0 |v_r|}{g(v_r)} z + v_r \\ F &= (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) F_n, \end{aligned} \tag{1}$$

where z denotes the virtual bristle deflection, v_r the relative velocity of the surfaces in contact, and F the resulting friction force between the surfaces. The normal force between the surfaces is denoted by F_n. The function

$$g(v_r) = \mu_C + (\mu_S - \mu_C) e^{-|\frac{v_r}{v_S}|^{\alpha}}, \tag{2}$$

parameterizes the static friction curve that is incorporated in the model, with μ_C being the Coulomb friction, μ_S the Stribeck friction, v_S the Stribeck velocity, and α a curve parameter that further tunes the Stribeck effect. The remaining terms parameterize the bristle dynamics. In the literature, σ_0 is used to denote the bristle stiffness coefficient, σ_1 the bristle damping coefficient, and σ_2 the viscous friction coefficient. These coefficients, however, are normalized by F_n and therefore do not have the appropriate units. To render the model physically more appealing, we introduce the following variables:

$$\begin{aligned} \hat{k}_0 &:= k_0(F_n) = \sigma_0 F_n \\ \hat{d}_1 &:= d_1(F_n) = \sigma_1 F_n \\ \hat{d}_2 &:= d_2(F_n) = \sigma_2 F_n \\ \hat{g}_0 &:= g_0(v_r, F_n) = \frac{|v_r|}{F_n g(v_r)}. \end{aligned} \tag{3}$$

We now have that $\hat{k}_0$ ([Nm^{-1}]) is a proper stiffness coefficient, $\hat{d}_1$ and $\hat{d}_2$ ([Nsm^{-1}]) are proper damping coefficients, while $\hat{g}_0$ ([$\mathrm{mN}^{-1}\mathrm{s}^{-1}$]) is a proper conductance coefficient. If we further eliminate $\dot{z}$ from the output equation, we get the following description of the LuGre model.

$$\begin{aligned} \dot{z} &= -\hat{g}_0 \hat{k}_0 z + v_r \\ F &= (1 - \hat{d}_1 \hat{g}_0) \hat{k}_0 z + (\hat{d}_1 + \hat{d}_2) v_r. \end{aligned} \tag{4}$$

An important characteristic of the LuGre model is that the virtual bristle displacement z is bounded according to

$$-\frac{\mu_S}{\sigma_0} \leqslant z \leqslant \frac{\mu_S}{\sigma_0}. \tag{5}$$

The fact that the state space does not consist of the whole of $\mathbb{R}$ turns out to be crucial for the passivity analysis of the model. We derive the passivity conditions in Section IV.

Another key feature of the model is the fact that the steady-state behaviour coincides with a commonly used static friction curve parametrization (see [1] and [3]), namely

$$F(v_r)|_{\dot{z}=0} = \Big(\frac{1}{\hat{g}_0} + \hat{d}_2\Big) v_r. \tag{6}$$

This is exactly the rationale behind the LuGre friction model, but at the same time it offers the opportunity to incorporate other friction curve parameterizations into the model by changing the conductance term $\hat{g}_0$ in (3). In Section VI we use this methodology to insert a continuously differentiable friction curve that is a close approximation of (6).

III. Port-Hamiltonian (PH) Formulation of LuGre Friction Model

In this section, we give a PH description of the LuGre friction model (4). First, we introduce the standard PH description of a system without direct feedthrough, as it is treated in, e.g., [14], after which we extend it to a more general form.

A. PH Systems

A basic PH system description is given as follows [14]

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial^T H}{\partial x}(x) + G(x) u \\ y &= G^T(x) \frac{\partial^T H}{\partial x}, \end{aligned} \tag{7}$$

where $x \in \mathcal{X}$ is the state, $H : \mathcal{X} \to \mathbb{R}$ the Hamiltonian, $G(x)$ the input distribution matrix, $J(x)$ the lossless interconnection structure matrix, satisfying $J(x) = -J^T(x)$, and $R(x)$ the dissipation structure matrix, satisfying $R(x) = R^T(x)$. Although this type of PH description is suitable for describing a large class of physical systems, for a LuGre model it turns out that a more generic form is needed.

B. PH Systems with Feedthrough and Modulation

A more general description of a PH system than (7) arises when a direct feedthrough channel is incorporated. This form is not yet widely used in the literature, but it can be found in [15] and [9]. If we furthermore also allow for modulations of the system matrices, we arrive at the following PH system description

$$\begin{aligned} \dot{x} &= [J(\cdot) - R(\cdot)] \frac{\partial^T H}{\partial x}(\cdot) + [G(\cdot) - P(\cdot)] u \\ y &= [G(\cdot) + P(\cdot)]^T \frac{\partial^T H}{\partial x}(\cdot) + [M(\cdot) + S(\cdot)] u, \end{aligned} \tag{8}$$

with $M(\cdot)$ being skew-symmetric, and $S(\cdot)$ symmetric. The lossless interconnection structure within the system is now described by $J(\cdot)$, $M(\cdot)$, and $G(\cdot)$, while the dissipation structure is described by $R(\cdot)$, $S(\cdot)$, and $P(\cdot)$. The dots in the arguments of the system matrices denote possible modulations to be specified. Often we have that the matrices and the Hamiltonian are functions of the state x, but it also occurs (e.g. in power converters [14]) that they are modulated by switches, the input, the output, or external variables. The description above is therefore a very generic form. Having this generic PH form, the question now is how to put the LuGre friction model in such a framework.

C. PH LuGre Model

Since we are dealing with a scalar system, the skew-symmetric interconnection terms are of course necessarily zero, i.e., $J_\ell = M_\ell = 0$, where we use the subscript ℓ to denote the LuGre model. The remaining terms have to be selected sequentially. We start by selecting the feedthrough matrix $S_\ell(\cdot)$. An obvious choice for this matrix is

$$S_\ell(F_n) = \hat{d}_1 + \hat{d}_2. \tag{9}$$

Next, we choose the Hamiltonian. The most natural candidate is the (virtual) elastic energy stored in the virtual bristles

$$H_\ell(z, F_n) = \frac{1}{2}\hat{k}_0 z^2. \tag{10}$$

Although we have the freedom to choose other Hamiltonians, the one above has the advantage of not being modulated by the relative velocity v_r and having a clear physical meaning.

Having set both $H_\ell(\cdot)$ and $S_\ell(\cdot)$, we proceed by selecting $G_\ell(\cdot)$ and $P_\ell(\cdot)$ from

$$\begin{aligned} & [G_\ell(\cdot) + P_\ell(\cdot)]^T \frac{\partial^T H_\ell}{\partial z}(z, F_n) = \hat{k}_0(1 - \hat{d}_1\hat{g}_0)z \\ \Rightarrow \quad & [G_\ell(v_r) + P_\ell(v_r)]^T = (1 - \hat{d}_1\hat{g}_0). \end{aligned} \tag{11}$$

We choose to have that

$$G_\ell(v_r) = 1 - \frac{1}{2}\hat{d}_1\hat{g}_0 \tag{12}$$

$$P_\ell(v_r) = -\frac{1}{2}\hat{d}_1\hat{g}_0. \tag{13}$$

The motivation for this particular form is given in Section V. The dissipation term $R_\ell(\cdot)$ is now derived as follows

$$\begin{aligned} & \dot{z} = v_r - \sigma_0\hat{g}_0 z = -R_\ell(\cdot)\hat{k}_0 z + [G_\ell(v_r) - P_\ell(v_r)]v_r \\ \Rightarrow \quad & v_r - \hat{k}_0\hat{g}_0 z = -R_\ell(\cdot)\hat{k}_0 z + v_r \\ \Rightarrow \quad & R_\ell(v_r, F_n) = \hat{g}_0. \end{aligned} \tag{14}$$

The result of the above considerations is the following PH description of the LuGre friction model

$$\begin{aligned} \dot{z} &= -R_\ell(v_r, F_n)\frac{\partial^T H_\ell}{\partial z}(z, F_n) + [G_\ell(v_r) - P_\ell(v_r)]v_r \\ F_\ell &= [G_\ell(v_r) + P_\ell(v_r)]\frac{\partial^T H_\ell}{\partial z}(z, F_n) + S_\ell(F_n)v_r, \end{aligned} \tag{15}$$

with $S_\ell(F_n)$, $H_\ell(z, F_n)$, $G_\ell(v_r)$, $P_\ell(v_r)$, and $F_\ell(v_r, F_n)$ given in (9), (10), (12), (13), and (14), respectively.

IV. Passivity Analysis of the LuGre Model via Dissipation Structure

As stated in the Introduction, passivity of the LuGre model in the $v_r \mapsto F$ mapping is crucial. Having derived the PH descriptions above, we can derive the passivity conditions for the LuGre friction model by analyzing the dissipation structure of the system. An important assumption that is needed in our specific case, with the Hamiltonian being a function of F_n, is that the normal force is constant. For the case that $F_n = 1$, the passivity condition for the LuGre model was already derived in [2] using direct analysis of the time integrals of supplied power. We are now able to give an alternative proof based on the PH description. Moreover, this proof holds for any constant F_n. Passivity conditions for the general case, where F_n is assumed to be time-varying, needs a different PH description and falls beyond the scope of this paper.

A. Passive Dynamical Systems

The theory concerning passivity and its more general form, dissipativity, can be found in the work of Willems [16] and Van der Schaft [14]. Passivity of a general dynamical input-state-output system is defined as follows.

Definition 4.1 (Passivity): A system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u), \end{aligned} \tag{16}$$

with $x \in \mathcal{X}$, $u, y \in \mathbb{R}^m$ is said to be passive if there exists a function $H : \mathcal{X} \to \mathbb{R}$ such that

$$H(x(t_1)) - H(x(t_0)) \leqslant \int_{t_0}^{t_1} u^T(t)y(t)dt. \tag{17}$$

for all $t_0, t_1 \in \mathbb{R}$, with $t_0 \leqslant t_1$, and all signals (u, x, y) that satisfy (16).

For PH systems of the form (8) it can be deduced that the power-balance equation can be written as

$$\frac{d}{dt}H(x(t)) = u^T y - \begin{pmatrix} \frac{\partial H}{\partial x} & u^T \end{pmatrix} \begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \begin{pmatrix} \frac{\partial^T H}{\partial x} \\ u \end{pmatrix}, \tag{18}$$

with the lefthand term denoting the evaluation of H along the trajectories of the system. Based on this power-balance equation, the necessary passivity inequality is given by

$$\begin{pmatrix} \frac{\partial H}{\partial x} & u^T \end{pmatrix} \begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \begin{pmatrix} \frac{\partial^T H}{\partial x} \\ u \end{pmatrix} \geqslant 0, \tag{19}$$

for all $x \in \mathcal{X}$ and admissible inputs $u : [0, t] \to \mathbb{R}^m$. This indeed shows that $R(\cdot)$, $S(\cdot)$, as well as $P(\cdot)$ determine the dissipation structure of the system.

B. Passivity Conditions for the LuGre Friction Model

Although it is often stated that the dissipation matrix in (19) has to be positive semi-definite in order to render the system passive, this is only true when the state space of the system is $\mathbb{R}^n$ itself. If the state space is actually a bounded subset $\mathcal{X}$ of $\mathbb{R}^n$, this is not a necessary condition and the inequality should only be satisfied on the domain of interest.

Taking into account the fact that the state space is bounded according to (5) and using the result of the previous section, the passivity condition for the LuGre friction model is given by

$$\begin{pmatrix} \hat{k}_0 z & v_r \end{pmatrix} \begin{pmatrix} \hat{g}_0 & -\frac{1}{2}\hat{d}_1\hat{g}_0 \\ -\frac{1}{2}\hat{d}_1\hat{g}_0 & \hat{d}_1 + \hat{d}_2 \end{pmatrix} \begin{pmatrix} \hat{k}_0 z \\ v_r \end{pmatrix} \geqslant 0$$

$$\forall\, z \in [\,-\tfrac{\mu_S}{\sigma_0}, \tfrac{\mu_S}{\sigma_0}\,],\ v_r \in \mathbb{R}. \tag{20}$$

We can now state the following proposition, which is a slight generalization of the result in [2].

Proposition 4.1: Under the assumption that F_n is constant, the LuGre friction model is passive if and only if

$$\hat{d}_1 \leqslant \hat{d}_2 \frac{\mu_C}{\mu_S - \mu_C}. \tag{21}$$

Proof. Writing out (20) results in

$$\hat{k}_0^2 \hat{g}_0 z^2 - \hat{k}_0 \hat{d}_1 \hat{g}_0 z v_r + (\hat{d}_1 + \hat{d}_2) v_r^2 \geqslant 0, \tag{22}$$

First of all we note that the left hand term is equal to zero for $v_r = 0$. Next to that, we have that (22) is equivalent to

$$\hat{k}_0^2 z^2 \frac{\hat{g}_0}{v_r^2} - \hat{k}_0 \hat{d}_1 z \frac{\hat{g}_0}{v_r} + (\hat{d}_1 + \hat{d}_2) \geqslant 0. \tag{23}$$

for all $v_r \in \mathbb{R}\backslash\{0\}$. Both $\hat{g}_0/v_r$ and $1/v_r$ are odd and their product is non-negative for all $v_r \in \mathbb{R}$ with

$$\lim_{|v_r|\to\infty} \left(\hat{k}_0^2 z^2 \frac{\hat{g}_0}{v_r^2} \right) = 0. \tag{24}$$

With z^2 also being non-negative, this results in

$$\inf \hat{k}_0^2 z^2 \frac{\hat{g}_0}{v_r^2} = 0. \tag{25}$$

For the second term in (23) the infimum is given by

$$\begin{aligned} &\inf_{\substack{v_r \in \mathbb{R} \\ -\frac{\mu_S}{\sigma_0} \leqslant z \leqslant \frac{\mu_S}{\sigma_0}}} \left(- \hat{k}_0 \hat{d}_1 z \frac{\hat{g}_0}{v_r} \right) = \\ &\lim_{|v_r|\to\infty} \left(- \hat{k}_0 \hat{d}_1 z \frac{\hat{g}_0}{v_r} \right) \Bigg|_{|z| = \frac{\mu_S}{\sigma_0}} = -\hat{d}_1 \mu_S \frac{1}{\mu_C}. \end{aligned} \tag{26}$$

This results in the passivity condition

$$-\hat{d}_1 \mu_S \frac{1}{\mu_C} + (\hat{d}_1 + \hat{d}_2) \geqslant 0, \tag{27}$$

which is equivalent to (21). □

V. PH Description of Mass Subject to Friction

In this section, a PH description of a mass subject to friction is presented. This interconnection shows how the PH description of the LuGre model can be used as a building block for physical modeling of systems in a PH framework. Moreover, due to the specific nature of this interconnection example, the way in which the dissipation structure of the PH LuGre building block carries over to the interconnected system, becomes particularly transparent.

A. Interconnection of PH System with PH LuGre Model.

Let us first introduce a standard (negative) feedback interconnection of a basic PH system (without feedthrough) with the PH Lugre model. If we denote the state, the input, and the output of the LuGre model by $x_\ell = z$, $u_\ell = v_r$, and $y_\ell = F_\ell$, respectively, and further omit the arguments of the matrices in (15), the PH LuGre model is described by

$$\Sigma_\ell : \begin{cases} \dot{x}_\ell = -R_\ell \dfrac{\partial^T H_\ell}{\partial x_\ell} + [G_\ell - P_\ell] u_\ell \\ y_\ell = [G_\ell + P_\ell]^T \dfrac{\partial^T H_\ell}{\partial x_\ell} + S_\ell u_\ell, \end{cases} \tag{28}$$

with R_ℓ, H_ℓ, G_ℓ, P_ℓ and S_ℓ given in (14), (10), (12), (13) and (9) respectively. The arguments are omitted for sake of brevity. The basic general PH system that we interconnect the PH LuGre model with is described by

$$\Sigma_p : \begin{cases} \dot{x}_p = [J_p - R_p] \dfrac{\partial^T H_p}{\partial x_p} + G_p u_p \\ y_p = G_p^T \dfrac{\partial^T H_p}{\partial x_p} . \end{cases} \tag{29}$$

The negative feedback interconnection is described by the following relation

$$\begin{pmatrix} u_p \\ y_p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_\ell \\ y_\ell \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \tag{30}$$

with u being an external input. The result is the following PH system

$$\Sigma : \begin{cases} \dot{x} = [J - R] \dfrac{\partial^T H}{\partial x} + G u \\ \\ y = G^T \dfrac{\partial^T H}{\partial x}, \end{cases} \tag{31}$$

with $x = \begin{pmatrix} x_\ell & x_p \end{pmatrix}^T$ and $G = \begin{pmatrix} G_p & 0 \end{pmatrix}^T$. The Hamiltonian is given by

$$H = H_p + H_\ell, \tag{32}$$

and the interconnection and dissipation structure matrices by

$$J = \begin{pmatrix} J_p & -G_p G_\ell^T \\ G_\ell G_p^T & 0 \end{pmatrix} \tag{33}$$

$$R = \begin{pmatrix} R_p + G_p S_\ell G_p^T & G_p P_\ell^T \\ P_\ell G_p^T & R_\ell \end{pmatrix}. \tag{34}$$

The description above clearly shows what terms of both systems (28) and (29) contribute to the lossless interconnection structure of the closed-loop system and what terms contribute to its dissipation structure.

B. Negative Feedback Interconnection of the PH LuGre Friction Model with Mass

Having described a general negative feedback interconnection of a PH system with the PH LuGre model, it is now straightforward to substitute the PH description of a mass m into this feedback loop. The PH description of the mass is given by

$$\Sigma_m : \begin{cases} \dot{x}_m = G_m u_m \\ y_m = G_m \dfrac{\partial^T H_m}{\partial x_m} x_m, \end{cases}$$

with state $x_m = p_m$ being the momentum of the mass, input $u_m = F$ the total force acting upon the mass, and output y_m the velocity of the mass. Furthermore, the Hamiltonian consists of the kinetic energy of the mass

$$H_m(p_m) = \frac{p_m^2}{2m}. \tag{35}$$

Of all the other terms in the PH description of the mass, only $G_m = 1$ is non-zero. Using again the power-preserving interconnection (30)

$$\begin{pmatrix} u_m \\ y_m \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_\ell \\ y_\ell \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_F, \tag{36}$$

where the external input u_F is an external force, the closed-loop system is given by

$$\begin{aligned} \begin{bmatrix} \dot{p}_m \\ \dot{z} \end{bmatrix} = \Bigg[&\begin{pmatrix} 0 & -1 + \frac{1}{2}\hat{d}_1\hat{g}_0 \\ 1 - \frac{1}{2}\hat{d}_1\hat{g}_0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} \hat{d}_1 + \hat{d}_2 & -\frac{1}{2}\hat{d}_1\hat{g}_0 \\ -\frac{1}{2}\hat{d}_1\hat{g}_0 & \hat{g}_0 \end{pmatrix} \Bigg] \begin{bmatrix} \frac{p_m}{m} \\ \hat{k}_0 z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_F, \end{aligned}$$

$$v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{p_m}{m} \\ \hat{k}_0 z \end{bmatrix} \tag{37}$$

with Hamiltonian

$$H(p_m, z) = \frac{p_m^2}{2m} + \frac{\hat{k}_0 z^2}{2}, \tag{38}$$

representing the sum of the kinetic energy of the mass and the elastic energy stored by the virtual bristles.

With the mass being a lossless system, the dissipation structure of the mass-friction interconnection is determined solely by the friction part. On the other hand, it can be shown that the chosen Hamiltonian (38) admits only one unique realization of both the lossless interconnection structure matrix and the dissipation structure matrix. The specific form of both $G_\ell(\cdot)$ and $P_\ell(\cdot)$, proposed in Section III, are actually chosen such that the PH LuGre description is consistent with the PH description of the mass-friction system.

VI. Re-parametrization of Friction Curve

In this section we present a friction curve parametrization that can be used to remove the discontinuities from the LuGre friction model. However, it can also be used as a stand-alone, static friction model.

As is stated in the introduction, static friction models give a functional relation between the friction force F and the relative velocity v_r between the two surfaces in contact. The most common parametrization of such a static friction curve is given by

$$F(v_r) = \left[\left(\mu_C + (\mu_S - \mu_C) e^{-|\frac{v_r}{v_S}|^\alpha} \right) \mathrm{sgn}(v_r) + \sigma_2 v_r \right] F_n. \tag{39}$$

In Section II it is discussed that this parametrization corresponds to the steady-state behaviour of the LuGre model, which is included via the conductance term $\hat{g}_0$. The parametrization above is not continuously differentiable however, which might be a serious obstacle for some controller design methodologies. In [10] a continuously differentiable static friction model is proposed that uses the hyperbolic tangent to deal with the signum and absolute-value functions, i.e., the authors use the fact that

$$\begin{aligned} \tanh(cx) &\approx \mathrm{sgn}(x) \\ x \tanh(cx) &\approx |x|, \end{aligned} \tag{40}$$

for some $c \gg 1$.

The static friction model that is proposed in [10] is given by

$$\begin{aligned} F(v_r) = \big[&\gamma_1 (\tanh(\gamma_2 v_r) - \tanh(\gamma_3 v_r)) \\ &+ \gamma_4 \tanh(\gamma_5 v_r) + \gamma_6 v_r \big] F_n, \end{aligned} \tag{41}$$

with friction parameters $\gamma_1, \ldots, \gamma_6$. Although this friction curve is presented in [10] as a static friction model, it can easily be extended to a LuGre type dynamical friction model by replacing the original conductivity term in (1) by

$$\hat{g}_0 = \frac{v_r}{\gamma_1(\tanh(\gamma_2 v_r) - \tanh(\gamma_3 v_r)) + \gamma_4 \tanh(\gamma_5 v_r)}. \tag{42}$$

The model (41) contains all relevant friction phenomena that can be described with a static friction curve (i.e., Coulomb friction, viscous friction, and the Stribeck effect), with the parameters γ_i providing some degree of freedom for shaping the curve. However, although γ_6 is exactly the viscous friction coefficient σ_2 in the original parametrization (39), the other parameters differ from the usual friction parameters such as μ_C, μ_S, and v_S, and therefore have no clear physical meaning in friction analysis. We therefore propose to use a continuously differentiable friction curve parametrization that, along the lines of [10], uses the hyperbolic tangent, but remains very close to the original parametrization (39), namely

$$\begin{aligned} F(v_r) = \Bigg[&\left(\mu_C + (\mu_S - \mu_C) e^{-\left(\frac{v_r}{v_S} \tanh(c v_r)\right)^\alpha} \right) \tanh(c v_r) \\ &+ \sigma_2 v_r \Bigg] F_n, \end{aligned} \tag{43}$$

for some constant $c \gg 1$. This static friction curve parametrization can now be inserted into the LuGre friction model by changing the conductance term according to

$$\hat{g}_0 = \frac{v_r \tanh(c v_r)}{F_n \left[\mu_C + (\mu_S - \mu_C) e^{-\left(\frac{v_r}{v_S} \tanh(c v_r)\right)^\alpha} \right]}. \tag{44}$$

It should be noted that the continuous transition through zero removes the pure stiction effect from the model. The resulting creeping motion, however, can be made arbitrarily small. Moreover, it is known that the pre-sliding regime is in any case not accurately described by the LuGre model [13].

VII. Conclusion

In this paper we have presented a port-Hamiltonian (PH) description of the LuGre friction model.

Having a PH description of the LuGre model enabled us to assess the dissipativity conditions in a straightforward manner and we gave an alternative proof for the results in [2]. Moreover, we showed that the passivity conditions can be given in case of any constant normal force between the surfaces in contact.

As a specific example of the use of the PH Lugre model, we presented an interconnection with a mass. It was shown that the lossless interconnection structure and dissipation structure of the port-Hamiltonian LuGre model are consistent with those of the interconnection, which are in turn uniquely determined by the choice of the Hamiltonian.

Additionally, to render the friction model continuously differentiable, a smooth re-parametrization of the friction curve is proposed that extends and simplifies the results of [10].

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